## EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Wednesday August 25, 2010. Time: 14h00–17h00. Place:

**Read this first!**

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes and calculator is allowed. The use of the problem companion, "opgaven- en tentamenbundel", is *not* allowed.
- You may provide your answers in Dutch or English.

*GOOD LUCK!*

(**25**) **1.** Norms and Inner Products

The figure below shows a (real-valued) greyvalue image $f$ consisting of 9 pixels, of which the numerical values are indicated.

| | | |
|---|---|---|
| 2 | 0 | 0 |
| 4 | −6 | 3 |
| 0 | 4 | 0 |

**a.** We define the $p$-norm of an $M \times N$ image $g$ as

$$\|g\|_p = \left( \sum_{i=1}^{M} \sum_{j=1}^{N} |g[i,j]|^p \right)^{\frac{1}{p}}$$

for $p \geq 1$. Compute the following norms for the above $3 \times 3$-image $f$:

($2\frac{1}{2}$) **a1.** $\|f\|_1$.

$\|f\|_1 = 2 + 0 + 0 + 4 + 6 + 3 + 0 + 4 + 0 = 19.$

($2\frac{1}{2}$) **a2.** $\|f\|_2$.

$\|f\|_2 = (2^2 + 0^2 + 0^2 + 4^2 + 6^2 + 3^2 + 0^2 + 4^2 + 0^2)^{\frac{1}{2}} = 9.$

**b.** We define furthermore the "∞-norm" of an $M \times N$ image $g$ as $\|g\|_\infty = \lim_{p\to\infty} \|g\|_p$ .

$(2\frac{1}{2})$ **b1.** Argue that $\|g\|_\infty = \max_{i=1,\ldots,M,j=1,\ldots,N} |g[i,j]|$.
(*Hint:* Consider the asymptotic behaviour of $(m^p+M^p)^{\frac{1}{p}} = M\left((\frac{m}{M})^p+1\right)^{\frac{1}{p}}$ for $0 \leq m \leq M$ as $p \to \infty$.)

Suppose $(i^*, j^*)$ is a grid point for which $g[i^*, j^*] \geq g[i,j]$ for all $i = 1,\ldots,M, j = 1,\ldots,N$. For ease of notation, let us write $S_p = |g[i^*, j^*]|^p$ and $s_p = \sum_{(i,j)\neq(i^*,j^*)} |g[i,j]|^p$, so

$$\|g\|_p = (s_p + S_p)^{\frac{1}{p}} = S_p^{\frac{1}{p}} \left(\frac{s_p}{S_p}+1\right)^{\frac{1}{p}} .$$

We now determine strict upper and lower bounds for this expression. From $S_p^{\frac{1}{p}} = |g[i^*, j^*]|$ and $0 \leq s_p = \sum_{(i,j)\neq(i^*,j^*)} |g[i,j]|^p \leq \sum_{(i,j)\neq(i^*,j^*)} |g[i^*,j^*]|^p = (MN-1)S_p$ it follows that $1 \leq \frac{s_p}{S_p}+1 \leq MN$, so that

$$|g[i^*, j^*]| \leq \|g\|_p \leq |g[i^*, j^*]|(MN)^{\frac{1}{p}} .$$

As $p \to \infty$ we have $(MN)^{\frac{1}{p}} \to 1$ and so $\|g\|_p \to |g[i^*, j^*]| = \max_{i=1,\ldots,M,j=1,\ldots,N} |g[i,j]|$.

$(2\frac{1}{2})$ **b2.** Compute $\|f\|_\infty$ for the given $3\times 3$-image $f$.

$\|f\|_\infty = 6$.

We define for an arbitrary $M \times N$ image $g$ the normalized image

$$g_p = \frac{g}{\|g\|_p} .$$

**c.** Determine for the given $3\times 3$ image $f$ respectively (you may use the *appendix*)

$(2\frac{1}{2})$ **c1.** $f_1$,

$(2\frac{1}{2})$ **c2.** $f_2$,

$(2\frac{1}{2})$ **c3.** $f_\infty$.

| | | |
|---|---|---|
| $\frac{2}{19}$ | 0 | 0 |
| $\frac{4}{19}$ | $-\frac{6}{19}$ | $\frac{3}{19}$ |
| 0 | $\frac{4}{19}$ | 0 |

(a) $f_1$

| | | |
|---|---|---|
| $\frac{2}{9}$ | 0 | 0 |
| $\frac{4}{9}$ | $-\frac{2}{3}$ | $\frac{1}{3}$ |
| 0 | $\frac{4}{9}$ | 0 |

(b) $f_2$

| | | |
|---|---|---|
| $\frac{1}{3}$ | 0 | 0 |
| $\frac{2}{3}$ | $-1$ | $\frac{1}{2}$ |
| 0 | $\frac{2}{3}$ | 0 |

(c) $f_\infty$

For arbitrary $M \times N$ images $g$ and $h$ we introduce the (real) standard inner product, as follows:

$$\langle g|h\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N} g[i,j]\, h[i,j] .$$

($2\frac{1}{2}$) **d.** Prove that $\langle g_p | h_q \rangle = \dfrac{\langle g|h\rangle}{\|g\|_p \|h\|_q}$.

Due to bilinearity we may extract scalar factors:

$$\langle g_p | h_q \rangle = \langle \tfrac{g}{\|g\|_p} | \tfrac{h}{\|h\|_q} \rangle = \frac{\langle g|h\rangle}{\|g\|_p \|h\|_q} \,.$$

In the case of discrete $M \times N$ images $g$ en $h$ Hölder's inequality reads as follows:

$$\|gh\|_1 \leq \|g\|_p \, \|h\|_q \,,$$

for each parameter pair $(p, q)$ for which $1 \leq p, q \leq \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

(5) **e.** Prove that for arbitrary $M \times N$ images $g$ and $h$ we have $\langle g_p | h_q \rangle \leq 1$. In this inequality the pair $(p, q)$ satisfies the conditions of Hölder's inequality.

Using the previous part and (in the final step below) Hölder's inequality we can make the following estimation:

$$\langle g_p | h_q \rangle \stackrel{\text{d}}{=} \frac{\langle g|h\rangle}{\|g\|_p \|h\|_q} \leq \frac{|\langle g|h\rangle|}{\|g\|_p \|h\|_q} \leq \frac{\|gh\|_1}{\|g\|_p \|h\|_q} \leq 1 \,.$$

P.S. The last step uses Hölder's inequality. The second last step follows from the fact that the absolute value of a sum of terms is always smaller than or equal to the sum of absolute values of those terms:

$$|\langle g|h\rangle| = \left| \sum_{i=1}^{M} \sum_{j=1}^{N} g[i,j]\, h[i,j] \right| \leq \sum_{i=1}^{M} \sum_{j=1}^{N} |g[i,j]\, h[i,j]| = \|gh\|_1 \,.$$

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## (35) 2. Linear Spaces and Projections

$C_0^2([0,1])$ is the class of twice continuously differentiable, real functions of the type $f : [0,1] \to \mathbb{R}$, for which $f(0) = f(1) = f'(0) = f'(1) = 0$. (P.S. With $f'(0)$ en $f'(1)$ we mean right, respectively left derivative at the corresponding point.) Without proof we conjecture that $C^\infty([0,1])$, the class of real-valued functions on the closed interval $[0,1]$ that are infinitely differentiable, constitutes a linear space. (P.S. Again the boundary derivatives $f^{(n)}(0)$ and $f^{(n)}(1)$ are defined in terms of single-sided limits.)

($7\frac{1}{2}$) **a.** Prove that $C_0^2([0,1])$ is a linear space.
(*Hint:* $C_0^2([0,1]) \subset C^\infty([0,1])$.)

Since $C_0^2([0,1]) \subset C^\infty([0,1])$, in which $C^\infty([0,1])$ is a linear space, it suffices to prove that $C_0^2([0,1])$ is closed w.r.t. vector addition and scalar multiplication. Suppose $f, g \in C_0^2([0,1])$ and $\lambda, \mu \in \mathbb{R}$ are arbitrary, then $\lambda f + \mu g$ is again twice continuously differentiable (since, by definition, $(\lambda f + \mu g)' = \lambda f' + \mu g'$, etc.). In particular we have $(\lambda f + \mu g)(r) = \lambda f(r) + \mu g(r) = 0$ and $(\lambda f + \mu g)'(r) = \lambda f'(r) + \mu g'(r) = 0$ for boundary points $r \in \{0, 1\}$, so $\lambda f + \mu g$ also satisfies the boundary conditions, therefore $\lambda f + \mu g \in C_0^2([0,1])$.

We endow the linear space $C_0^2([0,1])$ with a real inner product according to one of the definitions below. The subscript identifies the definition, therefore do not omit it in your notation.

**Definition 1:** For $f, g \in C_0^2([0,1])$,

$$\langle f|g\rangle_1 = \int_0^1 f(x)\, g(x)\, dx + \int_0^1 f'(x)\, g'(x)\, dx\,.$$

**Definition 2:** For $f, g \in C_0^2([0,1])$,

$$\langle f|g\rangle_2 = \int_0^1 f(x)\, g(x)\, dx - \frac{1}{2}\int_0^1 f''(x)\, g(x)\, dx - \frac{1}{2}\int_0^1 f(x)\, g''(x)\, dx\,.$$

(5) **b.** Show that Definition 1 is a good definition, i.e. that it indeed defines an inner product.

Suppose $f, g, h \in C_0^2([0,1])$ and $\lambda, \mu \in \mathbb{R}$. Then both $\int_0^1 f(x)\, g(x)\, dx$ and $\int_0^1 f'(x)\, g'(x)\, dx$ are well defined, so $\langle f|g\rangle_1 \in \mathbb{R}$. Moreover:

$$\begin{aligned}
\langle \lambda f + \mu g|h\rangle_1 &= \int_0^1 (\lambda f + \mu g)(x)\, h(x)\, dx + \int_0^1 (\lambda f + \mu g)'(x)\, h'(x)\, dx \\
&= \int_0^1 (\lambda f(x) + \mu g(x))\, h(x)\, dx + \int_0^1 (\lambda f'(x) + \mu g'(x))\, h'(x)\, dx \\
&= \lambda \left(\int_0^1 f(x)\, h(x)\, dx + \int_0^1 f'(x)\, h'(x)\, dx\right) + \mu \left(\int_0^1 g(x)\, h(x)\, dx + \int_0^1 g'(x)\, h'(x)\, dx\right) \\
&= \lambda \langle f|h\rangle_1 + \mu \langle g|h\rangle_1\,, \\
\langle f|\lambda g + \mu h\rangle_1 &= \int_0^1 f(x)\, (\lambda g + \mu h)(x)\, dx + \int_0^1 f'(x)\, (\lambda g + \mu h)'(x)\, dx \\
&= \int_0^1 f(x)\, (\lambda g(x) + \mu h(x))\, dx + \int_0^1 f'(x)\, (\lambda g'(x) + \mu h'(x))\, dx \\
&= \lambda \left(\int_0^1 f(x)\, g(x)\, dx + \int_0^1 f'(x)\, g'(x)\, dx\right) + \mu \left(\int_0^1 f(x)\, h(x)\, dx + \int_0^1 f'(x)\, h'(x)\, dx\right) \\
&= \lambda \langle f|g\rangle_1 + \mu \langle f|h\rangle_1\,, \\
\langle f|g\rangle_1 &= \int_0^1 f(x)\, g(x)\, dx + \int_0^1 f'(x)\, g'(x)\, dx \\
&= \langle g|h\rangle_1 \quad \text{commutativity of ordinary multiplication,} \\
\langle f|f\rangle_1 &= \int_0^1 (f(x))^2\, dx + \int_0^1 (f'(x))^2\, dx > 0 \quad \text{if } f \text{ is not the null function.}
\end{aligned}$$

(5) **c.** Prove that both definitions are equivalent.
(*Hint:* Partial integration.)

Using partial integration it follows that

$$\begin{aligned}
\int_0^1 f''(x)\, g(x)\, dx &= \left[f'(x)\, g(x)\right]_0^1 - \int_0^1 f'(x)\, g'(x)\, dx &= -\int_0^1 f'(x)\, g'(x)\, dx \quad \text{as well as} \\
\int_0^1 f(x)\, g''(x)\, dx &= \left[f(x)\, g'(x)\right]_0^1 - \int_0^1 f'(x)\, g'(x)\, dx &= -\int_0^1 f'(x)\, g'(x)\, dx\,.
\end{aligned}$$

The boundary terms cancel as a result of the boundary conditions satisfied by $f, g \in C_0^2([0,1])$. By substituting these equalities into Definition 2 it follows that $\langle f|g\rangle_2 = \langle f|g\rangle_1$.

By virtue of equivalence you may omit the subscript henceforth: $\langle f|g\rangle = \langle f|g\rangle_1 = \langle f|g\rangle_2$. With the help of this inner product we introduce, for arbitrary fixed $h \in C_0^2([0,1])$, the following linear mapping $P_h : C_0^2([0,1]) \to C_0^2([0,1])$:

**Definition:** $P_h(f) = \dfrac{\langle h|f\rangle}{\langle h|h\rangle}\, h$ .

(5) **d.** Show that $P_h \circ P_h = P_h$. The infix operator $\circ$ denotes composition.

Let $f \in C_0^2([0,1])$ be arbitrary. Then

$$(P_h \circ P_h)(f) = P_h(P_h(f)) = \frac{\langle h|P_h(f)\rangle}{\langle h|h\rangle}\, h = \frac{\langle h|\frac{\langle h|f\rangle}{\langle h|h\rangle}\, h\rangle}{\langle h|h\rangle}\, h = \frac{\langle h|f\rangle}{\langle h|h\rangle}\frac{\langle h|h\rangle}{\langle h|h\rangle}\, h = \frac{\langle h|f\rangle}{\langle h|h\rangle}\, h = P_h(f)\,.$$

The third equality uses linearity of the inner product w.r.t. the second argument, the rest follows from the definition of the composition operator $\circ$, resp. of $P_h$. Since this holds for all $f \in C_0^2([0,1])$ it follows that $P_h \circ P_h = P_h$ (idempotency).

(5) **e.** Show that $P_h^\dagger = P_h$, i.e. $\langle g|P_h f\rangle = \langle P_h g|f\rangle$ for all $f, g \in C_0^2([0,1])$.

Using bilinearity of the real inner product, the definition of $P_h$, and some elementary rewritings, we obtain

$$\langle g|P_h f\rangle = \langle g|\frac{\langle h|f\rangle}{\langle h|h\rangle}\, h\rangle = \frac{\langle h|f\rangle}{\langle h|h\rangle}\langle g|h\rangle = \frac{\langle h|g\rangle}{\langle h|h\rangle}\langle h|f\rangle = \langle\frac{\langle h|g\rangle}{\langle h|h\rangle}\, h|f\rangle = \langle P_h g|f\rangle\,.$$

General properties of the real inner product have been used in steps 2 (linearity w.r.t. second argument), 3 (symmetry), and 4 (linearity w.r.t. first argument).

Consider the following two functions (notice that $f(x) = f(1-x)$ and $g(x) = g(1-x)$):

$$f(x) = x^4 - 2x^3 + x^2 \quad (0 \le x \le 1) \quad \text{and} \quad g(x) = \begin{cases} -4x^3 + 3x^2 & (0 \le x \le \frac{1}{2}) \\ -4(1-x)^3 + 3(1-x)^2 & (\frac{1}{2} \le x \le 1) \end{cases}$$

($7\frac{1}{2}$) **f.** Show that $f, g \in C_0^2([0,1])$.

Polynomials are infinitely differentiable, so in particular it follows that $f$ is twice differentiable. For $g$ we have to inspect the "suspicious" point $x = \frac{1}{2}$ more closely.:

$$\begin{aligned} \lim_{x\uparrow\frac{1}{2}} g(x) &= \frac{1}{4} \\ \lim_{x\downarrow\frac{1}{2}} g(x) &= \frac{1}{4}\,. \end{aligned}$$

The function $g$ is therefore continuous (in $x = \frac{1}{2}$ and thus everywhere). Furthermore:

$$\begin{aligned} \lim_{x\uparrow\frac{1}{2}} g'(x) &= \lim_{x\uparrow\frac{1}{2}} (-12x^2 + 6x) &= 0 \\ \lim_{x\downarrow\frac{1}{2}} g'(x) &= \lim_{x\downarrow\frac{1}{2}} (12(1-x)^2 - 6(1-x)) &= 0\,. \end{aligned}$$

The function $g$ is therefore continuously differentiable in $x = \frac{1}{2}$ with $g'(\frac{1}{2}) = 0$. Moreover:

$$\begin{aligned} \lim_{x\uparrow\frac{1}{2}} g''(x) &= \lim_{x\uparrow\frac{1}{2}} (-24x + 6) &= -6 \\ \lim_{x\downarrow\frac{1}{2}} g''(x) &= \lim_{x\downarrow\frac{1}{2}} (-24(1-x) + 6) &= -6\,. \end{aligned}$$

The function $g'$ is therefore also continuously differentiable in $x = \frac{1}{2}$ with $g''(\frac{1}{2}) = -6$. All in all it follows that $g$ is twice continuously differentiable in $x = \frac{1}{2}$ and thus everywhere. Finally we have to check the boundary conditions: We have $f'(x) = 4x^3 - 6x^2 + 2x$ for all $0 \leq x \leq 1$, and $g'(x) = -12x^2 + 6x$ for $x < \frac{1}{2}$ and $g'(x) = -g'(1-x)$ for $x > \frac{1}{2}$, so $f(0) = f(1) = f'(0) = f'(1) = 0$ and likewise for $g$, with boundary derivatives defined as follows:

$$
\begin{aligned}
f'(0) &\stackrel{\text{def}}{=} \lim_{x \downarrow 0} f'(x) \\
f'(1) &\stackrel{\text{def}}{=} \lim_{x \uparrow 1} f'(x) .
\end{aligned}
$$

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## (20) 3. Partial Differential Equations and Fourier Transformation

Consider the following partial differential equation (p.d.e.):

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \qquad x \in \mathbb{R},\, t > 0 .$$

Here $u : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R} : (x,t) \mapsto u(x,t)$ is a real valued spatial filter for each constant value of the parameter $t \in \mathbb{R}^+$.

(5) **a.** Consider, for fixed $t$, the Fourier decomposition

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{u}(\omega,t)\, e^{i\omega x}\, d\omega \qquad \text{and thus} \qquad \widehat{u}(\omega,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx .$$

Show that with this definition the above p.d.e. for $u(x,t)$ can be reduced to the following ordinary differential equation for $\widehat{u}(\omega,t)$, in which $\omega \in \mathbb{R}$ can be interpreted as an arbitrary parameter:

$$\frac{d^2 \widehat{u}}{dt^2} - \omega^2\, \widehat{u} = 0 \qquad \omega \in \mathbb{R},\, t > 0 .$$

Substitution of $u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{u}(\omega,t)\, e^{i\omega x}\, d\omega$ into the p.d.e. yields, after interchanging differential and integral operators,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{d^2 \widehat{u}}{dt^2}(\omega,t) - \omega^2\, \widehat{u}(\omega,t) \right] e^{i\omega x}\, d\omega = 0 \qquad \omega \in \mathbb{R},\, t > 0 .$$

The part between square brackets on the left hand side is thus the Fourier transform of the null function (right hand side), so it is itself the null function. P.S.: Instead of "$d$" you may write "$\partial$". Note that the variable $\omega$ is considered as a constant parameter here (there is no differentiation w.r.t. $\omega$), so that we are actually dealing with an ordinary second order differential equation.

(5) **b.** Show that the general solution for $\widehat{u}(\omega,t)$ is given by

$$\widehat{u}(\omega,t) = A\, e^{-t|\omega|} + B\, e^{t|\omega|} .$$

Here, $A$ and $B$ are two integration constants yet to be determined.
(*Hint:* Stipulate a solution of type $\widehat{u}(t) = e^{\lambda t}$ and determine the possible values of $\lambda \in \mathbb{C}$ in terms of $\omega$.)

Stipulate a solution of the type $\widehat{u}(t) = e^{\lambda t}$ (the parameter $\omega$ has been omitted for ease of notation). Substitution yields $\lambda = \pm\omega$, so that the general solution is a linear combination of the form $\widehat{u}(\omega, t) = a\, e^{-t\omega} + b\, e^{t\omega}$. Subtlety: There are no absolute value signs! For the following reason we may however introduce those signs: The integration constants $a, b$ in general depend on the parameter $\omega$. To arrive at the given expression we reparametrize these constants as follows: If $\omega \geq 0$ we set $(A, B) = (a, b)$, and if $\omega < 0$ we take $(A, B) = (b, a)$. This produces the expression given.

**c.** Determine the constants $A$ en $B$ based on the following assumptions:

$(2\frac{1}{2})$ **c1.** $\lim_{t\to\infty} \widehat{u}(\omega, t) = 0$ for all $\omega \neq 0$.

$(2\frac{1}{2})$ **c2.** $\int_{-\infty}^{\infty} u(x, t)\, dx = 1$ for all $t > 0$.
(*Hint:* What does this normalization mean for $\widehat{u}(\omega, t)$?)

The limit $\lim_{t\to\infty} \widehat{u}(\omega, t)$ "explodes" for $\omega \neq 0$ unless $B = 0$. If $B = 0$ you obtain the desired limiting value, since $\lim_{t\to\infty} A\, e^{-t|\omega|} = 0$. Moreover, $1 = \int_{-\infty}^{\infty} u(x, t)\, dx = \widehat{u}(0, t) = A + B = A$. Therefore $(A, B) = (1, 0)$, regardless of $\omega$.

(5) **d.** Take $(A, B) = (1, 0)$, so $\widehat{u}(\omega, t) = e^{-t|\omega|}$. Determine $u(x, t)$.

Inverse Fourier transformation of $\widehat{u}(\omega, t) = e^{-t|\omega|}$ yields $u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\, e^{-t|\omega|}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{0} e^{(t+ix)\omega}\, d\omega + \int_{0}^{\infty} e^{(-t+ix)\omega}\, d\omega = \frac{1}{2\pi}\left\{\left[\frac{1}{t+ix}\, e^{(t+ix)\omega}\right]_{\omega\to-\infty}^{\omega=0} + \left[\frac{1}{-t+ix}\, e^{(-t+ix)\omega}\right]_{\omega=0}^{\omega\to\infty}\right\} = \frac{1}{2\pi}\left\{\frac{1}{t+ix} - \frac{1}{-t+ix}\right\} = \frac{t}{\pi}\frac{1}{x^2+t^2}$.

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**(20)** **4.** Distribution Theory

We consider the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$ given by

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$$

and its associated regular tempered distribution $T_f : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto T_f(\phi) = \int_{-\infty}^{\infty} f(x)\, \phi(x)\, dx$.

(10) **a.** Show that $f$ satisfies the o.d.e. (ordinary differential equation) $u' + u = 0$ *almost everywhere*, and explain what the annotation "almost everywhere" means in this case.

For $x < 0$ it is clear that $f$ is differentiable (with $f(x) = f'(x) = 0$) and trivially satisfies the o.d.e. For $x > 0$ $f$ is likewise differentiable, and we have $f'(x) = -e^{-x} = -f(x)$, which shows that also on this subdomain $f$ satisfies the o.d.e. $u' + u = 0$. However, at $x = 0$ $f$ is not differentiable, so this point needs to be excluded. This explains what is meant by the statement that $f$ satisfies the o.d.e. "almost everywhere".

(10) **b.** Show that, in distributional sense, $T_f$ satisfies the o.d.e. $u' + u = \delta$, in which the right hand side denotes the Dirac point distribution.
(*Hint:* What does it mean for $u' + u - \delta$ to be a distribution rather than a regular function?)

We have, respectively,

$$T_f(\phi) = \int_{-\infty}^{\infty} f(x)\, \phi(x)\, dx = \int_0^{\infty} e^{-x}\, \phi(x)\, dx\,,$$

and

$$T_f'(\phi) \overset{*}{=} -T_f(\phi') = -\int_{-\infty}^{\infty} f(x)\, \phi'(x)\, dx = -\int_0^{\infty} e^{-x}\, \phi'(x)\, dx \overset{\star}{=} -e^{-x}\, \phi(x)\,\Big|_0^{\infty} - \int_0^{\infty} e^{-x}\, \phi(x)\, dx = \phi(0) - T_f(\phi)\,.$$

The equality marked by $*$ holds by definition of distributional differentiation, the one marked by $\star$ follows by partial integration. Using the definition of the Dirac point distribution, $\delta(\phi) = \phi(0)$, we may rewrite the result as

$$T_f'(\phi) = \delta(\phi) - T_f(\phi)\,,$$

which shows that $T_f$ satisfies the inhomogeneous o.d.e. $u' + u = \delta$ in distributional sense. Notice that no restrictions on the domain of definition need to be imposed, and that the result is consistent with the "classical" result under a, since $\delta(x) = 0$ for $x \neq 0$.

## THE END