EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Friday January 23 2015. Time: 13:30–16:30. Place: AUD 16.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin. The ⏲ logo marks a somewhat lengthy item; you might want to postpone these until the end.
- Motivate your answers. The use of course notes is allowed. The use of problem companion ("opgaven-en tentamenbundel"), calculator, laptop, smartphone, or any other equipment, is *not* allowed.
- You may provide your answers in Dutch or English.

GOOD LUCK!

(**20**) **1.** Vector Space

Let $V \subset C(\mathbb{R})$ be the subset of real-valued continuous functions on $\mathbb{R}$ given as follows:

$$V = \{ f \in C(\mathbb{R}) \mid f(-1)f(1) = 0 \} \, .$$

You may take it for granted that $C(\mathbb{R})$ is a real vector space under the usual definitions of vector addition and scalar multiplication.

(5) **a.** Explain in terms of (a) formula(s) what is meant by "the usual definitions of vector addition and scalar multiplication" in this particular case. Your formula(s) should also explain what is meant by the attribute "real" in "real vector space".

(5) **b.** Is the subset V itself a real vector space? If so, prove it, otherwise provide a counterexample to one of the defining axioms.

Now consider $W \subset C(\mathbb{R})$ defined as follows: $W = \{ f \in C(\mathbb{R}) \mid f(-1) = 0 \text{ AND } f(1) = 0 \}$.

(5) **c.** Prove that W constitutes a real vector space.

(5) **d.** Is this still true if we replace the logical conjunction 'AND' in the definition of W by 'OR'?

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(**25**) **2.** Linear Operator (Homework Assignment, December 13, 2006, Problem 1)

In this problem V is a vector space over $\mathbb{R}$ equipped with a real inner product $\langle \_|\_ \rangle : V \times V \to \mathbb{R}$. Furthermore, $a \in V$ is a fixed unit vector: $\langle a|a \rangle = 1$.

(10) **a.** Show that the subset $V_a \subset V$ generated by a and defined as

$$V_a = \{ v \in V \mid \langle a|v \rangle = 0 \} ,$$

constitutes a linear subspace of V.

b. The vector a, moreover, induces a mapping $\phi_a : V \to V$, as follows:

$$\phi_a(v) = v - \langle a|v \rangle \, a \, .$$

(3) **b1.** Prove that ϕ_a is a linear map.

(3) **b2.** Prove that $\phi_a(v) \in V_a$ for all $v \in V$.

(3) **b3.** Prove that $\phi_a(\phi_a(v)) = \phi_a(v)$ for all $v \in V$.

(3) **b4.** Prove that $\langle \phi_a(v)|w \rangle = \langle v|\phi_a(w) \rangle$ for all $v, w \in V$.

(3) **b5.** Suppose $w \in V$ is such that $\langle \phi_a(v)|w \rangle = 0$ for all $v \in V$. Show that $w = \lambda \, a$ for some $\lambda \in \mathbb{R}$ and determine the value of λ in terms of a en w.
(*Hint:* Use the previous part and the defining properties of the inner product.)

(**25**) **3.** ALGEBRA

We introduce the set $\mathbb{M}$ of real-valued 2×2-matrices, as follows:

$$\mathbb{M} = \left\{ A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\} .$$

It is tacitly understood that the set $\mathbb{M} \equiv \{\mathbb{M}, +, \cdot\}$ is endowed with a real vector space structure, enabling matrix addition $(+)$ and scalar multiplication $(\cdot)$ in the usual way.

(5) **a.** Show that $\mathbb{M}$ is closed under matrix addition and scalar multiplication. (For this reason we do not make a notational distinction between the *set* $\mathbb{M}$ and the *vector space* $\mathbb{M} \equiv \{\mathbb{M}, +, \cdot\}$.)

On top of the vector space structure we also account for standard matrix multiplication, defining a new set $\mathscr{M} \equiv \{\mathbb{M}, +, \cdot, \circ\}$—in which $\circ$ indicates the infix matrix product operator—consisting of all linear combinations and matrix products of elements of $\mathbb{M}$. (You may conform to the usual habit of omitting explicit scalar and matrix multiplication signs $\cdot$ respectively $\circ$ in your notation.)

(5) **b.** Show that $\mathscr{M}$ contains *all* real 2×2 matrices (i.e. $\mathbb{M}$ is *not* closed under $\circ$).
(*Hint:* First consider how you can form any diagonal matrix from a product of two matrices in $\mathbb{M}$.)

Let $f \in C^\omega(\mathbb{R})$ be a real, analytical function, with function prototype $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$. For such a function we implement *function overloading*, by defining a function carrying the same name, but with a different prototype, viz. $f : \mathscr{M} \to \mathscr{M} : A \mapsto f(A)$, as follows. If $f(x) = \sum_{k=0}^\infty a_k x^k$ is the Taylor series of $f \in C^\omega(\mathbb{R})$, then

$$f(A) \stackrel{\text{def}}{=} \sum_{k=0}^\infty a_k A^k \,.$$

Here A^k is shorthand for the k-fold matrix autoproduct of A: $A^k = \underbrace{A \circ \ldots \circ A}_{k \text{ factors}}$, with $A^0 = I$.

In the problems below you may use the following standard Taylor series expansions:

$$\exp(x) \stackrel{\text{def}}{=} e^x = \sum_{k=0}^\infty \frac{1}{k!} x^k \,, \qquad \cosh(x) = \sum_{k=0}^\infty \frac{1}{(2k)!} x^{2k} \,, \qquad \sinh(x) = \sum_{k=0}^\infty \frac{1}{(2k+1)!} x^{2k+1} \,.$$

(5) **c1.** Compute $\exp(A)$ for $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathbb{M}$, $a \geq 0$, and for $A = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \in \mathbb{M}$, $b \geq 0$.

(5 ⏲) **c2.** Show that if $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \in \mathbb{M}$, with $a>0$, $b>0$, then $\exp(A) = \begin{pmatrix} \cosh(\sqrt{ab}) & \sqrt{\frac{a}{b}}\sinh(\sqrt{ab}) \\ \sqrt{\frac{b}{a}}\sinh(\sqrt{ab}) & \cosh(\sqrt{ab}) \end{pmatrix}$.
(*Hint:* Recall the Taylor expansions of cosh and sinh.)

Finally we consider the case of a *matrix-valued function* $F : \mathbb{R} \to \mathscr{M} : t \mapsto F(t)$ given by

$$F(t) = \begin{pmatrix} 0 & f(t) \\ g(t) & 0 \end{pmatrix} \,,$$

in which $f, g \in C^\omega(\mathbb{R})$ are smooth, positive real-valued functions of one variable.

(5 ⏲) **d.** Show that the chain rule for scalar-valued functions does not trivially carry over to matrix-valued functions, by showing that

$$\frac{d}{dt} \exp(F(t)) \neq \exp(F(t)) \, \frac{dF(t)}{dt} \,.$$

(*Hint:* Use the result in **c2**, and note that it suffices to show inequality for one corresponding entry of the matrices on left and hand right hand sides.)

(**30**) **4.** Fourier Transformation and Distribution Theory

We consider the so-called *sign* function in one dimension, denoted $\mathrm{sgn} : \mathbb{R} \to \mathbb{C}$, and given by

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & \text{if } x > 0 \end{cases}$$

For suitably defined functions $u : \mathbb{R} \to \mathbb{C} : x \mapsto u(x)$ we may use the following Fourier convention:

$$\widehat{u}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} u(x)\, dx\,.$$

(5) **a.** Argue why sgn is *not* a "suitably defined function", in the sense that the Fourier formula above cannot be used to compute its Fourier transform.
(*Hint:* Apply the intergral formula, and indicate where exactly "things go wrong".)

We nevertheless wish to obtain the Fourier transform $\widehat{\mathrm{sgn}} : \mathbb{R} \to \mathbb{C} : \omega \mapsto \widehat{\mathrm{sgn}}(\omega)$ of sgn. To this end we must generalize the definition of the Fourier transform. One way to achieve this is via a limiting procedure. Consider the ϵ-parametrized family of functions $\mathrm{sgn}_\epsilon : \mathbb{R} \to \mathbb{C}$, with $\epsilon \geq 0$, given by

$$\mathrm{sgn}_\epsilon(x) = \begin{cases} -\exp(\epsilon x) & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +\exp(-\epsilon x) & \text{if } x > 0 \end{cases}$$

Note that $\mathrm{sgn}(x) = \mathrm{sgn}_0(x) = \lim_{\epsilon \downarrow 0} \mathrm{sgn}_\epsilon(x)$ for every $x \in \mathbb{R}$.

(5) **b.** Show that sgn_ϵ does admit a Fourier transform $\widehat{\mathrm{sgn}}_\epsilon$ according to the integral formula as long as $\epsilon > 0$, and compute $\widehat{\mathrm{sgn}}_\epsilon(\omega)$.

Let us define

$$\widehat{\mathrm{sgn}}(\omega) \stackrel{\text{def}}{=} \lim_{\epsilon \downarrow 0} \widehat{\mathrm{sgn}}_\epsilon(\omega)\,.$$

(5) **c.** Show that, according to this definition, $\widehat{\mathrm{sgn}}(\omega) = \dfrac{2}{i\omega}$.

Alternatively we may interpret sgn as a tempered distribution. To avoid confusion we shall make a notational distinction between the regular tempered distribution $\mathrm{SGN} : \mathscr{S}(\mathbb{R}) \to \mathbb{C}$, given by

$$\mathrm{SGN} \stackrel{\text{def}}{=} T_{\mathrm{sgn}} : \mathscr{S}(\mathbb{R}) \to \mathbb{C} : \phi \mapsto \mathrm{SGN}(\phi) = \int_{-\infty}^{\infty} \mathrm{sgn}(x)\, \phi(x)\, dx\,,$$

and its associated "function under the integral" $\mathrm{sgn} : \mathbb{R} \to \mathbb{C}$.

(5) **d.** Show that the distributional derivative of SGN is given by the tempered distribution $\mathrm{SGN}' = 2\delta$, in which $\delta : \mathscr{S}(\mathbb{R}) \to \mathbb{C}$ denotes the Dirac point distribution at the origin.

The *distributional Fourier transform* of a general tempered distribution $T \in \mathscr{S}'(\mathbb{R})$ is defined as follows. Let $T : \mathscr{S}(\mathbb{R}) \to \mathbb{C}$ be a tempered distribution, then

$$\widehat{T} : \mathscr{S}(\mathbb{R}) \to \mathbb{C} : \phi \mapsto \widehat{T}(\phi) \stackrel{\text{def}}{=} T(\widehat{\phi})\,.$$

(5) **e.** Show with the help of this definition that $\widehat{\delta} = T_1$, i.e. the regular tempered distribution associated with the constant function $1 : \mathbb{R} \to \mathbb{C} : x \mapsto 1$.

(5) **f.** Show that $\widehat{\mathrm{SGN}'} = 2T_1$.
(*Hint:* Use **d** and **e**.)

THE END