EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Wednesday April 22, 2009. Time: 14h00–17h00. Place: HG 10.01 C.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student identification number on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes and calculator is allowed. The use of the problem companion, "opgaven- en tentamenbundel", is *not* allowed.
- You may provide your answers in Dutch or (preferably) in English.

GOOD LUCK!

(**35**) **1.** Consider the collection of square matrices,

$$\mathbb{M}_n = \left\{ X = \begin{pmatrix} X_{11} & \dots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \dots & X_{nn} \end{pmatrix} \middle| \; X_{ij} \in \mathbb{K} \right\}, \text{ in which } \mathbb{K} \text{ denotes either } \mathbb{R} \text{ or } \mathbb{C}.$$

($2\frac{1}{2}$) **a.** Provide explicit definitions for the operators $\otimes : \mathbb{K} \times \mathbb{M}_n \to \mathbb{M}_n$ ("scalar multiplication") and $\oplus : \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{M}_n$ ("vector addition") needed to turn this set into a linear space over $\mathbb{K}$. Make sure to use parentheses so as to avoid confusion on operator precedence, if necessary.

The necessary operators are scalar multiplication and vector addition. If $\lambda, \mu \in \mathbb{K}$ and $X, Y \in \mathbb{M}_n$, then

$$((\lambda \otimes X) \oplus (\mu \otimes Y))_{ij} = \lambda X_{ij} + \mu Y_{ij} \quad \text{for all } i, j = 1, \dots, n.$$

Henceforth we write λX instead of $\lambda \otimes X$ and $X + Y$ instead of $X \oplus Y$ for $\lambda \in \mathbb{K}$ and $X, Y \in \mathbb{M}_n$. Furthermore, let $\mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$ denote the linear space of linear operators on $\mathbb{M}_n$.

($2\frac{1}{2}$) **b.** Provide explicit definitions for the operators $\otimes : \mathbb{R} \times \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n) \to \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$ and $\oplus : \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n) \times \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n) \to \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$ that justifies the claim that $\mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$ is a linear space over $\mathbb{K}$. Make sure to use parentheses so as to avoid confusion on operator precedence, if necessary.

The necessary operators are scalar multiplication and vector addition. If $\lambda, \mu \in \mathbb{K}$ and $A, B \in \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$, then

$$((\lambda \otimes A) \oplus (\mu \otimes B))(X) = \lambda A(X) + \mu B(X) \quad \text{for all } X \in \mathbb{M}_n.$$

Again we write λA instead of $\lambda \otimes A$ and $A+B$ instead of $A \oplus B$ for $\lambda \in \mathbb{K}$ and $A, B \in \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$. With X as above, the transposed matrix X^{T} and the conjugate matrix $X^{\dagger}$ are defined as

$$X^{\mathrm{T}} = \begin{pmatrix} X_{11} & \dots & X_{n1} \\ \vdots & & \vdots \\ X_{1n} & \dots & X_{nn} \end{pmatrix} \quad \text{respectively} \quad X^{\dagger} = \begin{pmatrix} X_{11}^* & \dots & X_{n1}^* \\ \vdots & & \vdots \\ X_{1n}^* & \dots & X_{nn}^* \end{pmatrix} .$$

Here, $*$ denotes complex conjugation, i.e. if $z = x + iy$ for $x, y \in \mathbb{R}$, then $z^* = x - iy$.

Furthermore, the operators $\mathrm{P}_{\pm} : \mathbb{M}_n \to \mathbb{M}_n$ and $\mathrm{Q}_{\pm} : \mathbb{M}_n \to \mathbb{M}_n$ are defined by

$$\mathrm{P}_{\pm}(X) = \frac{1}{2}\left(X \pm X^{\mathrm{T}}\right) \quad \text{respectively} \quad \mathrm{Q}_{\pm}(X) = \frac{1}{2}\left(X \pm X^{\dagger}\right) .$$

($2\frac{1}{2}$) **c1.** Show that *matrix transposition*, $\mathrm{T} : \mathbb{M}_n \to \mathbb{M}_n : X \mapsto \mathrm{T}(X) \stackrel{\text{def}}{=} X^{\mathrm{T}}$, is a linear operator.

For all $\lambda, \mu \in \mathbb{K}$ and $X, Y \in \mathbb{M}_n$ we have $\mathrm{T}(\lambda X + \mu Y) \stackrel{\text{def}}{=} (\lambda X + \mu Y)^{\mathrm{T}} = \lambda X^{\mathrm{T}} + \mu Y^{\mathrm{T}} \stackrel{\text{def}}{=} \lambda \mathrm{T}(X) + \mu \mathrm{T}(Y)$.

($2\frac{1}{2}$) **c2.** Show that *matrix conjugation*, $\mathrm{C} : \mathbb{M}_n \to \mathbb{M}_n : X \mapsto \mathrm{C}(X) \stackrel{\text{def}}{=} X^{\dagger}$, is not a linear operator.

For all $\lambda, \mu \in \mathbb{K}$ and $X, Y \in \mathbb{M}_n$ we have $\mathrm{C}(\lambda X + \mu Y) \stackrel{\text{def}}{=} (\lambda X + \mu Y)^{\dagger} = \lambda^* X^{\dagger} + \mu^* Y^{\dagger} \stackrel{\text{def}}{=} \lambda^* \mathrm{C}(X) + \mu^* \mathrm{C}(Y) \neq \lambda \mathrm{C}(X) + \mu \mathrm{C}(Y)$. (If $\mathbb{K} = \mathbb{R}$ then $\mathrm{C} = \mathrm{T}$, so then it does define a linear operator.)

($2\frac{1}{2}$) **c3.** Show that $\mathrm{P}_{\pm} : \mathbb{M}_n \to \mathbb{M}_n$ are linear operators.

For all $\lambda, \mu \in \mathbb{K}$ and $X, Y \in \mathbb{M}_n$ we have $\mathrm{P}_{\pm}(\lambda X + \mu Y) \stackrel{\text{def}}{=} \frac{1}{2}\left(\lambda X + \mu Y \pm (\lambda X + \mu Y)^{\mathrm{T}}\right) = \lambda\left(\frac{1}{2}(X \pm X^{\mathrm{T}})\right) + \mu\left(\frac{1}{2}(Y \pm Y^{\mathrm{T}})\right) \stackrel{\text{def}}{=} \lambda \mathrm{P}_{\pm}(X) + \mu \mathrm{P}_{\pm}(Y)$.

($2\frac{1}{2}$) **c4.** Show that $\mathrm{Q}_{\pm} : \mathbb{M}_n \to \mathbb{M}_n$ are not linear operators.

For all $\lambda, \mu \in \mathbb{K}$ and $X, Y \in \mathbb{M}_n$ we have $\mathrm{Q}_{\pm}(\lambda X + \mu Y) \stackrel{\text{def}}{=} \frac{1}{2}\left(\lambda X + \mu Y \pm (\lambda X + \mu Y)^{\dagger}\right) = \frac{1}{2}(\lambda X \pm \lambda^* X^{\dagger}) + \frac{1}{2}(\mu Y \pm \mu^* Y^{\dagger}) \neq \lambda\left(\frac{1}{2}(X \pm X^{\dagger})\right) + \mu\left(\frac{1}{2}(Y \pm Y^{\dagger})\right) \stackrel{\text{def}}{=} \lambda \mathrm{Q}_{\pm}(X) + \mu \mathrm{Q}_{\pm}(Y)$. (If $\mathbb{K} = \mathbb{R}$ then $\mathrm{Q}_{\pm} = \mathrm{P}_{\pm}$, so then they do define linear operators.)

The null matrix in $\mathbb{M}_n$ is indicated by Ω, i.e. $\Omega_{ij} = 0$ for all $i, j = 1, \dots, n$. The identity matrix in $\mathbb{M}_n$ is indicated by I, i.e. $I_{ij} = 1$ if $i = j = 1, \dots, n$, otherwise $I_{ij} = 0$. The null operator $\mathrm{N} : \mathbb{M}_n \to \mathbb{M}_n$ is defined by $\mathrm{N}(X) = \Omega \in \mathbb{M}_n$ for all $X \in \mathbb{M}_n$. The identity operator $\mathrm{id} : \mathbb{M}_n \to \mathbb{M}_n$ is defined by $\mathrm{id}(X) = X$ for all $X \in \mathbb{M}_n$. Operator composition (i.e. successive application of operators in right-to-left order) is indicated by the infix operator $\circ$.

($2\frac{1}{2}$) **d1.** Show that $\mathrm{P}_+ + \mathrm{P}_- = \mathrm{id}$.

We have $(\mathrm{P}_+ + \mathrm{P}_-)(X) \stackrel{\text{def}}{=} \mathrm{P}_+(X) + \mathrm{P}_-(X) \stackrel{\text{def}}{=} \frac{1}{2}\left(X + X^{\mathrm{T}}\right) + \frac{1}{2}\left(X - X^{\mathrm{T}}\right) = X = \mathrm{id}(X)$ for all $X \in \mathbb{M}_n$, so $\mathrm{P}_+ + \mathrm{P}_- = \mathrm{id}$.

($2\frac{1}{2}$) **d2.** Show that $\mathrm{P}_+ - \mathrm{P}_- = \mathrm{T}$.

We have $(\mathrm{P}_+ - \mathrm{P}_-)(X) \stackrel{\text{def}}{=} \mathrm{P}_+(X) - \mathrm{P}_-(X) \stackrel{\text{def}}{=} \frac{1}{2}\left(X + X^{\mathrm{T}}\right) - \frac{1}{2}\left(X - X^{\mathrm{T}}\right) = X^{\mathrm{T}} = \mathrm{T}(X)$ for all $X \in \mathbb{M}_n$, so $\mathrm{P}_+ + \mathrm{P}_- = \mathrm{T}$.

($2\frac{1}{2}$) **d3.** Show that $\mathrm{P}_+ \circ \mathrm{P}_- = \mathrm{P}_- \circ \mathrm{P}_+ = \mathrm{N}$.

We have $(\mathrm{P}_+ \circ \mathrm{P}_-)(X) \stackrel{\text{def}}{=} \mathrm{P}_+(\mathrm{P}_-(X)) \stackrel{\text{def}}{=} \frac{1}{2}\left(\mathrm{P}_-(X) + \mathrm{P}_-(X)^{\mathrm{T}}\right) = \frac{1}{2}\left(\frac{1}{2}\left(X - X^{\mathrm{T}}\right) + \frac{1}{2}\left(X^{\mathrm{T}} - X\right)\right) = 0$ for all $X \in \mathbb{M}_n$, so $\mathrm{P}_+ \circ \mathrm{P}_- = \mathrm{N}$. By interchanging all $+$ and $-$ signs in the above argument, it follows that $\mathrm{P}_- \circ \mathrm{P}_+ = \mathrm{N}$.

($2\frac{1}{2}$) **d4.** Show that $\mathrm{P}_\pm \circ \mathrm{P}_\pm = \mathrm{P}_\pm$.

We have $(\mathrm{P}_\pm \circ \mathrm{P}_\pm)(X) \stackrel{\text{def}}{=} \mathrm{P}_\pm(\mathrm{P}_\pm(X)) \stackrel{\text{def}}{=} \frac{1}{2}\left(\mathrm{P}_\pm(X) \pm \mathrm{P}_\pm(X)^{\mathrm{T}}\right) = \frac{1}{2}\left(\frac{1}{2}\left(X \pm X^{\mathrm{T}}\right) \pm \frac{1}{2}\left(X^{\mathrm{T}} \pm X\right)\right) = \frac{1}{2}\left(X \pm X^{\mathrm{T}}\right) \stackrel{\text{def}}{=} \mathrm{P}_\pm(X)$ for all $X \in \mathbb{M}_n$, so $\mathrm{P}_\pm \circ \mathrm{P}_\pm = \mathrm{P}_\pm$.

Consider the following binary operator: $\langle\,|\,\rangle : \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{K} : (X, Y) \mapsto \langle X|Y\rangle \stackrel{\text{def}}{=} \mathrm{trace}(X^{\mathrm{T}}Y)$. Here, $\mathrm{trace} : \mathbb{M}_n \to \mathbb{K}$ is the (linear) *trace* operator, defined as summation of diagonal elements:

$$\mathrm{trace} X = \sum_{i=1}^{n} X_{ii}\,.$$

($2\frac{1}{2}$) **e1.** Show that if $\mathbb{K} = \mathbb{R}$ then $\langle\,|\,\rangle : \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{R}$ defines a real inner product.

We may exploit the fact that both trace as well as transposition T are linear operators. Let $X, Y, Z \in \mathbb{M}_n$ and $\lambda, \mu \in \mathbb{K}$ be arbitrary. Then we have

- $\langle X|Y\rangle \stackrel{\text{def}}{=} \mathrm{trace}(X^{\mathrm{T}}Y) \stackrel{\text{def}}{=} \sum_{i=1}^n (X^{\mathrm{T}}Y)_{ii} = \sum_{i=1}^n \sum_{j=1}^n X_{ij}Y_{ij} = \sum_{i=1}^n (Y^{\mathrm{T}}X)_{ii} = \langle Y|X\rangle$.
- $\langle \lambda X + \mu Y|Z\rangle \stackrel{\text{def}}{=} \mathrm{trace}((\lambda X + \mu Y)^{\mathrm{T}}Z) = \mathrm{trace}((\lambda X^{\mathrm{T}} + \mu Y^{\mathrm{T}})Z) = \lambda \mathrm{trace}(X^{\mathrm{T}}Z) + \mu \mathrm{trace}(Y^{\mathrm{T}}Z) \stackrel{\text{def}}{=} \lambda\langle X|Z\rangle + \mu\langle Y|Z\rangle$.
- $\langle X|X\rangle \stackrel{\text{def}}{=} \mathrm{trace}(X^{\mathrm{T}}X) \stackrel{\text{def}}{=} \sum_{i=1}^n (X^{\mathrm{T}}X)_{ii} = \sum_{i=1}^n \sum_{j=1}^n X_{ij}X_{ij} \geq 0$, and clearly the sum of squares equals zero iff $X = \Omega$, the zero matrix.

($2\frac{1}{2}$) **e2.** Show that if $\mathbb{K} = \mathbb{C}$ then $\langle\,|\,\rangle : \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{C}$ does not define a complex inner product.

Take $\lambda \in \mathbb{C}\backslash\mathbb{R}$, and $X, Y \in \mathbb{M}_n$, then $\langle \lambda X|Y\rangle \stackrel{\text{def}}{=} \mathrm{trace}((\lambda X)^{\mathrm{T}}Y) = \mathrm{trace}(\lambda X^{\mathrm{T}}Y) = \lambda \mathrm{trace}(X^{\mathrm{T}}Y) \stackrel{\text{def}}{=} \lambda\langle X|Y\rangle \neq \lambda^*\langle X|Y\rangle$.

($2\frac{1}{2}$) **f.** How would you modify the definition of $\langle\,|\,\rangle : \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{C}$ in the complex case, $\mathbb{K} = \mathbb{C}$, such that it does define a complex inner product? You may state your definition without proof.

We must use conjugation instead of transposition: $\langle\,|\,\rangle : \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{C} : (X, Y) \mapsto \langle X|Y\rangle \stackrel{\text{def}}{=} \mathrm{trace}(X^\dagger Y)$.

In the remainder of this problem we restrict ourselves to $\mathbb{K} = \mathbb{R}$. In particular, we consider the case of the real inner product, recall **e1**.

Operator transposition, $\mathrm{T} : \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n) \to \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$, is implicitly defined by the identity

$$\langle A^{\mathrm{T}}(X)|Y\rangle \stackrel{\text{def}}{=} \langle X|A(Y)\rangle \quad \text{for all } A \in \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n) \text{ and } X, Y \in \mathbb{M}_n.$$

($2\frac{1}{2}$) **g.** Show that $\mathrm{P}_\pm^{\mathrm{T}} = \mathrm{P}_\pm$. (Together with **d4** this shows that $\mathrm{P}_\pm$ are orthogonal projections.)

Let $X, Y \in \mathbb{M}_n$. Using the properties of the inner product and previous definitions we find $\langle X|\mathrm{P}_\pm(Y)\rangle \stackrel{\text{def}}{=} \langle X|\frac{1}{2}(Y \pm Y^{\mathrm{T}})\rangle = \frac{1}{2}\langle X|Y\rangle \pm \frac{1}{2}\langle X|Y^{\mathrm{T}}\rangle \stackrel{*}{=} \frac{1}{2}\langle X|Y\rangle \pm \frac{1}{2}\langle X^{\mathrm{T}}|Y\rangle = \langle \frac{1}{2}(X \pm X^{\mathrm{T}})|Y\rangle \stackrel{\text{def}}{=} \langle \mathrm{P}_\pm(X)|Y\rangle$. In $*$ we have used the fact that $\langle X|Y^{\mathrm{T}}\rangle = \mathrm{trace}(X^{\mathrm{T}}Y^{\mathrm{T}}) = \mathrm{trace}((YX)^{\mathrm{T}}) = \mathrm{trace}(YX) = \mathrm{trace}(XY) = \langle X^{\mathrm{T}}|Y\rangle$, which in turn follows from the properties $\mathrm{trace}(X) = \mathrm{trace}(X^{\mathrm{T}})$, $\mathrm{trace}(XY) = \mathrm{trace}(YX)$, and $X^{\mathrm{TT}} = X$.

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($32\frac{1}{2}$) 2. Consider the Laplace equation for the function $u \in C^\infty(\mathbb{R}^2)$:

$$\Delta u = 0 \quad \text{in which } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \,.$$

Our aim is to find a first order partial differential equation, such that its solutions also satisfy the Laplace equation. To this end we introduce a real linear space V, the dimension $n \geq 2$ of which is yet to be determined, and furnish it with an additional operator henceforth referred to as "multiplication". The product of $v, w \in V$ is then simply written as $vw \in V$. In this way V is turned into a so-called algebra, for which we stipulate the following algebraic axioms, viz. for all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$:

- $(uv)w = u(vw)$,
- $u(v + w) = uv + uw$,
- $(u + v)w = uw + vw$,
- $\lambda(uv) = (\lambda u)v = u(\lambda v)$,

(Multiplication takes precedence over vector addition unless parentheses indicate otherwise.)

We now attempt to decompose the Laplacian operator as follows:

$$\Delta = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) .$$

Here $a, b \in V$ are two fixed, independent elements (vectors). For consistency we assume that $u(x, y) \in V$ (instead of our original assumption $u(x, y) \in \mathbb{R}$).

a. Show that V is *not* commutative, and that it must possess an identity element $1 \in V$ that has to be formally identified with the scalar number $1 \in \mathbb{R}$, by showing that

($2\frac{1}{2}$) **a1.** $ab + ba = 0$,

($2\frac{1}{2}$) **a2.** $a^2 = b^2 = 1$.

Expanding

$$\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) = a^2\frac{\partial^2}{\partial x^2} + (ab + ba)\frac{\partial^2}{\partial x \partial y} + b^2\frac{\partial^2}{\partial y^2} \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

shows that we must set $ab + ba = 0$ and $a^2 = b^2 = 1$. Notice that we may *not* assume that $ab = ba$.

(5) **b.** Show that the (unordered) pair (a, b) satisfying the conditions of **a1** and **a2** is not unique. (*Hint:* Suppose $a' = a_1 a + a_2 b$, $b' = b_1 a + b_2 b$ for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$.)

Following the hint we subject the transformed pair (a', b') to the required conditions of **a1**–**a2**. This leads to the following system of equations for the coefficients $a_1, a_2, b_1, b_2 \in \mathbb{R}$:

$$\begin{cases} a_1^2 + a_2^2 &= 1 \\ b_1^2 + b_2^2 &= 1 \\ a_1 b_1 + a_2 b_2 &= 0 \,. \end{cases}$$

In other words, apart from the choice $(a_1, a_2) = (1, 0)$, $(b_1, b_2) = (0, 1)$ corresponding to the solution under **a1–a2** we may take any pair of mutually perpendicular unit vectors $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$. (This ambiguity is not surprising, since the Laplacian is invariant under orthogonal transformations of (x, y)-coordinates.)

c. Suppose $v, w \in \text{span}\{a, b\} \subset V$ are such that, say, $v = v_1 a + v_2 b$ and $w = w_1 a + w_2 b$ for some $v_1, v_2, w_1, w_2 \in \mathbb{R}$. Compute

($2\frac{1}{2}$) **c1.** $vw + wv$,

($2\frac{1}{2}$) **c2.** v^2.

We find $vw + wv = (v_1 a + v_2 b)(w_1 a + w_2 b) = 2v_1 w_1 a^2 + (v_1 w_2 + v_2 w_1)(ab + ba) + 2v_2 w_2 b^2 \stackrel{\textbf{a1–a2}}{=} 2(v_1 w_1 + v_2 w_2)$, from which it follows, by setting $v_1 = w_1$ and $v_2 = w_2$, that $v^2 = v_1^2 + v_2^2$.

($2\frac{1}{2}$) **d.** Show that $\dim V > 2$.
(*Hint:* Alternatively, show that $\text{span}\{a, b\} \subset V$ is not closed under multiplication.)

Take $v = v_1 a + v_2 b \in V$, with $v_1, v_2 \in \mathbb{R}$ arbitrary. Suppose $1 \in \text{span}\{a, b\}$, i.e. there exist $e_1, e_2 \in \mathbb{R}$, at least one of which is nonzero, such that $1 = e_1 a + e_2 b$, then, again using **a1–a2** and the definition of the unit element $1 \in V$,

$$v_1 a + v_2 b = v \stackrel{\text{def}}{=} 1v = e_1 v_1 + e_2 v_2 + (e_1 v_2 - e_2 v_1) ab \quad \text{for all } v_1, v_2 \in \mathbb{R}.$$

However, $a, b, ab \notin \mathbb{R}$, whereas $e_1 v_1 + e_2 v_2 \in \mathbb{R}$, so that we conclude that $e_1 v_1 + e_2 v_2 = 0$ for all $v_1, v_2 \in \mathbb{R}$, implying $e_1 = e_2 = 0$, which is a contradiction. (To see that $a, b, ab \notin \mathbb{R}$, consider **a1–a2**, from which it is immediately obvious that $a, b \notin \mathbb{R}$. That $ab \notin \mathbb{R}$ follows e.g. by taking its square: $(ab)^2 = abab = -ab^2 a = -a^2 = -1$.)

We add two more independent elements to the set $\{a, b\}$, viz. the unit element 1 and the element ab, and define $V = \text{span}\{1, a, b, ab\}$. Instead of $\lambda\, 1 + \mu\, a + \nu\, b + \rho\, ab$ $(\lambda, \mu, \nu, \rho \in \mathbb{R})$ we write $\lambda + \mu\, a + \nu\, b + \rho\, ab$ for an arbitrary element of V.

(5) **e.** Show that V is closed under multiplication, i.e. show that if $v = v_0 + v_1 a + v_2 b + v_3 ab \in V$, $w = w_0 + w_1 a + w_2 b + w_3 ab \in V$, then also $vw \in V$.

The easiest answer is as follows. Multiplication of v and w, expanded as given, will generate $4 \times 4 = 16$ terms, each of which is a multiple of any of the following product forms: $1, a, b, ab, a^2, b^2, ba, a^2 b, ab^2, aba, bab, (ab)^2$. With the properties **a1–a2** of a and b these can all be reduced to multiples involving only $1, a, b, ab$ (for we have $a^2 = 1$, $b^2 = 1$, $ba = -ab$, $a^2 b = b$, $ab^2 = a$, $aba = -a^2 b = -b$, $bab = -b^2 a = -a$, and finally $(ab)^2 = -ab^2 a = -a^2 = -1$). The more cumbersome answer is to write out the product explicitly, and to apply the above simplification rules, yielding $vw = (v_0 + v_1 a + v_2 b + v_3 ab)(w_0 + w_1 a + w_2 b + w_3 ab) = v_0 w_0 + v_1 w_1 + v_2 w_2 - v_3 w_3 + (v_0 w_1 + v_1 w_0 - v_2 w_3 + v_3 w_2) a + (v_0 w_2 + v_1 w_3 + v_2 w_0 - v_3 w_1) b + (v_0 w_3 + v_1 w_2 - v_2 w_1 + v_3 w_0) ab \in V$.

We now try to realize elements of V in terms of real-valued 2×2-matrices. To this end we hypothesize that

$$V \subset \mathbb{M}_2 = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \,\middle|\, a_{ij} \in \mathbb{R} \right\}.$$

(5) **f.** Construct explicit matrices $A, B \in \mathbb{M}_2$ corresponding to $a, b \in V$ in the sense that

- $AB + BA = \Omega$,

- $A^2 = B^2 = I$.

Here $\Omega \in \mathbb{M}_2$ is the null matrix, corresponding to the null element $0 \in V$, and $I \in \mathbb{M}_2$ is the identity matrix, corresponding to the identity element $1 \in V$.
(*Hint:* As an ansatz, stipulate a diagonal matrix A, and show that B must then be anti-diagonal.)

Some examples of $a \sim A$ and $b \sim B$, and $ab \sim AB$, and the element $1 \sim I$:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Many other choices are possible, recall **b**. Examples such as the above are found by trial and error. Following the hint ($a_{12} = a_{21} = 0$), setting $A = \text{diag}\{a_{11}, a_{22}\}$, and taking B arbitrary, one finds by writing out the conditions $A^2 = I$ and $AB + BA = \Omega$ that $a_{11} = \pm 1$, $a_{22} = \mp 1$, and that B must have zeros on its diagonal, and writing out $B^2 = I$ reveals that its remaining elements must equal $b_{12} = b_{21} = \pm 1$.

($2\frac{1}{2}$) **g.** Given A and B as determined under **f**, what is the matrix form $v_0 + v_1 A + v_2 B + v_3 AB \in \mathbb{M}_2$ corresponding to a general element $v_0 + v_1 a + v_2 b + v_3 ab \in V$?

From **f** it follows by superposition $X = v_0 I + v_1 A + v_2 B + v_3 AB$ that

$$X = \begin{pmatrix} v_0 + v_1 & v_2 + v_3 \\ v_2 - v_3 & v_0 - v_1 \end{pmatrix} .$$

Note that *any* matrix $Y \in \mathbb{M}_2$ can be realized in this way, viz. by setting

$$v_0 = \frac{1}{2}(Y_{11} + Y_{22}) \quad v_1 = \frac{1}{2}(Y_{11} - Y_{22}) \quad v_2 = \frac{1}{2}(Y_{12} + Y_{21}) \quad v_3 = \frac{1}{2}(Y_{12} - Y_{21}) .$$

($2\frac{1}{2}$) **h.** Show that if $u : \mathbb{R}^2 \to V$ satisfies the first order partial differential equation

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0 ,$$

then it is also a solution of $\Delta u = 0$.

Recall **a1–a2**. By construction we now have

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\left(a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}\right) = 0 \quad \text{since} \quad \left(a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}\right) = 0 .$$

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(**5**) **3.** Consider a strictly monotonic, continuously differentiable function $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$, with $f'(x) > 0$ for all $x \in \mathbb{R}$, and $f(\pm\infty) = \pm\infty$. The inverse function theorem states that such a function has an inverse, $f^{-1} : \mathbb{R} \to \mathbb{R} : y \mapsto f^{-1}(y)$, such that $(f^{-1} \circ f)(x) = x$ for all $x \in \mathbb{R}$, and $(f \circ f^{-1})(y) = y$ for all $y \in \mathbb{R}$.

($2\frac{1}{2}$) **a.** Argue why the equation $f(x) = 0$ has precisely one solution for $x \in \mathbb{R}$ ($x = a$, say).
(*Hint:* Sketch the graph of such a function f.)

Suppose there are two distinct solutions, $f(a) = f(b) = 0$, with $a < b$, say. According to the mean value theorem there exists a c in-between a and b, such that $f'(c) = (f(b) - f(a))/(b - a) = 0$ (average slope on (a, b)). This is a contradiction. Rephrased in terms of the graph of f, the argument is as follows. Since f is continuous, the graph of f is uninterrupted. Since $f' > 0$, the graph is monotonically increasing, so it can intersect the x-axis at most once. It actually does intersect the x-axis exactly once, since f is continuous and f assumes both negative and positive values, whence it must assume all intermediate values, in particular $f(a) = 0$ for some unique $a \in \mathbb{R}$.

($2\frac{1}{2}$) **b.** Show that $\delta(f(x)) = \dfrac{\delta(x-a)}{f'(a)}$, in which $a \in \mathbb{R}$ is the unique point for which $f(a) = 0$.
(*Hint:* Evaluate the distribution corresponding to the Dirac function on the left hand side on an arbitrary test function $\phi \in \mathscr{S}(\mathbb{R})$, i.e. $\int_{-\infty}^{\infty} \delta(f(x))\, \phi(x)\, dx$, and apply substitution of variables.)

Consider

$$\int_{-\infty}^{\infty} \delta(f(x))\, \phi(x)\, dx \stackrel{*}{=} \int_{-\infty}^{\infty} \delta(y)\, \phi(f^{-1}(y))\, \frac{dy}{f'(f^{-1}(y))} \stackrel{\star}{=} \frac{\phi(f^{-1}(0))}{f'(f^{-1}(0))} \stackrel{\circ}{=} \frac{\phi(a)}{f'(a)} \stackrel{\star}{=} \int_{-\infty}^{\infty} \frac{\delta(x-a)}{f'(a)}\, \phi(x)\, dx\,.$$

In $*$ substitution $y = f(x)$ has been carried out, which implies $dy = f'(x)\, dx$ and, by the inverse function theorem, $x = f^{-1}(y)$. Boundary values remain unchanged because $f(\pm\infty) = \pm\infty$. In $\star$ the definition of the Dirac-δ "function" has been used. In $\circ$ the identity $a = f^{-1}(0)$ (the equivalent of $f(a) = 0$, cf. **a**) has been invoked. Since this holds for all $\phi \in \mathscr{S}(\mathbb{R})$ we conclude that $\delta(f(x)) = \dfrac{\delta(x-a)}{f'(a)}$.

♣

(**$27\frac{1}{2}$**) **4.** In this problem we consider a synthetic signal $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined as follows:

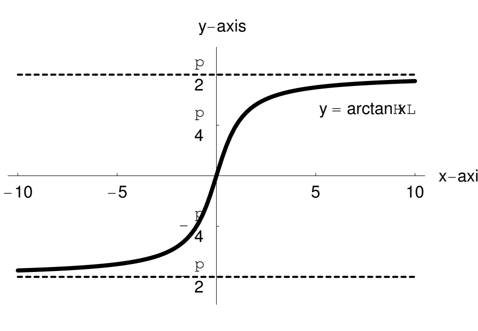
$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Furthermore, we define the so-called Poisson filter $\phi_\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ for $\sigma > 0$ as

$$\phi_\sigma(x) = \frac{1}{\pi} \frac{\sigma}{x^2 + \sigma^2}\,.$$

In this problem you may use the following standard formulas (cf. the graph shown below):

$$\int \frac{1}{1+x^2}\, dx = \arctan x + c \qquad \text{resp.} \qquad \lim_{x \to \pm\infty} \arctan x = \pm\frac{\pi}{2}\,.$$



Throughout this problem we employ the following Fourier convention:

$$\widehat{u}(\omega) = \mathcal{F}(u)(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}\, u(x)\, dx \quad \text{whence} \quad u(x) = \mathcal{F}^{-1}(\widehat{u})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\, \widehat{u}(\omega)\, d\omega\,.$$

($2\frac{1}{2}$) **a.** Show that $\int_{-\infty}^{\infty} \phi_\sigma(x)\, dx = 1$ regardless of the value of σ.

$\int_{-\infty}^{\infty} \phi_\sigma(x)\, dx = \frac{\sigma}{\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+\sigma^2}\, dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+y^2}\, dy = \frac{1}{\pi}\left[\arctan y\right]_{y\to-\infty}^{y\to+\infty} = \frac{1}{\pi}(\frac{\pi}{2}+\frac{\pi}{2}) = 1.$ Hierin is de volgende substitutie van variabelen toegepast: $y = x/\sigma$.

($2\frac{1}{2}$) **b1.** Prove that for any, sufficiently smooth, integrable filter ϕ we have:

$$(f * \phi)(x) = \int_{-\infty}^{x} \phi(\xi)\, d\xi\,.$$

$(f * \phi)(x) = \int_{-\infty}^{\infty} f(y)\, \phi(x-y)\, dy = \int_0^{\infty} \phi(x-y)\, dy = \int_{-\infty}^{x} \phi(\xi)\, d\xi$. In de laatste stap is substitutie van variabelen, $\xi = x - y$, toegepast.

($2\frac{1}{2}$) **b2.** Show by explicit computation that the convolution product $f * \phi_\sigma$ is given by

$$(f * \phi_\sigma)(x) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{x}{\sigma}\,.$$

$(f * \phi_\sigma)(x) = \int_{-\infty}^{\infty} f(y)\, \phi_\sigma(x-y)\, dy = \frac{\sigma}{\pi}\int_0^{\infty} \frac{1}{(x-y)^2+\sigma^2}\, dy = \int_{-\infty}^{x/\sigma} \frac{1}{1+\xi^2}\, d\xi = \frac{1}{\pi}\left[\arctan \xi\right]_{\xi\to-\infty}^{\xi=x/\sigma} = \frac{1}{\pi}(\arctan\frac{x}{\sigma} + \frac{\pi}{2}) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{x}{\sigma}$. Hierin is de volgende substitutie van variabelen toegepast: $\xi = (x-y)/\sigma$. Gebruik van onderdeel **b1** leidt uiteraard tot hetzelfde resultaat.

(5) **c.** Show that the Fourier transform $\widehat{f} = \mathcal{F}(f)$ of f is given by

$$\widehat{f}(\omega) = \frac{1}{i\omega}\,.$$

(*Hint:* From part **b1** it follows that $\frac{d}{dx}(f * \phi)(x) = \phi(x)$. Subject this to Fourier transformation.)

Uit $\frac{d}{dx}(f * \phi_\sigma)(x) = \phi_\sigma(x)$ volgt dat $\mathcal{F}\left(\frac{d}{dx}(f * \phi_\sigma)\right)(\omega) = \mathcal{F}(\phi_\sigma)(\omega)$. Met behulp van Result 6 op blz. 97 en Theorem 17 op blz. 104 volgt dat dit equivalent is met $i\omega\mathcal{F}(f)(\omega)\mathcal{F}(\phi_\sigma)(\omega) = \mathcal{F}(\phi_\sigma)(\omega)$. Ervan uitgaand dat $\mathcal{F}(\phi_\sigma)(\omega) \neq 0$ voor (bijna) alle ω concluderen we dat $\mathcal{F}(f)(\omega) = \frac{1}{i\omega}$.

d. Without proof we state the Fourier transform $\widehat{\phi}_\sigma = \mathcal{F}(\phi_\sigma)$ of ϕ_σ:

$$\widehat{\phi}_\sigma(\omega) = e^{-\sigma|\omega|}\,.$$

($2\frac{1}{2}$) **d1.** Explain the behaviour of $\widehat{\phi}_\sigma(\omega)$ for $\omega \to 0$ in terms of properties of the corresponding spatial filter $\phi_\sigma(x)$.
(*Hint:* Cf. problem **a**.)

($2\frac{1}{2}$) **d2.** Explain the behaviour of $\widehat{f}(\omega)$ for $\omega \to 0$ in terms of properties of the corresponding spatial signal $f(x)$.

Zie de definitie van $\widehat{u}(\omega)$: Indien $\widehat{u}(0)$ bestaat dan moet hiervoor kennelijk gelden $\widehat{u}(0) = \int_{-\infty}^{\infty} u(x)\, dx$. Dit verklaart zowel het gedrag van $\widehat{\phi}_\sigma(\omega)$ als van $\widehat{f}(\omega)$ voor $\omega \to 0$: $\widehat{\phi}_\sigma(0) = \int_{-\infty}^{\infty} \phi_\sigma(x)\, dx = 1$ (zie **a**), terwijl $\widehat{f}(0)$ niet bestaat omdat $\int_{-\infty}^{\infty} f(x)\, dx = \int_0^\infty dx$ niet convergeert.

(5) **e.** Determine the function $\mathcal{F}(f * \phi_\sigma)$.

Noteren we gemakshalve $f_\sigma = f * \phi_\sigma$ dan geldt hiervoor in de gehanteerde Fourierconventie $\widehat{f_\sigma} = \mathcal{F}(f * \phi_\sigma) = \mathcal{F}(f)\,\mathcal{F}(\phi_\sigma) = \widehat{f}\,\widehat{\phi}_\sigma$, dus $\widehat{f_\sigma}(\omega) = \frac{1}{i\omega}\, e^{-\sigma|\omega|}$.

($2\frac{1}{2}$) **f.** Show that $\lim\limits_{\sigma\to 0} f * \phi_\sigma = f$. Hint: Take the "Fourier route".

Schrijf wederom $f_\sigma = f * \phi_\sigma$. Uit **e** blijkt dat in het Fourierdomein geldt $\lim_{\sigma\to 0} \widehat{f_\sigma}(\omega) = \lim_{\sigma\to 0} \frac{1}{i\omega}\, e^{-\sigma|\omega|} = \frac{1}{i\omega} = \widehat{f}(\omega)$, dus volgt $\lim_{\sigma\to 0} f_\sigma(x) = f(x)$.

($2\frac{1}{2}$) **g.** Prove the claim under **d**: $\widehat{\phi}_\sigma(\omega) = e^{-\sigma|\omega|}$.
(*Hint:* Apply the inverse Fourier transform.)

Following the hint we get

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\, e^{-\sigma|\omega|}\, d\omega &= \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{(\sigma+ix)\omega}\, d\omega + \int_0^\infty e^{(-\sigma+ix)\omega}\, d\omega\right) = \frac{1}{2\pi}\left(\left[\frac{e^{(\sigma+ix)\omega}}{\sigma+ix}\right]_{-\infty}^{0} + \left[\frac{e^{(-\sigma+ix)\omega}}{-\sigma+ix}\right]_0^{\infty}\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{\sigma+ix} + \frac{1}{\sigma-ix}\right) = \frac{1}{\pi}\,\frac{\sigma}{x^2+\sigma^2} \stackrel{\text{def}}{=} \phi_\sigma(x)\,.
\end{aligned}$$

THE END

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