EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Friday January 25, 2013. Time: 14h00–17h00. Place: MF lecture room 07

Read this first!

- Write your name and student ID on each paper.
- The exam consists of 3 problems. Maximum credits are indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of any additional material or equipment, including the problem companion ("opgaven- en tentamenbundel"), is *not* allowed.
- You may provide your answers in Dutch or English.
- Do not hesitate to ask questions on linguistic matters or if you suspect an erroneous problem formulation.

Good luck!

(**30**) 1. Group Theory

In this problem we consider the set of 2-parameter transformations on $\mathbb{L}_2(\mathbb{R})$ defined by

$$G = \{T_{a,b} : \mathbb{L}_2(\mathbb{R}) \to \mathbb{L}_2(\mathbb{R}) : f \mapsto T_{a,b}(f) \mid T_{a,b}(f)(x) = bf(x+a),\, a \in \mathbb{R},\, b \in \mathbb{R}^+\} .$$

By $T_{a,b}(f)(x)$ we mean $(T_{a,b}(f))\,(x)$. We furnish the set G with the usual composition operator, indicated by the infix symbol $\circ$:

$$\circ : G \times G \to G : (T_{a,b}, T_{c,d}) \mapsto T_{a,b} \circ T_{c,d} ,$$

i.e. $(T_{a,b} \circ T_{c,d})(f) = T_{a,b}(T_{c,d}(f))$.

a. Show that this is a good definition by proving the following claims for $a, c \in \mathbb{R}$, $b, d \in \mathbb{R}^+$:

(5) **a1.** If $f \in \mathbb{L}_2(\mathbb{R})$, then $T_{a,b}(f) \in \mathbb{L}_2(\mathbb{R})$. (Closure of $\mathbb{L}_2(\mathbb{R})$ under the mapping $T_{a,b}$.)

Suppose $f \in \mathbb{L}_2(\mathbb{R})$, then $\|T_{a,b}f\|_2^2 = \int_{-\infty}^{\infty} |bf(x+a)|^2 dx \overset{*}{=} b^2 \int_{-\infty}^{\infty} |f(y)|^2 dy = b^2 \|f\|_2^2 < \infty$. In $*$ we have changed variables: $y = x + a$, the other identities rely on definitions.

(5) **a2.** If $T_{a,b} \in G$, then $T_{a,b} \circ T_{c,d} = T_{a+c,bd}$. (Closure of G under composition $\circ$.)

For arbitrary $f \in \mathbb{L}_2(\mathbb{R})$ and $x \in \mathbb{R}$ we have $(T_{a,b} \circ T_{c,d})(f)(x) = T_{a,b}(T_{c,d}(f))(x) = bT_{c,d}(f)(x+a) = bdf(x+a+c) = T_{a+c,bd}(f)(x)$, so that we may conclude that $T_{a,b} \circ T_{c,d} = T_{a+c,bd}$.

(10) **b.** Show that $\{G, \circ\}$ constitutes a commutative group, and give explicit expressions for the identity element $e \in G$, and for the inverse element $T_{a,b}^{\text{inv}} \in G$ corresponding to $T_{a,b} \in G$.

Closure has been proven. Let $a, c, e \in \mathbb{R}$ and $b, d, f \in \mathbb{R}^+$ be arbitrary. Commutativity follows from $T_{a,b} \circ T_{c,d} = T_{a+c,bd} = T_{c+a,db} = T_{c,d} \circ T_{a,b}$. Associativity follows from the fact that composition is, by construction, associative.

Alternatively, exploit the identity proven under a2: $(T_{a,b} \circ T_{c,d}) \circ T_{e,f} = T_{a+c,bd} \circ T_{e,f} = T_{(a+c)+e,(bd)f} = T_{a+(c+e),b(df)} = T_{a,b} \circ (T_{c+e,df}) = T_{a,b} \circ (T_{c,d} \circ T_{e,f})$. The identity element is $T_{0,1} \in G$, since $T_{a,b} \circ T_{0,1} = T_{0,1} \circ T_{a,b} = T_{0+a,1b} = T_{a,b}$. The inverse element $T_{a,b}^{-1}$ of $T_{a,b} \in G$ is given by $T_{a,b}^{-1} = T_{-a,1/b} \in G$, since $T_{a,b} \circ T_{-a,1/b} = T_{-a,1/b} \circ T_{a,b} = T_{-a+a,b/b} = T_{0,1}$.

(10) **c.** Show that $G_1 = \{T_{a,b} \in G \mid a \in \mathbb{R},\, b = 1\}$ is a subgroup of G.
☞ HINT: EXPLOIT THE FACT THAT G IS A GROUP AND $G_1 \subset G$.

$T_{a,1} \circ T_{c,1} = T_{a+c,1} \in G_1$, so $G_1 \subset G$ is closed under $\circ$. Moreover, $T_{a,1}^{-1} = T_{-a,1} \in G$, so $G_1 \subset G$ is also closed under inversion. Hence $G_1 \subset G$ is a subgroup.

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(**30**) **2.** DISTRIBUTION THEORY

Recall the Dirac point distribution,

$$\delta : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto \delta(\phi) = \phi(0)\,,$$

and its derivative,

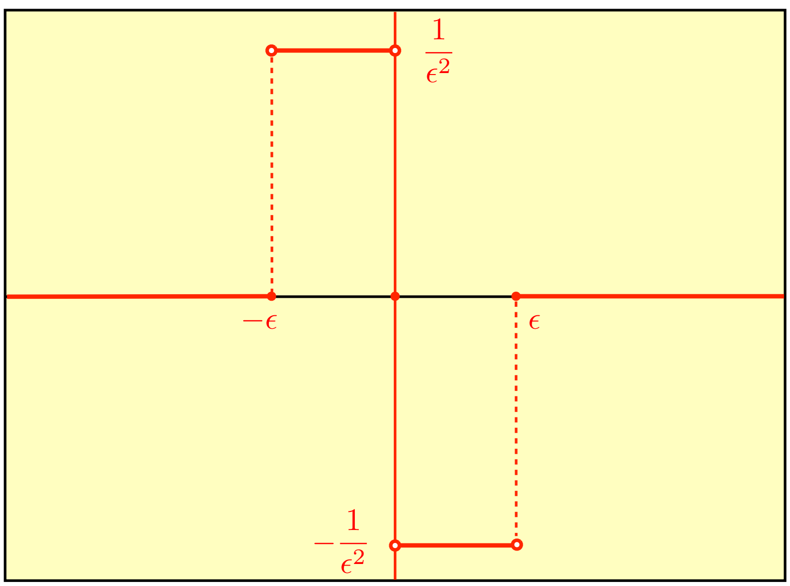
$$\delta' : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto \delta'(\phi) = -\phi'(0)\,.$$

In this problem we consider an approximation of δ' in the form of a 1-parameter family of functions, given by

$$f_\epsilon : \mathbb{R} \to \mathbb{R} : x \mapsto f_\epsilon(x) = \begin{cases} 0 & \text{if } x \leq -\epsilon \\ 1/\epsilon^2 & \text{if } -\epsilon < x < 0 \\ 0 & \text{if } x = 0 \\ -1/\epsilon^2 & \text{if } 0 < x < \epsilon \\ 0 & \text{if } x \geq \epsilon \end{cases}$$

with $\epsilon > 0$.

(5) **a.** Draw the graph of $y = f_\epsilon(x)$, indicating relevant values (in terms of ϵ) on each axis.



Consider the regular tempered distribution T_{f_ϵ} associated with the function f_ϵ, i.e.

$$T_{f_\epsilon} : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto T_{f_\epsilon}(\phi) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)dx\,.$$

(10) **b.** Show that $T_{f_\epsilon}(\phi) = \frac{1}{\epsilon^2}\int_{-\epsilon}^{0} \phi(x)dx - \frac{1}{\epsilon^2}\int_{0}^{\epsilon} \phi(x)dx.$

We may replace the integration boundaries $\pm\infty$ by the boundaries $\pm\epsilon$ of the compact domain of support of f_ϵ. Subsequently we may split the integral $\int_{-\epsilon}^{\epsilon} = \int_{-\epsilon}^{0} + \int_{0}^{\epsilon}$, and substitute the definition of f_ϵ.

Let $\phi \in \mathscr{S}(\mathbb{R})$ be an analytical test function, and recall Taylor's theorem:

$$\phi(x) = \phi(0) + \phi'(0)x + \frac{1}{2}\phi''(\xi)x^2 ,$$

in which ξ is some number between 0 and x.

(5) **c.** Using this Taylor expansion, argue (mathematically) why we may replace the Lagrange remainder term $\frac{1}{2}\phi''(\xi)x^2$ in the expression for $T_{f_\epsilon}(\phi)$ by a term of order $\mathcal{O}(\epsilon^2)$ as $\epsilon \downarrow 0$.

Both ξ and x are of order $\mathcal{O}(\epsilon)$ on the effective support interval, so $\phi''(\xi) = \mathcal{O}(1)$, whence we have for the Lagrange remainder $\frac{1}{2}\phi''(\xi)x^2 = \mathcal{O}(\epsilon^2)$.

(10) **d.** Show that, for any analytical test function $\phi \in \mathscr{S}(\mathbb{R})$, $\lim_{\epsilon\downarrow 0} T_{f_\epsilon}(\phi) = \delta'(\phi)$.

We have

$$\lim_{\epsilon\downarrow 0} T_{f_\epsilon}(\phi) = \lim_{\epsilon\downarrow 0}\left[\frac{1}{\epsilon^2}\int_{-\epsilon}^{0}\phi(x)dx - \frac{1}{\epsilon^2}\int_{0}^{\epsilon}\phi(x)dx\right] = \lim_{\epsilon\downarrow 0}\left[\frac{1}{\epsilon^2}\int_{-\epsilon}^{0}\left(\phi(0)+\phi'(0)x+\mathcal{O}(\epsilon^2)\right)dx - \frac{1}{\epsilon^2}\int_{0}^{\epsilon}\left(\phi(0)+\phi'(0)x+\mathcal{O}(\epsilon^2)\right)dx\right] .$$

Observe the following:

- The zeroth order terms in both integrals cancel eachother.
- The first order terms in both integrals add up: $\frac{1}{\epsilon^2}\int_{-\epsilon}^{0}\phi'(0)xdx - \frac{1}{\epsilon^2}\int_{0}^{\epsilon}\phi'(0)xdx = (-\frac{1}{2}-\frac{1}{2})\phi'(0) = -\phi'(0)$.
- The $\mathcal{O}(\epsilon^2)$ term in each integral yields, after integration, and taking into account the factor $1/\epsilon^2$, an $\mathcal{O}(\epsilon)$ outcome.

Thus

$$\lim_{\epsilon\downarrow 0} T_{f_\epsilon}(\phi) = \lim_{\epsilon\downarrow 0}\left[-\phi'(0)+\mathcal{O}(\epsilon)\right] = -\phi'(0) \stackrel{\text{def}}{=} \delta'(\phi) .$$

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(**40**) **3.** Fourier Transformation (Exam January 17, 2007, Problem 3)

In this problem we employ the following Fourier convention:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \quad \text{with, consequently,} \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{i\omega x}\, d\omega\,.$$

We also define the following complex valued one-dimensional signal $f_{a,b} : \mathbb{R} \to \mathbb{C} : x \mapsto f(x)$, as follows:

$$f_{a,b}(x) = e^{(a+bi)\,|x|}\,.$$

In this expression, $a, b \in \mathbb{R}$ are constant parameters and $|x|$ denotes the absolute value of $x \in \mathbb{R}$.

(10) **a.** Determine $\widehat{f_{a,b}}(\omega)$ and state the necessary conditions that $a, b \in \mathbb{R}$ have to fulfill in order for this function to be well-defined (in a classical, i.e. non-distributional sense).

$$\widehat{f}_{a,b}(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} e^{-i\omega x}\, f_{a,b}(x)\, dx = \int_{-\infty}^{\infty} e^{-i\omega x + (a+bi)|x|}\, dx = \int_{-\infty}^{0} e^{(-a-(b+\omega)i)x}\, dx + \int_{0}^{\infty} e^{(a+(b-\omega)i)x}\, dx =$$

$$\frac{-1}{a+(b+\omega)i} \left[e^{-ax}\, e^{-(b+\omega)ix} \right]_{-\infty}^{0} + \frac{1}{a+(b-\omega)i} \left[e^{ax}\, e^{(b-\omega)ix} \right]_{0}^{\infty} = \left(-\frac{1}{a+(b+\omega)i} - \frac{1}{a+(b-\omega)i} \right) \stackrel{*}{=} -\frac{2(a+bi)}{(a+bi)^2 + \omega^2}\,,$$

under the necessary condition that $a < 0$. The last step $(*)$ and the associated condition follow from the limits

$$\lim_{x \to -\infty} e^{-ax}\, e^{-(b+\omega)ix} = \lim_{x \to \infty} e^{ax}\, e^{(b-\omega)ix} = 0 \quad \text{iff } a < 0.$$

Note that the oscillating factors $e^{-(b+\omega)ix}$ and $e^{(b-\omega)ix}$ are irrelevant in these limiting cases, as their absolute values are equal to 1.

The convolution of two functions $f, g : \mathbb{R} \to \mathbb{C}$ is given by

$$(f * g)\,(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy\,.$$

b. Prove that convolution is associative, i.e. that for all $f, g, h : \mathbb{R} \to \mathbb{C}$ for which the expressions below are well-defined we have

(5) **b1.** $f * (g * h) = (f * g) * h$, and

Repetitive use of the theorem $\mathcal{F}(f * g) = \mathcal{F}(f)\, \mathcal{F}(g)$, or $f * g = \mathcal{F}^{-1}(\mathcal{F}(f)\, \mathcal{F}(g))$, yields

$$f * (g * h) = \mathcal{F}^{-1}(\mathcal{F}(f)\, \mathcal{F}(g * h)) = \mathcal{F}^{-1}(\mathcal{F}(f)\, (\mathcal{F}(g)\, \mathcal{F}(h))) \stackrel{*}{=} \mathcal{F}^{-1}((\mathcal{F}(f)\, \mathcal{F}(g))\, \mathcal{F}(h)) = \mathcal{F}^{-1}(\mathcal{F}(f * g)\, \mathcal{F}(h)) = (f * g) * h\,.$$

In $*$ associativity of the usual function product has been used.

(5) **b2.** $f * g = g * f$.

By the same token it follows that

$$f * g = \mathcal{F}^{-1}(\mathcal{F}(f)\, \mathcal{F}(g)) \stackrel{*}{=} \mathcal{F}^{-1}(\mathcal{F}(g)\, \mathcal{F}(f)) = g * f\,.$$

In $*$ commutativity of the usual function product has been used.

By the symbol $*^n$ we denote n-fold convolution, i.e.

$$f *^n f \stackrel{\text{def}}{=} f * \ldots * f \quad \text{with } n{+}1 \text{ factors } f.$$

(10) **c.** Determine the explicit form of the function $\widehat{f}_{a,b} *^n \widehat{f}_{a,b}$ for those $a, b \in \mathbb{R}$ for which $\widehat{f}_{a,b}$ is well-defined (recall part a).

Write $\mathcal{F}(f) = \widehat{f}$, then the product of two functions $f, g : \mathbb{R} \to \mathbb{C}$ satisfies

$$\mathcal{F}(f\,g) = \frac{1}{2\pi}\left(\mathcal{F}(f) * \mathcal{F}(g)\right) \quad \text{so that} \quad \widehat{f} * \widehat{g} = 2\pi\,\mathcal{F}(f\,g)\,.$$

In particular it follows that

$$\widehat{f}_{a,b} *^n \widehat{f}_{a,b} = (2\pi)^n\,\mathcal{F}(f_{a,b}^{n+1}) \stackrel{*}{=} (2\pi)^n\,\mathcal{F}(f_{(n+1)a,(n+1)b}) = (2\pi)^n\,\widehat{f}_{(n+1)a,(n+1)b}\,.$$

In $*$ the explicit form of $f_{a,b}$ has been used, which shows that $f_{a,b}^n(x) = (f_{a,b}(x))^n = (e^{(a+bi)|x|})^n = e^{(na+nbi)|x|} = f_{na,nb}(x)$ for all $x \in \mathbb{R}$. Conclusion:

$$\left(\widehat{f}_{a,b} *^n \widehat{f}_{a,b}\right)(\omega) = (2\pi)^n\,\widehat{f}_{(n+1)a,(n+1)b}(\omega) \stackrel{\text{a}}{=} -(2\pi)^n\,\frac{2(n{+}1)(a{+}bi)}{(n{+}1)^2(a+bi)^2 + \omega^2}\,.$$

(10) **d.** Let $g(x) = x\,e^{-|x|}$. Find $\widehat{g}(\omega)$.

$$\widehat{g}(\omega) = \int_{-\infty}^{\infty} x\,e^{-i\omega x-|x|}\,dx = i\frac{d}{d\omega}\int_{-\infty}^{\infty} e^{-i\omega x-|x|}\,dx = i\frac{d}{d\omega}\widehat{f}_{a=-1,b=0}(\omega) \stackrel{\text{a}}{=} i\frac{d}{d\omega}\frac{2}{1+\omega^2} = \frac{-4i\omega}{(1+\omega^2)^2}\,.$$

THE END