EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Monday January 23, 2012. Time: 09h00–12h00. Place: AUD 14.

Read this first!

- Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of problem companion ("opgaven- en tentamenbundel"), calculator, laptop, or any other equipment, is *not* allowed.
- You may provide your answers in Dutch or English.
- Feel free to ask questions on linguistic matters or if you suspect an erroneous problem formulation.

Good luck!

(**30**) **1.** Group Theory

In this problem we consider sets of real-valued 2×2-matrices generated by the following matrix:

$$A \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Nonnegative integer matrix powers are defined as repetitive matrix products:

$$X^k \stackrel{\text{def}}{=} \underbrace{X \ldots X}_{k \text{ factors}} \qquad (k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}) .$$

By convention the empty product yields the 2×2 identity matrix, $X^0 \stackrel{\text{def}}{=} I$. Consider the set

$$L \stackrel{\text{def}}{=} \left\{ A^k \;\middle|\; k \in \mathbb{N}_0 \right\} .$$

L is furnished with an internal operator of type $L\times L \to L$, viz. standard matrix multiplication.

($2\frac{1}{2}$) **a1.** Show that L is closed under matrix multiplication.

Let $a, b \in L$. By definition there exist $k, \ell \in \mathbb{N}_0$ such that $a = A^k$, $b = A^\ell$, hence $ab = A^k A^\ell = A^{k+\ell} \in L$, since $k+\ell \in \mathbb{N}_0$.

($2\frac{1}{2}$) **a2.** Show that L contains exactly 4 distinct elements, and compute their matrix representations.

By either definition or straightforward computation we find $A^0 = I$, $A^1 = A$, $A^2 = -I$, $A^3 = -A$. Consequently, $A^4 = I = A^0$, and, more generally, $A^{4+k} = A^k$ for all $k \in \mathbb{N}_0$. This 4-periodicity property implies that the only distinct elements of L are those A^k given by $k = 0, 1, 2, 3$.

($2\frac{1}{2}$) **a3.** Prove that L is a commutative group by providing its 4×4 group multiplication table.

From the fact that $A^k A^\ell = A^{(k+\ell) \bmod 4}$ the multiplication table is evident:

	A^0	A^1	A^2	A^3
A^0	A^0	A^1	A^2	A^3
A^1	A^1	A^2	A^3	A^0
A^2	A^3	A^0	A^1	A^2
A^3	A^0	A^1	A^2	A^3

Explicit formulas for the matrix powers in the table are given in a2.

We furthermore define the linear space $\mathscr{L} \stackrel{\text{def}}{=} \operatorname{span} L$.

($2\frac{1}{2}$) **a4.** Show that $\dim \mathscr{L} = 2$, in other words, that there are two linearly independent elements $X_1, X_2 \in \mathscr{L}$ such that every element $X \in \mathscr{L}$ can be written as a linear combination of the form $X = \lambda_1 X_1 + \lambda_2 X_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

Let $X = \sum_{k=0}^{3} \lambda_k A^k \in \mathscr{L}$. With the explicit forms for A^k computed in a2 we observe that $X = (\lambda_0 - \lambda_2)I + (\lambda_1 - \lambda_3)A$, so $\mathscr{L} = \operatorname{span}\{I, A\}$, and since I and A are linearly independent we conclude that $\{I, A\}$ is a basis of $\mathscr{L}$, i.e. $\dim \mathscr{L} = 2$.

The exponential function can be applied to square matrices via its formal Taylor series:

$$\exp X \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{k!} X^k = I + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{24}X^4 + \ldots$$

Based on this we define the set $\mathscr{R} = \{\exp(\theta A) \mid \theta \in \mathbb{R}\}$, and furnish it with standard matrix multiplication. Without proof we state that

$$\exp X \exp Y = \exp(X + Y) \quad \text{if } [X, Y] \stackrel{\text{def}}{=} XY - YX = 0.$$

b. Show that $\mathscr{R}$ is a commutative group by proving the following properties.

(2) **b1.** $\mathscr{R}$ is closed, i.e. $\exp(\eta A)\exp(\theta A) \in \mathscr{R}$. Specify this element for given $\eta, \theta \in \mathbb{R}$.

Since $[\eta A, \theta A] = \eta\theta[A, A] = 0$ we may apply the given multiplication formula (see $*$ below). For $\exp(\eta A), \exp(\theta A) \in \mathscr{R}$ we have $\exp(\eta A)\exp(\theta A) \stackrel{*}{=} \exp((\eta + \theta)A) \in \mathscr{R}$.

(2) **b2.** $\mathscr{R}$ is associative.

For all $\alpha, \beta, \gamma \in \mathbb{R}$ we have

$$(\exp(\alpha A)\exp(\beta A))\exp(\gamma A) = (\exp((\alpha + \beta)A))\exp(\gamma A) = \exp(((\alpha + \beta) + \gamma)A)$$

which clearly equals, by associativity of ordinary number multiplication,

$$\exp(\alpha + (\beta + \gamma)A) = \exp(\alpha A)(\exp((\beta + \gamma)A)) = \exp(\alpha A)(\exp(\beta A)\exp(\gamma A)) \,.$$

(2) **b3.** $\mathscr{R}$ has a unit element. Specify this element.

The unit element is $I = \exp(0A) \in \mathscr{R}$.

(2) **b4.** Every element of $\mathscr{R}$ has an inverse. Specify the inverse of $\exp(\theta A)$ for given $\theta \in \mathbb{R}$.

The inverse of $\exp(\theta A) \in \mathscr{R}$ is $\exp(-\theta A) \in \mathscr{R}$, since $\exp(\theta A)\exp(-\theta A) = \exp((\theta-\theta)A) = \exp(0A) = I$.

(2) **b5.** $\mathscr{R}$ is commutative.

This follows from commutativity of ordinary number addition: $\exp(\eta A)\exp(\theta A) = \exp((\eta+\theta)A) = \exp((\theta+\eta)A) = \exp(\theta A)\exp(\eta A)$ for all $\eta, \theta \in \mathbb{R}$.

(10) **c.** Show that $\mathscr{R}$ is actually the group of 2×2 rotation matrices, i.e.

$$\mathscr{R} = \left\{ R(\theta) \stackrel{\text{def}}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \,\middle|\, \theta \in \mathbb{R} \right\}.$$

☞ Hint: Recall the Taylor expansions of $\cos\theta$ and $\sin\theta$.

From a2 it follows that $A^{2k} = (-1)^k I$ and $A^{2k+1} = (-1)^k A$ for all $k \in \mathbb{N}_0$, therefore split the following sum (in step $*$) as follows:

$$\exp(\theta A) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{k!}(\theta A)^k = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} A^k \stackrel{*}{=} \sum_{k=0}^{\infty} \frac{\theta^{2k}}{(2k)!} A^{2k} + \sum_{k=0}^{\infty} \frac{\theta^{2k+1}}{(2k+1)!} A^{2k+1} \stackrel{\text{a2}}{=} \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} I + \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} A.$$

Here we recognize the trigonometric Taylor expansions:

$$\cos\theta = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} \quad \text{and} \quad \sin\theta = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!},$$

so that apparently

$$\exp(\theta A) = \cos\theta I + \sin\theta A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

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(**25**) **2.** Vector Space

We consider the class V of positive definite functions $f : \mathbb{R} \to \mathbb{R}^+ : x \mapsto f(x) > 0$ for all $x \in \mathbb{R}$. We endow V with a binary infix operator

$$\oplus : V \times V \to V : (f, g) \mapsto f \oplus g \quad \text{defined such that } (f \oplus g)(x) \stackrel{\text{def}}{=} f(x)g(x) \text{ for all } x \in \mathbb{R}.$$

We also provide a scalar multiplication operator

$$\otimes : \mathbb{R} \times V \to V : (\lambda, f) \mapsto \lambda \otimes f \quad \text{defined such that } (\lambda \otimes f)(x) \stackrel{\text{def}}{=} f(x)^\lambda \text{ for all } x \in \mathbb{R}.$$

Show that V constitutes a vector space, and provide explicit formulas for the neutral element $0 \in V$ as well as for the inverse element $(-f) \in V$ for any $f \in V$. Start by proving the closure properties implied by the above notation, and proceed by verifying all vector space axioms. It is mandatory to adhere to the symbols $\oplus$ and $\otimes$ in your notation wherever appropriate.

☛ Caveat: $0(x) \neq 0$, $(-f)(x) \neq -f(x)$. No confusion will arise if you use $\oplus/\otimes$ consistently.

Closure: If $f, g \in V$, $\lambda, \mu \in \mathbb{R}$, then, with henceforth $\otimes$ taking precedence over $\oplus$, $(\lambda \otimes f \oplus \mu \otimes g)(x) \stackrel{\text{def}}{=} f(x)^\lambda g(x)^\mu > 0$ for all $x \in \mathbb{R}$, whence $\lambda \otimes f \oplus \mu \otimes g \in V$. Furthermore, let $f, g, h \in V$, $\lambda, \mu, \nu \in \mathbb{R}$ be given in all that follows, and $x \in \mathbb{R}$ arbitrary. We have

- $((f \oplus g) \oplus h)(x) \stackrel{\text{def}}{=} (f \oplus g)(x)h(x) \stackrel{\text{def}}{=} (f(x)g(x))h(x) = f(x)(g(x)h(x)) \stackrel{\text{def}}{=} f(x)(g \oplus h)(x) \stackrel{\text{def}}{=} (f \oplus (g \oplus h))(x)$, i.e. $(f \oplus g) \oplus h = f \oplus (g \oplus h)$.
- Let $0 \in V$ be the function given by $0(x) \stackrel{\text{def}}{=} 1 > 0$, then $(0 \oplus f)(x) \stackrel{\text{def}}{=} 0(x)f(x) = 1f(x) = f(x)$, i.e. $0 \oplus f = f$.
- Let $(-f) \in V$ be the function given by $(-f)(x) \stackrel{\text{def}}{=} f(x)^{-1} > 0$, or $(-f) \stackrel{\text{def}}{=} (-1) \otimes f$, then $((-f) \oplus f)(x) \stackrel{\text{def}}{=} (-f)(x)f(x) \stackrel{\text{def}}{=} f(x)^{-1}f(x) = 1 \stackrel{\text{def}}{=} 0(x)$.
- $(f \oplus g)(x) \stackrel{\text{def}}{=} f(x)g(x) = g(x)f(x) \stackrel{\text{def}}{=} (g \oplus f)(x)$, i.e. $f \oplus g = g \oplus f$.
- $(\lambda \otimes (f \oplus g))(x) \stackrel{\text{def}}{=} (f \oplus g)(x)^\lambda \stackrel{\text{def}}{=} (f(x)g(x))^\lambda = f(x)^\lambda g(x)^\lambda \stackrel{\text{def}}{=} (\lambda \otimes f)(x)(\lambda \otimes g)(x) \stackrel{\text{def}}{=} (\lambda \otimes f \oplus \lambda \otimes g)(x)$, i.e. $\lambda \otimes (f \oplus g) = \lambda \otimes f \oplus \lambda \otimes g$.
- $((\lambda+\mu)\otimes f)(x) \stackrel{\text{def}}{=} f(x)^{\lambda+\mu} = f(x)^\lambda f(x)^\mu \stackrel{\text{def}}{=} (\lambda\otimes f)(x)(\mu\otimes f)(x) \stackrel{\text{def}}{=} (\lambda\otimes f\oplus\mu\otimes f)(x)$, i.e. $(\lambda+\mu)\otimes f = \lambda\otimes f\oplus\mu\otimes f$.
- $((\lambda\mu) \otimes f)(x) \stackrel{\text{def}}{=} f(x)^{\lambda\mu} = (f(x)^\mu)^\lambda \stackrel{\text{def}}{=} (\mu \otimes f)(x)^\lambda \stackrel{\text{def}}{=} (\lambda \otimes (\mu \otimes f))(x)$, i.e. $(\lambda\mu) \otimes f = \lambda \otimes (\mu \otimes f)$.
- $(1 \otimes f)(x) \stackrel{\text{def}}{=} f(x)^1 = f(x)$, i.e. $1 \otimes f = f$.

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(**15**) **3.** Distribution Theory

In this problem we insist on a notational distinction between the Dirac point distribution and its formal integral representation involving a corresponding "Dirac function". We shall write

$$T_\delta : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto T_\delta(\phi) \stackrel{\text{def}}{=} \phi(0)\,,$$

for the distribution proper, respectively

$$\delta : \mathbb{R} \to \mathbb{R} : x \mapsto \delta(x)\,,$$

for the virtual function "under the integral", so $T_\delta(\phi) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx$.

The goal of this problem will be to prove that $\delta \notin L^p(\mathbb{R})$ for any $p>1$, including $p=\infty$.

One can show that there exist so-called "bump functions" $\psi \in \mathscr{S}(\mathbb{R})$ such that $\psi(x)=0$ outside an arbitrarily chosen support interval. In particular we consider the subfamily $\mathscr{B}_\epsilon(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$ of bump functions defined for given $\epsilon>0$ as follows:

$$\mathscr{B}_\epsilon(\mathbb{R}) = \left\{ \psi \in \mathscr{S}(\mathbb{R}) \mid \psi(x)=0 \text{ outside the interval } (-\tfrac{\epsilon}{2}, \tfrac{\epsilon}{2}), \text{ and } \max_{x\in\mathbb{R}} |\psi(x)|=\psi(0)=1 \right\}.$$

(5) **a.** Show that $\mathscr{B}_\epsilon(\mathbb{R}) \subset L^q(\mathbb{R})$ for any $q \geq 1$, $\epsilon>0$, by proving that $\|\psi\|_q \leq \sqrt[q]{\epsilon}$ for $\psi \in \mathscr{B}_\epsilon(\mathbb{R})$.

$\|\psi\|_q^q = \int_{-\epsilon/2}^{\epsilon/2} |\psi(x)|^q dx \leq \epsilon \max |\psi(x)|^q = \epsilon$.

(10) **b.** Use this fact to disprove the hypothesis that $\delta \in L^p(\mathbb{R})$ for some $p>1$ or $p=\infty$.

☞ Hint: Consider the hypothesis and subproblem a in the context of Hölder's inequality.

For fixed $p>1$ or $p=\infty$ take $q\geq 1$ such that it satisfies Hölder's condition ($1/p+1/q=1$), then $1=\psi(0)=\int_{-\infty}^{\infty}\delta(x)\psi(x)dx\leq \|\delta\|_p\|\psi\|_q\leq \sqrt[q]{\epsilon}\|\delta\|_p$. This is a contradiction, since the r.h.s. can be made arbitrarily small. (Note that this argument fails for $p=1$, i.e. $q=\infty$.)

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(**30**) 4. Fourier Transformation (Exam March 21, 2007, Problem 3)

In this problem we use the following Fourier convention for $f\in\mathscr{S}'(\mathbb{R})$:

$$\widehat{f}(\omega)\stackrel{\text{def}}{=}\int_{-\infty}^{\infty}f(x)\,e^{-i\omega x}\,dx\,.$$

As a result we have for the inverse:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\omega)\,e^{i\omega x}\,d\omega\,.$$

Here we permit ourselves the sloppiness of identifying a regular tempered distribution f with its Riesz representant ("function under the integral") with function definition $f(x)$. The Dirac δ-distribution is identified with the "Dirac delta function" with function definition $\delta(x)$.

($7\frac{1}{2}$) **a.** Given $\widehat{f}(\omega)=\delta(\omega-a)$ for some constant $a\in\mathbb{R}$. Determine $f(x)$.

Substitution into the definition of the inverse Fourier transform yields

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\omega)\,e^{i\omega x}\,d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-a)\,e^{i\omega x}\,d\omega\stackrel{*}{=}\frac{e^{iax}}{2\pi}\,.$$

The last step $*$ exploits the definition of the Dirac delta function.

($7\frac{1}{2}$) **b.** Given $g(x)=2\cos^2 x$. (With $\cos^2 x$ we mean $(\cos x)^2$.) Determine $\widehat{g}(\omega)$.

☞ Hint: You can check your result with the help of subproblem a.

Note that $g(x)=2\cos^2 x=\cos(2x)+1$. Therefore

$$\widehat{g}(\omega)=\int_{-\infty}^{\infty}g(x)\,e^{-i\omega x}\,dx=\int_{-\infty}^{\infty}\left(\cos(2x)+1\right)\,e^{-i\omega x}\,dx\,.$$

Since

$$\cos(2x)=\frac{e^{2ix}+e^{-2ix}}{2}\,,$$

we have

$$\begin{aligned}\widehat{g}(\omega)&=\int_{-\infty}^{\infty}\left(\frac{1}{2}e^{-i(\omega-2)x}+\frac{1}{2}e^{-i(\omega+2)x}+e^{-i\omega x}\right)dx=\frac{1}{2}\int_{-\infty}^{\infty}e^{-i(\omega-2)x}\,dx+\frac{1}{2}\int_{-\infty}^{\infty}e^{-i(\omega+2)x}\,dx+\int_{-\infty}^{\infty}e^{-i\omega x}\,dx\\&=\pi\,\delta(\omega-2)+\pi\,\delta(\omega+2)+2\pi\,\delta(\omega)\,.\end{aligned}$$

Check, via a:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{g}(\omega)\,e^{i\omega x}\,d\omega&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\pi\,\delta(\omega-2)+\pi\,\delta(\omega+2)+2\pi\,\delta(\omega)\right)\,e^{i\omega x}\,d\omega\\&=\frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega-2)\,e^{i\omega x}\,d\omega+\frac{1}{2}\int_{-\infty}^{\infty}\delta(\omega+2)\,e^{i\omega x}\,d\omega+\int_{-\infty}^{\infty}\delta(\omega)\,e^{i\omega x}\,d\omega\\&=\frac{1}{2}e^{2ix}+\frac{1}{2}e^{-2ix}+1=\cos(2x)+1=g(x)\,.\end{aligned}$$

In the following part you may use the standard integral

$$\int_{-\infty}^{\infty} e^{-(x+iy)^2}\,dx = \sqrt{\pi} \quad \text{regardless of the value of } y \in \mathbb{R}.$$

($7\frac{1}{2}$) **c.** Given $\phi(x) = \dfrac{1}{\sqrt{\pi}}\, e^{-x^2}$. Determine $\widehat{\phi}(\omega)$.

$$\widehat{\phi}(\omega) = \int_{-\infty}^{\infty} \phi(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2 - i\omega x}\, dx = e^{-\frac{1}{4}\omega^2}\, \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}i\omega)^2}\, dx \overset{*}{=} e^{-\frac{1}{4}\omega^2}\,.$$

In the last step $*$ we have used the given standard integral.

($7\frac{1}{2}$) **d.** Given $h(x) = \dfrac{2}{\sqrt{\pi}} \cos^2 x\, e^{-x^2}$. Determine $\widehat{h}(\omega)$.

☞ HINT: NOTE THAT $h = g\,\phi$, RECALL SUBPROBLEMS B AND C.

With the help of the hint we determine

$$\begin{aligned}
\widehat{h}(\omega) &= \widehat{(g\,\phi)}(\omega) \overset{*}{=} \frac{1}{2\pi}\, (\widehat{g} * \widehat{\phi})(\omega) = \int_{-\infty}^{\infty} \widehat{g}(\omega')\, \widehat{\phi}(\omega - \omega')\, d\omega' \\
&\overset{\text{b}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} \big(\pi\, \delta(\omega' - 2) + \pi\, \delta(\omega' + 2) + 2\pi\, \delta(\omega')\big)\, \widehat{\phi}(\omega - \omega')\, d\omega' \\
&\overset{\star}{=} \frac{1}{2}\, \widehat{\phi}(\omega - 2) + \frac{1}{2}\, \widehat{\phi}(\omega + 2) + \widehat{\phi}(\omega) \overset{\text{c}}{=} \frac{1}{2}\, e^{-\frac{1}{4}(\omega-2)^2} + \frac{1}{2}\, e^{-\frac{1}{4}(\omega+2)^2} + e^{-\frac{1}{4}\omega^2}\,.
\end{aligned}$$

In $*$ the theorem for the Fourier transform of a product of two functions has been used. In $\star$ we have used the definition of the Dirac delta function.

THE END