EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Friday January 24, 2014. Time: 14h00–17h00. Place: PAV SH2 E

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of problem companion ("opgaven- en tentamenbundel"), calculator, laptop, or other equipment, is *not* allowed.
- You may provide your answers in Dutch or English.

GOOD LUCK!

(25) **1.** VECTOR SPACE

We introduce the set $V = \mathbb{R}^2$ and furnish it with an addition and scalar multiplication operator, as follows. For all $(x, y) \in \mathbb{R}^2$, $(u, v) \in \mathbb{R}^2$, and $\lambda \in \mathbb{R}$ we define

$$(x, y) + (u, v) = (x + u, y + v) \quad \text{and} \quad \lambda \cdot (x, y) = (\lambda x, 0)\,.$$

(10) **a.** Show that, given these definitions, V does *not* constitute a vector space.

Next we consider the set $V = C^1(\mathbb{R})$ of continuously differentiable, real-valued functions with domain $\mathbb{R}$. You may take it for granted that V is a linear space given the usual definitions of vector addition and scalar multiplication for functions. Let $W \subset V$ be the subset of functions defined as follows:

$$W = \left\{ f \in V \mid f'(x) = f(0) \right\}$$

(5) **b.** If $\varnothing$ denotes the empty set, show that $W \neq \varnothing$.

(10) **c.** Show that W is a one-dimensional linear subspace, and provide an explicit basis function.

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(35) **2.** LINEAR OPERATOR

We consider the linear space $V = C^1([0, 1]) \cap L^1([0, 1])$ of real-valued, continuously differentiable, integrable functions, with the usual vector space structure.

(5) **a.** Give a precise mathematical definition of "the usual vector space structure".

Consider the operator $A : V \to W : f \mapsto A(f)$, with W a suitably defined function space, and

$$A(f)(x) = \int_0^x f(t)\, dt \quad \text{for } x \in [0, 1].$$

(5) **b.** Show that W is a subset of V by arguing that, for all $f \in V$, $A(f)$ is continuously differentiable and $\|A(f)\|_1 \leq \|f\|_1$.

(5) **c1.** Show that A is a linear operator.

(5) **c2.** Show that $W \subset V$ is a linear subspace.

A function $f \in V$ is called a *fixed point* of A if $A(f) = f$.

(5) **d.** Show that the only fixed point of A is the zero function.
(*Hint:* Differentiate the fixed point equation.)

We furnish the linear space of linear operators on V, $\mathscr{L}(V,V)$, with an algebraic structure by defining "multiplication" in terms of operator composition $\circ : \mathscr{L}(V,V) \times \mathscr{L}(V,V) \to \mathscr{L}(V,V)$, i.e. if $A, B \in \mathscr{L}(V,V)$, then $A \circ B \in \mathscr{L}(V,V)$ is the linear operator given by

$$(A \circ B)(f) = A(B(f)) \quad \text{for all } f \in V.$$

(5) **e.** Explain what we mean by the operator exponential $e^A \in \mathscr{L}(V,V)$ for $A \in \mathscr{L}(V,V)$, in terms of this algebraic structure.
(*Hint:* Use the algebraic analogy with the familiar expansion $e^a = \sum_{k=0}^{\infty} \frac{1}{k!} a^k \in \mathbb{R}$ for numbers $a \in \mathbb{R}$.)

(5) **f.** Show that $u(x,t) = (e^{tA} f)(x)$ satisfies the following initial value problem for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$:

$$\begin{cases} \dfrac{\partial u}{\partial t} & = & Au \\ u(x,0) & = & f(x)\,. \end{cases}$$

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(**20**) **3.** Distribution Theory

Let $U \in \mathscr{S}'(\mathbb{R})$ be a tempered distribution satisfying the following "distributional ordinary differential equation" (distributional o.d.e.):

$$U'' = \delta\,,$$

in which $\delta \in \mathscr{S}'(\mathbb{R})$ is the Dirac point distribution given by $\delta : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto \delta(\phi) = \phi(0)$.

(5) **a.** Argue why this differential equation does not have a solution in $C^2(\mathbb{R})$.

We postulate that $U = T_u \in \mathscr{S}'(\mathbb{R})$ is a regular tempered distribution corresponding to some function $u : \mathbb{R} \to \mathbb{R}$. If $U = T_u$ satisfies the distributional o.d.e. above, then we shall refer to both u as well as U as a "distributional solution".

(10) **b.** Show that $u : \mathbb{R} \to \mathbb{R} : x \mapsto u(x) = \dfrac{1}{2}|x|$ is a distributional solution.

(5) **c.** Show that the solution in problem b is not unique.

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(**20**) **4.** Fourier Analysis (Exam June 15, 2009, Problem 4)

For each $n \in \mathbb{N}$ we define the function $f_n : \mathbb{R} \to \mathbb{R}$ as follows:

$$f_n(x) \stackrel{\text{def}}{=} \frac{1}{x^n}\,.$$

We employ the following Fourier convention:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \quad \text{with, as a result,} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{i\omega x}\, d\omega\,.$$

Without proof we state the Fourier transform of the function f_1, viz. $\widehat{f}_1(\omega) = -i\,\pi\,\mathrm{sgn}\,(\omega)$. Here, $\mathrm{sgn}\,(\omega) = -1$ for $\omega < 0$, $\mathrm{sgn}\,(0) = 0$, and $\mathrm{sgn}\,(\omega) = +1$ for $\omega > 0$.

The convolution product of two functions f and g is defined as

$$(f * g)(x) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy\,,$$

provided the integral on the right hand side exists. If this is not the case, but the functions f and g do permit Fourier transformation, we employ the following *implicit definition* for the convolution product ($\mathcal{F}(u)$ is here synonymous for $\widehat{u}$):

$$\mathcal{F}(f * g) = \mathcal{F}(f)\,\mathcal{F}(g)\,.$$

(5) **a.** Show that the function $\widehat{f}_n$ is purely imaginary for odd $n \in \mathbb{N}$, and real for even $n \in \mathbb{N}$. (*Hint:* Use the (anti-)symmetry property $f_n(x) = (-1)^n f_n(-x)$ for all $x \in \mathbb{R}$.)

b. Prove the following recursions for the functions f_n, respectively $\widehat{f}_n$:

($2\frac{1}{2}$) **b1.** $f_{n+1}(x) = -\dfrac{1}{n}\, f_n'(x)$, $n \in \mathbb{N}$.

($2\frac{1}{2}$) **b2.** $\widehat{f}_{n+1}(\omega) = -\dfrac{1}{n}\, i\omega\, \widehat{f}_n(\omega)$, $n \in \mathbb{N}$.

(5) **c.** Determine $\widehat{f}_n(\omega)$ for each $n \in \mathbb{N}$, given that $\widehat{f}_1(\omega) = -i\,\pi\,\mathrm{sgn}\,(\omega)$.

(5) **d.** Prove: $\widehat{f}_n * \widehat{f}_m = 2\pi\, \widehat{f}_{n+m}$ for all $n, m \in \mathbb{N}$.

THE END