EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Wednesday April 22, 2009. Time: 14h00–17h00. Place: HG 10.01 C.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student identification number on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes and calculator is allowed. The use of the problem companion, "opgaven- en tentamenbundel", is *not* allowed.
- You may provide your answers in Dutch or (preferably) in English.

GOOD LUCK!

(**35**) **1.** Consider the collection of square matrices,

$$\mathbb{M}_n = \left\{ X = \begin{pmatrix} X_{11} & \dots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \dots & X_{nn} \end{pmatrix} \,\middle|\, X_{ij} \in \mathbb{K} \right\}, \text{ in which } \mathbb{K} \text{ denotes either } \mathbb{R} \text{ or } \mathbb{C}.$$

($2\frac{1}{2}$) **a.** Provide explicit definitions for the operators $\otimes : \mathbb{K} \times \mathbb{M}_n \to \mathbb{M}_n$ ("scalar multiplication") and $\oplus : \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{M}_n$ ("vector addition") needed to turn this set into a linear space over $\mathbb{K}$. Make sure to use parentheses so as to avoid confusion on operator precedence, if necessary.

Henceforth we write λX instead of $\lambda \otimes X$ and $X + Y$ instead of $X \oplus Y$ for $\lambda \in \mathbb{K}$ and $X, Y \in \mathbb{M}_n$. Furthermore, let $\mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$ denote the linear space of linear operators on $\mathbb{M}_n$.

($2\frac{1}{2}$) **b.** Provide explicit definitions for the operators $\otimes : \mathbb{R} \times \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n) \to \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$ and $\oplus : \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n) \times \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n) \to \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$ that justifies the claim that $\mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$ is a linear space over $\mathbb{K}$. Make sure to use parentheses so as to avoid confusion on operator precedence, if necessary.

Again we write λA instead of $\lambda \otimes A$ and $A + B$ instead of $A \oplus B$ for $\lambda \in \mathbb{K}$ and $A, B \in \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$. With X as above, the transposed matrix X^{T} and the conjugate matrix $X^{\dagger}$ are defined as

$$X^{\mathrm{T}} = \begin{pmatrix} X_{11} & \dots & X_{n1} \\ \vdots & & \vdots \\ X_{1n} & \dots & X_{nn} \end{pmatrix} \quad \text{respectively} \quad X^{\dagger} = \begin{pmatrix} X_{11}^* & \dots & X_{n1}^* \\ \vdots & & \vdots \\ X_{1n}^* & \dots & X_{nn}^* \end{pmatrix}.$$

Here, $*$ denotes complex conjugation, i.e. if $z = x + iy$ for $x, y \in \mathbb{R}$, then $z^* = x - iy$.

Furthermore, the operators $\mathrm{P}_\pm : \mathbb{M}_n \to \mathbb{M}_n$ and $\mathrm{Q}_\pm : \mathbb{M}_n \to \mathbb{M}_n$ are defined by

$$\mathrm{P}_\pm(X) = \frac{1}{2}\left(X \pm X^{\mathrm{T}}\right) \quad \text{respectively} \quad \mathrm{Q}_\pm(X) = \frac{1}{2}\left(X \pm X^{\dagger}\right).$$

($2\frac{1}{2}$) **c1.** Show that *matrix transposition*, $\mathrm{T} : \mathbb{M}_n \to \mathbb{M}_n : X \mapsto \mathrm{T}(X) \stackrel{\text{def}}{=} X^{\mathrm{T}}$, is a linear operator.

($2\frac{1}{2}$) **c2.** Show that *matrix conjugation*, $\mathrm{C} : \mathbb{M}_n \to \mathbb{M}_n : X \mapsto \mathrm{C}(X) \stackrel{\text{def}}{=} X^{\dagger}$, is not a linear operator.

($2\frac{1}{2}$) **c3.** Show that $\mathrm{P}_{\pm} : \mathbb{M}_n \to \mathbb{M}_n$ are linear operators.

($2\frac{1}{2}$) **c4.** Show that $\mathrm{Q}_{\pm} : \mathbb{M}_n \to \mathbb{M}_n$ are not linear operators.

The null matrix in $\mathbb{M}_n$ is indicated by Ω, i.e. $\Omega_{ij} = 0$ for all $i, j = 1, \ldots, n$. The identity matrix in $\mathbb{M}_n$ is indicated by I, i.e. $I_{ij} = 1$ if $i = j = 1, \ldots, n$, otherwise $I_{ij} = 0$. The null operator $\mathrm{N} : \mathbb{M}_n \to \mathbb{M}_n$ is defined by $\mathrm{N}(X) = \Omega \in \mathbb{M}_n$ for all $X \in \mathbb{M}_n$. The identity operator $\mathrm{id} : \mathbb{M}_n \to \mathbb{M}_n$ is defined by $\mathrm{id}(X) = X$ for all $X \in \mathbb{M}_n$. Operator composition (i.e. successive application of operators in right-to-left order) is indicated by the infix operator $\circ$.

($2\frac{1}{2}$) **d1.** Show that $\mathrm{P}_+ + \mathrm{P}_- = \mathrm{id}$.

($2\frac{1}{2}$) **d2.** Show that $\mathrm{P}_+ - \mathrm{P}_- = \mathrm{T}$.

($2\frac{1}{2}$) **d3.** Show that $\mathrm{P}_+ \circ \mathrm{P}_- = \mathrm{P}_- \circ \mathrm{P}_+ = \mathrm{N}$.

($2\frac{1}{2}$) **d4.** Show that $\mathrm{P}_{\pm} \circ \mathrm{P}_{\pm} = \mathrm{P}_{\pm}$.

Consider the following binary operator: $\langle\ |\ \rangle : \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{K} : (X, Y) \mapsto \langle X|Y\rangle \stackrel{\text{def}}{=} \mathrm{trace}(X^{\mathrm{T}} Y)$. Here, $\mathrm{trace} : \mathbb{M}_n \to \mathbb{K}$ is the (linear) *trace* operator, defined as summation of diagonal elements:

$$\mathrm{trace} X = \sum_{i=1}^{n} X_{ii}\,.$$

($2\frac{1}{2}$) **e1.** Show that if $\mathbb{K} = \mathbb{R}$ then $\langle\ |\ \rangle : \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{R}$ defines a real inner product.

($2\frac{1}{2}$) **e2.** Show that if $\mathbb{K} = \mathbb{C}$ then $\langle\ |\ \rangle : \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{C}$ does not define a complex inner product.

($2\frac{1}{2}$) **f.** How would you modify the definition of $\langle\ |\ \rangle : \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{C}$ in the complex case, $\mathbb{K} = \mathbb{C}$, such that it does define a complex inner product? You may state your definition without proof.

In the remainder of this problem we restrict ourselves to $\mathbb{K} = \mathbb{R}$. In particular, we consider the case of the real inner product, recall **e1**.

Operator transposition, $\mathrm{T} : \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n) \to \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n)$, is implicitly defined by the identity

$$\langle A^{\mathrm{T}}(X)|Y\rangle \stackrel{\text{def}}{=} \langle X|A(Y)\rangle \quad \text{for all } A \in \mathscr{L}(\mathbb{M}_n, \mathbb{M}_n) \text{ and } X, Y \in \mathbb{M}_n.$$

($2\frac{1}{2}$) **g.** Show that $\mathrm{P}_{\pm}^{\mathrm{T}} = \mathrm{P}_{\pm}$. (Together with **d4** this shows that $\mathrm{P}_{\pm}$ are orthogonal projections.)

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($32\frac{1}{2}$) 2. Consider the Laplace equation for the function $u \in C^\infty(\mathbb{R}^2)$:

$$\Delta u = 0 \quad \text{in which } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,.$$

Our aim is to find a first order partial differential equation, such that its solutions also satisfy the Laplace equation. To this end we introduce a real linear space V, the dimension $n \geq 2$ of which is yet to be determined, and furnish it with an additional operator henceforth referred to as "multiplication". The product of $v, w \in V$ is then simply written as $vw \in V$. In this way V is turned into a so-called algebra, for which we stipulate the following algebraic axioms, viz. for all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$:

- $(uv)w = u(vw)$,
- $u(v + w) = uv + uw$,
- $(u + v)w = uw + vw$,
- $\lambda(uv) = (\lambda u)v = u(\lambda v)$,

(Multiplication takes precedence over vector addition unless parentheses indicate otherwise.)

We now attempt to decompose the Laplacian operator as follows:

$$\Delta = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\,.$$

Here $a, b \in V$ are two fixed, independent elements (vectors). For consistency we assume that $u(x, y) \in V$ (instead of our original assumption $u(x, y) \in \mathbb{R}$).

a. Show that V is *not* commutative, and that it must possess an identity element $1 \in V$ that has to be formally identified with the scalar number $1 \in \mathbb{R}$, by showing that

($2\frac{1}{2}$) **a1.** $ab + ba = 0$,

($2\frac{1}{2}$) **a2.** $a^2 = b^2 = 1$.

(5) **b.** Show that the (unordered) pair (a, b) satisfying the conditions of **a1** and **a2** is not unique.
(*Hint:* Suppose $a' = a_1 a + a_2 b$, $b' = b_1 a + b_2 b$ for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$.)

c. Suppose $v, w \in \text{span}\{a, b\} \subset V$ are such that, say, $v = v_1 a + v_2 b$ and $w = w_1 a + w_2 b$ for some $v_1, v_2, w_1, w_2 \in \mathbb{R}$. Compute

($2\frac{1}{2}$) **c1.** $vw + wv$,

($2\frac{1}{2}$) **c2.** v^2.

($2\frac{1}{2}$) **d.** Show that $\dim V > 2$.
(*Hint:* Alternatively, show that $\text{span}\{a, b\} \subset V$ is not closed under multiplication.)

We add two more independent elements to the set $\{a, b\}$, viz. the unit element 1 and the element ab, and define $V = \text{span}\{1, a, b, ab\}$. Instead of $\lambda 1 + \mu a + \nu b + \rho ab$ $(\lambda, \mu, \nu, \rho \in \mathbb{R})$ we write $\lambda + \mu a + \nu b + \rho ab$ for an arbitrary element of V.

(5) **e.** Show that V is closed under multiplication, i.e. show that if $v = v_0 + v_1 a + v_2 b + v_3 ab \in V$, $w = w_0 + w_1 a + w_2 b + w_3 ab \in V$, then also $vw \in V$.

We now try to realize elements of V in terms of real-valued 2×2-matrices. To this end we hypothesize that

$$V \subset \mathbb{M}_2 = \left\{ \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \middle| \, a_{ij} \in \mathbb{R} \right\} .$$

(5) **f.** Construct explicit matrices $A, B \in \mathbb{M}_2$ corresponding to $a, b \in V$ in the sense that

- $AB + BA = \Omega$,
- $A^2 = B^2 = I$.

Here $\Omega \in \mathbb{M}_2$ is the null matrix, corresponding to the null element $0 \in V$, and $I \in \mathbb{M}_2$ is the identity matrix, corresponding to the identity element $1 \in V$.
(*Hint:* As an ansatz, stipulate a diagonal matrix A, and show that B must then be anti-diagonal.)

($2\frac{1}{2}$) **g.** Given A and B as determined under **f**, what is the matrix form $v_0 + v_1 A + v_2 B + v_3 AB \in \mathbb{M}_2$ corresponding to a general element $v_0 + v_1 a + v_2 b + v_3 ab \in V$?

($2\frac{1}{2}$) **h.** Show that if $u : \mathbb{R}^2 \to V$ satisfies the first order partial differential equation

$$a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} = 0 \, ,$$

then it is also a solution of $\Delta u = 0$.

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(**5**) **3.** Consider a strictly monotonic, continuously differentiable function $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$, with $f'(x) > 0$ for all $x \in \mathbb{R}$, and $f(\pm\infty) = \pm\infty$. The inverse function theorem states that such a function has an inverse, $f^{-1} : \mathbb{R} \to \mathbb{R} : y \mapsto f^{-1}(y)$, such that $(f^{-1} \circ f)(x) = x$ for all $x \in \mathbb{R}$, and $(f \circ f^{-1})(y) = y$ for all $y \in \mathbb{R}$.

($2\frac{1}{2}$) **a.** Argue why the equation $f(x) = 0$ has precisely one solution for $x \in \mathbb{R}$ ($x = a$, say).
(*Hint:* Sketch the graph of such a function f.)

($2\frac{1}{2}$) **b.** Show that $\delta(f(x)) = \dfrac{\delta(x - a)}{f'(a)}$, in which $a \in \mathbb{R}$ is the unique point for which $f(a) = 0$.
(*Hint:* Evaluate the distribution corresponding to the Dirac function on the left hand side on an arbitrary test function $\phi \in \mathscr{S}(\mathbb{R})$, i.e. $\int_{-\infty}^{\infty} \delta(f(x)) \, \phi(x) \, dx$, and apply substitution of variables.)

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($27\frac{1}{2}$) 4. In this problem we consider a synthetic signal $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined as follows:

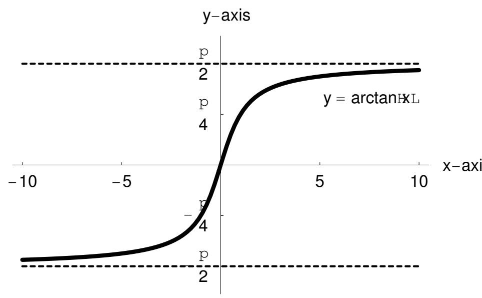
$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Furthermore, we define the so-called Poisson filter $\phi_\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ for $\sigma > 0$ as

$$\phi_\sigma(x) = \frac{1}{\pi} \frac{\sigma}{x^2 + \sigma^2} .$$

In this problem you may use the following standard formulas (cf. the graph shown below):

$$\int \frac{1}{1+x^2}\, dx = \arctan x + c \qquad \text{resp.} \qquad \lim_{x \to \pm\infty} \arctan x = \pm \frac{\pi}{2} .$$



Throughout this problem we employ the following Fourier convention:

$$\widehat{u}(\omega) = \mathcal{F}(u)(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}\, u(x)\, dx \quad \text{whence} \quad u(x) = \mathcal{F}^{-1}(\widehat{u})(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x}\, \widehat{u}(\omega)\, d\omega .$$

($2\frac{1}{2}$) **a.** Show that $\int_{-\infty}^{\infty} \phi_\sigma(x)\, dx = 1$ regardless of the value of σ.

($2\frac{1}{2}$) **b1.** Prove that for any, sufficiently smooth, integrable filter ϕ we have:

$$(f * \phi)(x) = \int_{-\infty}^{x} \phi(\xi)\, d\xi .$$

($2\frac{1}{2}$) **b2.** Show by explicit computation that the convolution product $f * \phi_\sigma$ is given by

$$(f * \phi_\sigma)(x) = \frac{1}{2} + \frac{1}{\pi} \arctan \frac{x}{\sigma} .$$

(5) **c.** Show that the Fourier transform $\widehat{f} = \mathcal{F}(f)$ of f is given by

$$\widehat{f}(\omega) = \frac{1}{i\omega} .$$

(*Hint:* From part **b1** it follows that $\frac{d}{dx}(f * \phi)(x) = \phi(x)$. Subject this to Fourier transformation.)

d. Without proof we state the Fourier transform $\widehat{\phi}_\sigma = \mathcal{F}(\phi_\sigma)$ of ϕ_σ:

$$\widehat{\phi}_\sigma(\omega) = e^{-\sigma|\omega|}\,.$$

($2\frac{1}{2}$) **d1.** Explain the behaviour of $\widehat{\phi}_\sigma(\omega)$ for $\omega \to 0$ in terms of properties of the corresponding spatial filter $\phi_\sigma(x)$.
(*Hint:* Cf. problem **a**.)

($2\frac{1}{2}$) **d2.** Explain the behaviour of $\widehat{f}(\omega)$ for $\omega \to 0$ in terms of properties of the corresponding spatial signal $f(x)$.

(5) **e.** Determine the function $\mathcal{F}(f * \phi_\sigma)$.

($2\frac{1}{2}$) **f.** Show that $\lim_{\sigma \to 0} f * \phi_\sigma = f$. Hint: Take the "Fourier route".

($2\frac{1}{2}$) **g.** Prove the claim under **d**: $\widehat{\phi}_\sigma(\omega) = e^{-\sigma|\omega|}$.
(*Hint:* Apply the inverse Fourier transform.)

THE END