EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Friday January 24, 2014. Time: 14h00–17h00. Place: PAV SH2 E

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of problem companion ("opgaven- en tentamenbundel"), calculator, laptop, or other equipment, is *not* allowed.
- You may provide your answers in Dutch or English.

GOOD LUCK!

(25) **1.** VECTOR SPACE

We introduce the set $V = \mathbb{R}^2$ and furnish it with an addition and scalar multiplication operator, as follows. For all $(x,y) \in \mathbb{R}^2$, $(u,v) \in \mathbb{R}^2$, and $\lambda \in \mathbb{R}$ we define

$$(x,y) + (u,v) = (x+u, y+v) \quad \text{and} \quad \lambda \cdot (x,y) = (\lambda x, 0)\,.$$

(10) **a.** Show that, given these definitions, V does *not* constitute a vector space.

The axiom that fails to hold is the requirement $1 \cdot v = v$ for all $v \in V$. Indeed, taking $v = (x,y) \in V$ with $y \neq 0$ we obtain $1 \cdot v = 1 \cdot (x,y) = (1\,x, 0) = (x,0) \neq (x,y) = v$.

Next we consider the set $V = C^1(\mathbb{R})$ of continuously differentiable, real-valued functions with domain $\mathbb{R}$. You may take it for granted that V is a linear space given the usual definitions of vector addition and scalar multiplication for functions. Let $W \subset V$ be the subset of functions defined as follows:

$$W = \left\{ f \in V \mid f'(x) = f(0) \right\}$$

(5) **b.** If $\varnothing$ denotes the empty set, show that $W \neq \varnothing$.

The set W evidently contains the zero function, thus $W \neq \varnothing$.

(10) **c.** Show that W is a one-dimensional linear subspace, and provide an explicit basis function.

Note that $W \neq \varnothing$. Furthemore, if $f, g \in W$, $\lambda, \mu \in \mathbb{R}$, then we have $(\lambda f + \mu g)'(x) = \lambda f'(x) + \mu g'(x) = \lambda f(0) + \mu g(0) = (\lambda f + \mu g)(0)$, whence $\lambda f + \mu g \in W$ (closure). A basis is obtained by solving the differential equation for $f \in W$. Clearly we have $f(x) = f(0)x + c$, in which $c \in \mathbb{R}$ is a constant. By substituting $x = 0$ we see that $f(0) = c$, so that $f(x) = c(x+1)$. Thus W is spanned by the single function $b \in W$ given by $b(x) = x + 1$, i.e. $\mathscr{B} = \{b\}$ is a basis.

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(**35**) **2.** LINEAR OPERATOR

We consider the linear space $V = C^1([0,1]) \cap L^1([0,1])$ of real-valued, continuously differentiable, integrable functions, with the usual vector space structure.

(5) **a.** Give a precise mathematical definition of "the usual vector space structure".

Let $f, g \in V$, $\lambda, \mu \in \mathbb{R}$, then for all $x \in [0,1]$ we have $(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x)$.

Consider the operator $A : V \to W : f \mapsto A(f)$, with W a suitably defined function space, and

$$A(f)(x) = \int_0^x f(t)\,dt \quad \text{for } x \in [0,1].$$

(5) **b.** Show that W is a subset of V by arguing that, for all $f \in V$, $A(f)$ is continuously differentiable and $\|A(f)\|_1 \leq \|f\|_1$.

We must show that if $f \in V$, then also $A(f) \in V$. Note that $A(f)$ is a primitive (or antiderivative) of f, and thus certainly $A(f) \in C^1([0,1])$. Moreover,

$$\|A(f)\|_1 = \int_0^1 |A(f)(x)|\,dx = \int_0^1 |\int_0^x f(t)\,dt|\,dx \leq \int_0^1 \int_0^x |f(t)|\,dt\,dx \leq \int_0^1 \int_0^1 |f(t)|\,dt\,dx = \int_0^1 |f(t)|\,dt = \|f\|_1\,,$$

whence $A(f) \in L^1([0,1])$.

(5) **c1.** Show that A is a linear operator.

For any $f, g \in V$, $\lambda, \mu \in \mathbb{R}$ we have

$$A(\lambda f + \mu g) = \int_0^x (\lambda f + \mu g)(t)\,dt = \int_0^x \lambda f(t) + \mu g(t)\,dt = \lambda \int_0^x f(t)\,dt + \mu \int_0^x g(t)\,dt = \lambda A(f) + \mu A(g)\,.$$

Thus $A \in \mathscr{L}(V, W)$.

(5) **c2.** Show that $W \subset V$ is a linear subspace.

We need to show closure of W. Let $F, G \in W$, say $F = A(f)$, $G = A(g)$, with $f, g \in V$, then for any $\lambda, \mu \in \mathbb{R}$ we have $\lambda F + \mu G = \lambda A(f) + \mu A(g) = A(\lambda f + \mu g) \in W$. In the final step we have used the previous result, viz. $A \in \mathscr{L}(V, W)$.

A function $f \in V$ is called a *fixed point* of A if $A(f) = f$.

(5) **d.** Show that the only fixed point of A is the zero function.
(*Hint:* Differentiate the fixed point equation.)

Suppose $A(f) = f$, then differentiation yields $f = f'$, whence $f(x) = ce^x$. Moreover, from its definition it follows that $A(f)(0) = 0$, whence $f(0) = 0$. This initial condition implies $c = 0 \in \mathbb{R}$, thus $f = 0 \in V$. This is indeed a fixed point of A.

We furnish the linear space of linear operators on V, $\mathscr{L}(V, V)$, with an algebraic structure by defining "multiplication" in terms of operator composition $\circ : \mathscr{L}(V,V) \times \mathscr{L}(V,V) \to \mathscr{L}(V,V)$, i.e. if $A, B \in \mathscr{L}(V,V)$, then $A \circ B \in \mathscr{L}(V,V)$ is the linear operator given by

$$(A \circ B)(f) = A(B(f)) \quad \text{for all } f \in V.$$

(5) **e.** Explain what we mean by the operator exponential $e^A \in \mathscr{L}(V,V)$ for $A \in \mathscr{L}(V,V)$, in terms of this algebraic structure.
(*Hint:* Use the algebraic analogy with the familiar expansion $e^a = \sum_{k=0}^{\infty} \frac{1}{k!} a^k \in \mathbb{R}$ for numbers $a \in \mathbb{R}$.)

For $A \in \mathscr{L}(V,V)$, define $A^k = A \circ \ldots \circ A$ for $k \in \mathbb{N}_0$, with exactly k instances of A. Subsequently define $e^A \in \mathscr{L}(V,V)$ as

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \in \mathscr{L}(V,V)\,.$$

(5) **f.** Show that $u(x,t) = (e^{tA} f)(x)$ satisfies the following initial value problem for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$:

$$\begin{cases} \dfrac{\partial u}{\partial t} &= Au \\ u(x,0) &= f(x)\,. \end{cases}$$

From the previous problem it follows that $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$. Term by term differentiation w.r.t. t yields

$$\frac{d}{dt} e^{tA} = \frac{d}{dt} \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} k \frac{t^{k-1}}{k!} A^k = \sum_{j=0}^{\infty} \frac{t^j}{j!} A^{j+1} = A \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j = A e^{tA}\,.$$

The p.d.e. for $u(x,t)$ follows from this operator identity:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial (e^{tA} f)(x)}{\partial t} = \frac{d}{dt} e^{tA} f(x) = A e^{tA} f(x) = Au(x,t)\,.$$

The initial condition follows from the fact that $e^{tA} = I$, the identity operator, if $t = 0$, implying $u(x,0) = f(x)$.

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(**20**) **3.** Distribution Theory

Let $U \in \mathscr{S}'(\mathbb{R})$ be a tempered distribution satisfying the following "distributional ordinary differential equation" (distributional o.d.e.):

$$U'' = \delta\,,$$

in which $\delta \in \mathscr{S}'(\mathbb{R})$ is the Dirac point distribution given by $\delta : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto \delta(\phi) = \phi(0)$.

(5) **a.** Argue why this differential equation does not have a solution in $C^2(\mathbb{R})$.

If $U \in C^2(\mathbb{R})$, then $U'' \in C^0(\mathbb{R})$, contradicting the fact that $\delta \notin C^0(\mathbb{R})$.

We postulate that $U = T_u \in \mathscr{S}'(\mathbb{R})$ is a regular tempered distribution corresponding to some function $u : \mathbb{R} \to \mathbb{R}$. If $U = T_u$ satisfies the distributional o.d.e. above, then we shall refer to both u as well as U as a "distributional solution".

(10) **b.** Show that $u : \mathbb{R} \to \mathbb{R} : x \mapsto u(x) = \frac{1}{2}|x|$ is a distributional solution.

Substituting $U(\phi) = T_u(\phi) = \int_{-\infty}^{\infty} u(x)\phi(x)dx$ in the distributional o.d.e. yields $U''(\phi) = U(\phi'') = \int_{-\infty}^{\infty} u(x)\phi''(x)dx = \int_{-\infty}^{\infty} |x|\phi''(x)dx = -\frac{1}{2}\int_{-\infty}^{0} x\phi''(x)dx + \frac{1}{2}\int_{0}^{\infty} x\phi''(x)dx$. Integration by parts yields $U''(\phi) = -\frac{1}{2} x\phi'(x)\Big|_{-\infty}^{0} + \frac{1}{2}\int_{-\infty}^{0} \phi'(x)dx +$

$\frac{1}{2}x\phi'(x)\Big|_0^\infty - \frac{1}{2}\int_0^\infty \phi'(x)dx \stackrel{*}{=} \frac{1}{2}\phi(x)\Big|_{-\infty}^0 + \frac{1}{2}\phi(x)\Big|_0^\infty \stackrel{\star}{=} \phi(0) = \delta(\phi)$. In $*$ and $\star$ we have used $\lim_{x\to\pm\infty} x\phi'(x) = 0$, respectively $\lim_{x\to\pm\infty}\phi(x) = 0$. Since this holds for all $\phi \in \mathscr{S}(\mathbb{R})$ we have $U'' = \delta$.

(5) **c.** Show that the solution in problem b is not unique.

We may always add to u a "classical" solution $h : \mathbb{R} \to \mathbb{R}$ of the homogeneous o.d.e. Thus $u_h(x) = \frac{1}{2}|x| + h(x)$ is a solution for every $C^2(\mathbb{R})$-function h with $h'' = 0$. Clearly $h(x) = ax + b$. It is easily verified that h also has a vanishing second order derivative in distributional sense: $T_h''(\phi) = T_h(\phi'') = \int_{-\infty}^\infty h(x)\phi''(x)dx = \int_{-\infty}^\infty h''(x)\phi(x)dx = 0$. In the final step we have used two-fold partial integration.

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(**20**) **4.** Fourier Analysis (Exam June 15, 2009, Problem 4)

For each $n \in \mathbb{N}$ we define the function $f_n : \mathbb{R} \to \mathbb{R}$ as follows:

$$f_n(x) \stackrel{\text{def}}{=} \frac{1}{x^n}\,.$$

We employ the following Fourier convention:

$$\widehat{f}(\omega) = \int_{-\infty}^\infty f(x)\,e^{-i\omega x}\,dx \quad \text{with, as a result,} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \widehat{f}(\omega)\,e^{i\omega x}\,d\omega\,.$$

Without proof we state the Fourier transform of the function f_1, viz. $\widehat{f_1}(\omega) = -i\,\pi\,\mathrm{sgn}\,(\omega)$. Here, $\mathrm{sgn}\,(\omega) = -1$ for $\omega < 0$, $\mathrm{sgn}\,(0) = 0$, and $\mathrm{sgn}\,(\omega) = +1$ for $\omega > 0$.

The convolution product of two functions f and g is defined as

$$(f * g)(x) \stackrel{\text{def}}{=} \int_{-\infty}^\infty f(y)\,g(x-y)\,dy\,,$$

provided the integral on the right hand side exists. If this is not the case, but the functions f and g do permit Fourier transformation, we employ the following *implicit definition* for the convolution product ($\mathcal{F}(u)$ is here synonymous for $\widehat{u}$):

$$\mathcal{F}(f * g) = \mathcal{F}(f)\,\mathcal{F}(g)\,.$$

(5) **a.** Show that the function $\widehat{f_n}$ is purely imaginary for odd $n \in \mathbb{N}$, and real for even $n \in \mathbb{N}$. (*Hint:* Use the (anti-)symmetry property $f_n(x) = (-1)^n f_n(-x)$ for all $x \in \mathbb{R}$.)

If $z = a + bi \in \mathbb{C}$ we write the complex conjugate as $z^* = a - bi$, $a, b \in \mathbb{R}$. For $\omega \in \mathbb{R}$ arbitrary we have

$$\widehat{f_n}(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^\infty f_n(x)\,e^{-i\omega x}\,dx \stackrel{\text{hint}}{=} (-1)^n \int_{-\infty}^\infty f_n(-x)\,e^{-i\omega x}\,dx \stackrel{*}{=} (-1)^n \int_{-\infty}^\infty f_n(y)\,e^{i\omega y}\,dy \stackrel{\star}{=} (-1)^n \left(\int_{-\infty}^\infty f_n(y)\,e^{-i\omega y}\,dy\right)^*$$
$$= (-1)^n\,\widehat{f}_n^*(\omega)\,.$$

In $*$ substitution of variables, $x = -y$, has been used. In $\star$ the fact that $f_n(y) \in \mathbb{R}$ for all $y \in \mathbb{R}$ has been used, as well as the fact that $\int_\Omega f^*(x)\,dx = \left(\int_\Omega f(x)\,dx\right)^*$ for any integration domein $\Omega \subset \mathbb{R}$. Conclusion: For even n we have $\widehat{f_n}(\omega) = \widehat{f}_n^*(\omega)$, i.e. $\widehat{f_n}(\omega) \in \mathbb{R}$. For odd n we have $\widehat{f_n}(\omega) = -\widehat{f}_n^*(\omega)$, i.e. $\widehat{f_n}(\omega) \in i\mathbb{R}$, i.e. purely imaginary.

b. Prove the following recursions for the functions f_n, respectively $\widehat{f_n}$:

$(2\frac{1}{2})$ **b1.** $f_{n+1}(x) = -\dfrac{1}{n}\, f_n'(x)$, $n \in \mathbb{N}$.

Straightforward differentiation yields $f_n'(x) \stackrel{\text{def}}{=} \left[x^{-n}\right]' = -n\, x^{-n-1} \stackrel{\text{def}}{=} -n\, f_{n+1}(x)$, from which the conjecture follows.

$(2\frac{1}{2})$ **b2.** $\widehat{f}_{n+1}(\omega) = -\dfrac{1}{n}\, i\omega\, \widehat{f}_n(\omega)$, $n \in \mathbb{N}$.

We have $\mathcal{F}(f_{n+1})(\omega) \stackrel{*}{=} -\dfrac{1}{n}\, \mathcal{F}(f_n')(\omega) \stackrel{\star}{=} -\dfrac{1}{n}\, i\omega\, \mathcal{F}(f_n)(\omega)$. In $*$ problem b1 has been used together with linearity of Fourier transformation. In $\star$ the following property has been used: $\mathcal{F}(f')(\omega) = i\omega\mathcal{F}(f)(\omega)$.

(5) **c.** Determine $\widehat{f}_n(\omega)$ for each $n \in \mathbb{N}$, given that $\widehat{f}_1(\omega) = -i\, \pi\, \mathrm{sgn}\,(\omega)$.

Claim (induction hypothesis): $\widehat{f}_n(\omega) = \dfrac{\pi}{i}\, \dfrac{(-i\omega)^{n-1}}{(n-1)!}\, \mathrm{sgn}(\omega)$. Proof by induction: For $n = 1$ this result agrees with the one given. Furthermore, $\widehat{f}_{n+1}(\omega) \stackrel{\text{b2}}{=} -\dfrac{1}{n}\, i\omega\, \widehat{f}_n(\omega) \stackrel{*}{=} -\dfrac{1}{n}\, i\omega\, \dfrac{\pi}{i}\, \dfrac{(-i\omega)^{n-1}}{(n-1)!}\, \mathrm{sgn}(\omega) = \dfrac{\pi}{i}\, \dfrac{(-i\omega)^{n}}{n!}\, \mathrm{sgn}(\omega)$. In $*$ the induction hypothesis has been invoked for $\widehat{f}_n(\omega)$.

(5) **d.** Prove: $\widehat{f}_n * \widehat{f}_m = 2\pi\, \widehat{f}_{n+m}$ for all $n, m \in \mathbb{N}$.

It is evident that $f_n\, f_m = f_{n+m}$ ($\star$), as for all $x \in \mathbb{R}$ we have $f_n(x)\, f_m(x) = x^{-n}\, x^{-m} = x^{-(n+m)} = f_{n+m}(x)$. Consequently: $\widehat{f}_n * \widehat{f}_m = \mathcal{F}(f_n) * \mathcal{F}(f_m) \stackrel{*}{=} 2\pi\mathcal{F}(f_n\, f_m) \stackrel{\star}{=} 2\pi\, \mathcal{F}(f_{n+m}) = 2\pi\, \widehat{f}_{n+m}$. In $*$ we have used the fact that for two functions u_1 en u_2 we have, provided left and right hand sides exist, $\mathcal{F}(u_1\, u_2) = \dfrac{1}{2\pi}\, \mathcal{F}(u_1) * \mathcal{F}(u_2)$. In $\star$ we have used the first observation above.

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