EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Friday January 25, 2013. Time: 14h00–17h00. Place: MF lecture room 07

Read this first!

- Write your name and student ID on each paper.
- The exam consists of 3 problems. Maximum credits are indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of any additional material or equipment, including the problem companion ("opgaven- en tentamenbundel"), is *not* allowed.
- You may provide your answers in Dutch or English.
- Do not hesitate to ask questions on linguistic matters or if you suspect an erroneous problem formulation.

Good luck!

(**30**) **1.** Group Theory

In this problem we consider the set of 2-parameter transformations on $\mathbb{L}_2(\mathbb{R})$ defined by

$$G = \{T_{a,b} : \mathbb{L}_2(\mathbb{R}) \to \mathbb{L}_2(\mathbb{R}) : f \mapsto T_{a,b}(f) \mid T_{a,b}(f)(x) = bf(x+a),\, a \in \mathbb{R},\, b \in \mathbb{R}^+\} .$$

By $T_{a,b}(f)(x)$ we mean $(T_{a,b}(f))\,(x)$. We furnish the set G with the usual composition operator, indicated by the infix symbol $\circ$:

$$\circ : G \times G \to G : (T_{a,b}, T_{c,d}) \mapsto T_{a,b} \circ T_{c,d} ,$$

i.e. $(T_{a,b} \circ T_{c,d})(f) = T_{a,b}(T_{c,d}(f))$.

a. Show that this is a good definition by proving the following claims for $a, c \in \mathbb{R}$, $b, d \in \mathbb{R}^+$:

(5) **a1.** If $f \in \mathbb{L}_2(\mathbb{R})$, then $T_{a,b}(f) \in \mathbb{L}_2(\mathbb{R})$. (Closure of $\mathbb{L}_2(\mathbb{R})$ under the mapping $T_{a,b}$.)

(5) **a2.** If $T_{a,b} \in G$, then $T_{a,b} \circ T_{c,d} = T_{a+c,bd}$. (Closure of G under composition $\circ$.)

(10) **b.** Show that $\{G, \circ\}$ constitutes a commutative group, and give explicit expressions for the identity element $e \in G$, and for the inverse element $T^{\text{inv}}_{a,b} \in G$ corresponding to $T_{a,b} \in G$.

(10) **c.** Show that $G_1 = \{T_{a,b} \in G \mid a \in \mathbb{R},\, b = 1\}$ is a subgroup of G.
☞ Hint: Exploit the fact that G is a group and $G_1 \subset G$.

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(**30**) **2.** Distribution Theory

Recall the Dirac point distribution,

$$\delta : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto \delta(\phi) = \phi(0)\,,$$

and its derivative,

$$\delta' : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto \delta'(\phi) = -\phi'(0)\,.$$

In this problem we consider an approximation of δ' in the form of a 1-parameter family of functions, given by

$$f_\epsilon : \mathbb{R} \to \mathbb{R} : x \mapsto f_\epsilon(x) = \begin{cases} 0 & \text{if } x \leq -\epsilon \\ 1/\epsilon^2 & \text{if } -\epsilon < x < 0 \\ 0 & \text{if } x = 0 \\ -1/\epsilon^2 & \text{if } 0 < x < \epsilon \\ 0 & \text{if } x \geq \epsilon \end{cases}$$

with $\epsilon > 0$.

(5) **a.** Draw the graph of $y = f_\epsilon(x)$, indicating relevant values (in terms of ϵ) on each axis.

Consider the regular tempered distribution T_{f_ϵ} associated with the function f_ϵ, i.e.

$$T_{f_\epsilon} : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto T_{f_\epsilon}(\phi) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f_\epsilon(x)\phi(x)dx\,.$$

(10) **b.** Show that $T_{f_\epsilon}(\phi) = \dfrac{1}{\epsilon^2}\displaystyle\int_{-\epsilon}^{0} \phi(x)dx - \dfrac{1}{\epsilon^2}\int_{0}^{\epsilon} \phi(x)dx$.

Let $\phi \in \mathscr{S}(\mathbb{R})$ be an analytical test function, and recall Taylor's theorem:

$$\phi(x) = \phi(0) + \phi'(0)x + \frac{1}{2}\phi''(\xi)x^2\,,$$

in which ξ is some number between 0 and x.

(5) **c.** Using this Taylor expansion, argue (mathematically) why we may replace the Lagrange remainder term $\frac{1}{2}\phi''(\xi)x^2$ in the expression for $T_{f_\epsilon}(\phi)$ by a term of order $\mathcal{O}(\epsilon^2)$ as $\epsilon \downarrow 0$.

(10) **d.** Show that, for any analytical test function $\phi \in \mathscr{S}(\mathbb{R})$, $\lim_{\epsilon \downarrow 0} T_{f_\epsilon}(\phi) = \delta'(\phi)$.

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(**40**) **3.** FOURIER TRANSFORMATION (EXAM JANUARY 17, 2007, PROBLEM 3)

In this problem we employ the following Fourier convention:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \quad \text{with, consequently,} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{i\omega x}\, d\omega\,.$$

We also define the following complex valued one-dimensional signal $f_{a,b} : \mathbb{R} \to \mathbb{C} : x \mapsto f(x)$, as follows:

$$f_{a,b}(x) = e^{(a+bi)\,|x|}\,.$$

In this expression, $a, b \in \mathbb{R}$ are constant parameters and $|x|$ denotes the absolute value of $x \in \mathbb{R}$.

(10) **a.** Determine $\widehat{f}_{a,b}(\omega)$ and state the necessary conditions that $a, b \in \mathbb{R}$ have to fulfill in order for this function to be well-defined (in a classical, i.e. non-distributional sense).

The convolution of two functions $f, g : \mathbb{R} \to \mathbb{C}$ is given by

$$(f * g)\,(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy\,.$$

b. Prove that convolution is associative, i.e. that for all $f, g, h : \mathbb{R} \to \mathbb{C}$ for which the expressions below are well-defined we have

(5) **b1.** $f * (g * h) = (f * g) * h$, and

(5) **b2.** $f * g = g * f$.

By the symbol $*^n$ we denote n-fold convolution, i.e.

$$f *^n f \stackrel{\text{def}}{=} f * \ldots * f \quad \text{with } n{+}1 \text{ factors } f.$$

(10) **c.** Determine the explicit form of the function $\widehat{f}_{a,b} *^n \widehat{f}_{a,b}$ for those $a, b \in \mathbb{R}$ for which $\widehat{f}_{a,b}$ is well-defined (recall part a).

(10) **d.** Let $g(x) = x\, e^{-|x|}$. Find $\widehat{g}(\omega)$.

THE END