EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Monday January 23, 2012. Time: 09h00–12h00. Place: AUD 14.

Read this first!

- Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of problem companion ("opgaven- en tentamenbundel"), calculator, laptop, or any other equipment, is *not* allowed.
- You may provide your answers in Dutch or English.
- Feel free to ask questions on linguistic matters or if you suspect an erroneous problem formulation.

Good luck!

(**30**) **1.** Group Theory

In this problem we consider sets of real-valued 2×2-matrices generated by the following matrix:

$$A \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Nonnegative integer matrix powers are defined as repetitive matrix products:

$$X^k \stackrel{\text{def}}{=} \underbrace{X \ldots X}_{k \text{ factors}} \qquad (k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}).$$

By convention the empty product yields the 2×2 identity matrix, $X^0 \stackrel{\text{def}}{=} I$. Consider the set

$$L \stackrel{\text{def}}{=} \left\{ A^k \;\middle|\; k \in \mathbb{N}_0 \right\}.$$

L is furnished with an internal operator of type $L \times L \to L$, viz. standard matrix multiplication.

($2\frac{1}{2}$) **a1.** Show that L is closed under matrix multiplication.

($2\frac{1}{2}$) **a2.** Show that L contains exactly 4 distinct elements, and compute their matrix representations.

($2\frac{1}{2}$) **a3.** Prove that L is a commutative group by providing its 4×4 group multiplication table.

We furthermore define the linear space $\mathscr{L} \stackrel{\text{def}}{=} \operatorname{span} L$.

($2\frac{1}{2}$) **a4.** Show that $\dim \mathscr{L} = 2$, in other words, that there are two linearly independent elements $X_1, X_2 \in \mathscr{L}$ such that every element $X \in \mathscr{L}$ can be written as a linear combination of the form $X = \lambda_1 X_1 + \lambda_2 X_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

The exponential function can be applied to square matrices via its formal Taylor series:

$$\exp X \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{k!} X^k = I + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{24}X^4 + \ldots$$

Based on this we define the set $\mathscr{R} = \{\exp(\theta A) \mid \theta \in \mathbb{R}\}$, and furnish it with standard matrix multiplication. Without proof we state that

$$\exp X \exp Y = \exp(X+Y) \quad \text{if } [X,Y] \stackrel{\text{def}}{=} XY - YX = 0.$$

b. Show that $\mathscr{R}$ is a commutative group by proving the following properties.

(2) **b1.** $\mathscr{R}$ is closed, i.e. $\exp(\eta A)\exp(\theta A) \in \mathscr{R}$. Specify this element for given $\eta, \theta \in \mathbb{R}$.

(2) **b2.** $\mathscr{R}$ is associative.

(2) **b3.** $\mathscr{R}$ has a unit element. Specify this element.

(2) **b4.** Every element of $\mathscr{R}$ has an inverse. Specify the inverse of $\exp(\theta A)$ for given $\theta \in \mathbb{R}$.

(2) **b5.** $\mathscr{R}$ is commutative.

(10) **c.** Show that $\mathscr{R}$ is actually the group of 2×2 rotation matrices, i.e.

$$\mathscr{R} = \left\{ R(\theta) \stackrel{\text{def}}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \;\middle|\; \theta \in \mathbb{R} \right\}.$$

☞ HINT: RECALL THE TAYLOR EXPANSIONS OF $\cos\theta$ AND $\sin\theta$.

♣

(**25**) **2.** VECTOR SPACE

We consider the class V of positive definite functions $f : \mathbb{R} \to \mathbb{R}^+ : x \mapsto f(x) > 0$ for all $x \in \mathbb{R}$. We endow V with a binary infix operator

$$\oplus : V \times V \to V : (f,g) \mapsto f \oplus g \quad \text{defined such that } (f \oplus g)(x) \stackrel{\text{def}}{=} f(x)g(x) \text{ for all } x \in \mathbb{R}.$$

We also provide a scalar multiplication operator

$$\otimes : \mathbb{R} \times V \to V : (\lambda, f) \mapsto \lambda \otimes f \quad \text{defined such that } (\lambda \otimes f)(x) \stackrel{\text{def}}{=} f(x)^\lambda \text{ for all } x \in \mathbb{R}.$$

Show that V constitutes a vector space, and provide explicit formulas for the neutral element $0 \in V$ as well as for the inverse element $(-f) \in V$ for any $f \in V$. Start by proving the closure properties implied by the above notation, and proceed by verifying all vector space axioms. It is mandatory to adhere to the symbols $\oplus$ and $\otimes$ in your notation wherever appropriate.

☛ CAVEAT: $0(x) \neq 0$, $(-f)(x) \neq -f(x)$. NO CONFUSION WILL ARISE IF YOU USE $\oplus/\otimes$ CONSISTENTLY.

♣

(**15**) **3.** Distribution Theory

In this problem we insist on a notational distinction between the Dirac point distribution and its formal integral representation involving a corresponding "Dirac function". We shall write

$$T_\delta : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto T_\delta(\phi) \stackrel{\text{def}}{=} \phi(0)\,,$$

for the distribution proper, respectively

$$\delta : \mathbb{R} \to \mathbb{R} : x \mapsto \delta(x)\,,$$

for the virtual function "under the integral", so $T_\delta(\phi) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx$.

The goal of this problem will be to prove that $\delta \notin L^p(\mathbb{R})$ for any $p>1$, including $p=\infty$.

One can show that there exist so-called "bump functions" $\psi \in \mathscr{S}(\mathbb{R})$ such that $\psi(x)=0$ outside an arbitrarily chosen support interval. In particular we consider the subfamily $\mathscr{B}_\epsilon(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$ of bump functions defined for given $\epsilon>0$ as follows:

$$\mathscr{B}_\epsilon(\mathbb{R}) = \left\{ \psi \in \mathscr{S}(\mathbb{R}) \mid \psi(x)=0 \text{ outside the interval } (-\tfrac{\epsilon}{2}, \tfrac{\epsilon}{2}), \text{ and } \max_{x\in\mathbb{R}} |\psi(x)| = \psi(0) = 1 \right\}.$$

(5) **a.** Show that $\mathscr{B}_\epsilon(\mathbb{R}) \subset L^q(\mathbb{R})$ for any $q \geq 1$, $\epsilon>0$, by proving that $\|\psi\|_q \leq \sqrt[q]{\epsilon}$ for $\psi \in \mathscr{B}_\epsilon(\mathbb{R})$.

(10) **b.** Use this fact to disprove the hypothesis that $\delta \in L^p(\mathbb{R})$ for some $p>1$ or $p=\infty$.

☞ Hint: Consider the hypothesis and subproblem a in the context of Hölder's inequality.

♣

(**30**) **4.** Fourier Transformation (Exam March 21, 2007, problem 3)

In this problem we use the following Fourier convention for $f \in \mathscr{S}'(\mathbb{R})$:

$$\widehat{f}(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx\,.$$

As a result we have for the inverse:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{i\omega x}\, d\omega\,.$$

Here we permit ourselves the sloppiness of identifying a regular tempered distribution f with its Riesz representant ("function under the integral") with function definition $f(x)$. The Dirac δ-distribution is identified with the "Dirac delta function" with function definition $\delta(x)$.

($7\frac{1}{2}$) **a.** Given $\widehat{f}(\omega) = \delta(\omega - a)$ for some constant $a \in \mathbb{R}$. Determine $f(x)$.

($7\frac{1}{2}$) **b.** Given $g(x) = 2\cos^2 x$. (With $\cos^2 x$ we mean $(\cos x)^2$.) Determine $\widehat{g}(\omega)$.

☞ Hint: You can check your result with the help of subproblem a.

In the following part you may use the standard integral

$$\int_{-\infty}^{\infty} e^{-(x+iy)^2}\,dx = \sqrt{\pi} \quad \text{regardless of the value of } y \in \mathbb{R}.$$

($7\frac{1}{2}$) **c.** Given $\phi(x) = \dfrac{1}{\sqrt{\pi}}\, e^{-x^2}$. Determine $\widehat{\phi}(\omega)$.

($7\frac{1}{2}$) **d.** Given $h(x) = \dfrac{2}{\sqrt{\pi}}\, \cos^2 x\, e^{-x^2}$. Determine $\widehat{h}(\omega)$.

☞ Hint: Note that $h = g\,\phi$, recall subproblems b and c.

THE END