**EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS**

Course code: 8D020. Date: Wednesday August 25, 2010. Time: 14h00–17h00. Place:

**Read this first!**

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes and calculator is allowed. The use of the problem companion, "opgaven- en tentamenbundel", is *not* allowed.
- You may provide your answers in Dutch or English.

*GOOD LUCK!*

(**25**) **1.** Norms and Inner Products

The figure below shows a (real-valued) greyvalue image $f$ consisting of 9 pixels, of which the numerical values are indicated.

| | | |
|---|---|---|
| 2 | 0 | 0 |
| 4 | −6 | 3 |
| 0 | 4 | 0 |

**a.** We define the $p$-norm of an $M \times N$ image $g$ as

$$\|g\|_p = \left( \sum_{i=1}^{M} \sum_{j=1}^{N} |g[i,j]|^p \right)^{\frac{1}{p}}$$

for $p \geq 1$. Compute the following norms for the above $3 \times 3$-image $f$:

($2\frac{1}{2}$) **a1.** $\|f\|_1$.

($2\frac{1}{2}$) **a2.** $\|f\|_2$.

**b.** We define furthermore the "$\infty$-norm" of an $M \times N$ image $g$ as $\|g\|_\infty = \lim_{p\to\infty} \|g\|_p$ .

($2\frac{1}{2}$) **b1.** Argue that $\|g\|_\infty = \max_{i=1,\ldots,M, j=1,\ldots,N} |g[i,j]|$.
(*Hint:* Consider the asymptotic behaviour of $(m^p + M^p)^{\frac{1}{p}} = M\left((\frac{m}{M})^p + 1\right)^{\frac{1}{p}}$ for $0 \leq m \leq M$ as $p \to \infty$.)

($2\frac{1}{2}$) **b2.** Compute $\|f\|_\infty$ for the given $3\times3$-image $f$.

We define for an arbitrary $M\times N$ image $g$ the normalized image

$$g_p = \frac{g}{\|g\|_p}.$$

**c.** Determine for the given $3\times3$ image $f$ respectively (you may use the *appendix*)

($2\frac{1}{2}$) **c1.** $f_1$,

($2\frac{1}{2}$) **c2.** $f_2$,

($2\frac{1}{2}$) **c3.** $f_\infty$.

For arbitrary $M\times N$ images $g$ and $h$ we introduce the (real) standard inner product, as follows:

$$\langle g|h\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N} g[i,j]\,h[i,j]\,.$$

($2\frac{1}{2}$) **d.** Prove that $\langle g_p|h_q\rangle = \dfrac{\langle g|h\rangle}{\|g\|_p\|h\|_q}$.

In the case of discrete $M\times N$ images $g$ en $h$ Hölder's inequality reads as follows:

$$\|gh\|_1 \leq \|g\|_p\,\|h\|_q\,,$$

for each parameter pair $(p,q)$ for which $1\leq p,q\leq\infty$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$.

(5) **e.** Prove that for arbitrary $M\times N$ images $g$ and $h$ we have $\langle g_p|h_q\rangle \leq 1$. In this inequality the pair $(p,q)$ satisfies the conditions of Hölder's inequality.

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## (35) 2. Linear Spaces and Projections

$C_0^2([0,1])$ is the class of twice continuously differentiable, real functions of the type $f:[0,1]\to\mathbb{R}$, for which $f(0)=f(1)=f'(0)=f'(1)=0$. (P.S. With $f'(0)$ en $f'(1)$ we mean right, respectively left derivative at the corresponding point.) Without proof we conjecture that $C^\infty([0,1])$, the class of real-valued functions on the closed interval $[0,1]$ that are infinitely differentiable, constitutes a linear space. (P.S. Again the boundary derivatives $f^{(n)}(0)$ and $f^{(n)}(1)$ are defined in terms of single-sided limits.)

($7\frac{1}{2}$) **a.** Prove that $C_0^2([0,1])$ is a linear space.
(*Hint:* $C_0^2([0,1])\subset C^\infty([0,1])$.)

We endow the linear space $C_0^2([0,1])$ with a real inner product according to one of the definitions below. The subscript identifies the definition, therefore do not omit it in your notation.

**Definition 1:** For $f, g \in C_0^2([0,1])$,

$$\langle f|g\rangle_1 = \int_0^1 f(x)\, g(x)\, dx + \int_0^1 f'(x)\, g'(x)\, dx\,.$$

**Definition 2:** For $f, g \in C_0^2([0,1])$,

$$\langle f|g\rangle_2 = \int_0^1 f(x)\, g(x)\, dx - \frac{1}{2}\int_0^1 f''(x)\, g(x)\, dx - \frac{1}{2}\int_0^1 f(x)\, g''(x)\, dx\,.$$

(5) **b.** Show that Definition 1 is a good definition, i.e. that it indeed defines an inner product.

(5) **c.** Prove that both definitions are equivalent.
(*Hint:* Partial integration.)

By virtue of equivalence you may omit the subscript henceforth: $\langle f|g\rangle = \langle f|g\rangle_1 = \langle f|g\rangle_2$. With the help of this inner product we introduce, for arbitrary fixed $h \in C_0^2([0,1])$, the following linear mapping $P_h : C_0^2([0,1]) \to C_0^2([0,1])$:

**Definition:** $P_h(f) = \dfrac{\langle h|f\rangle}{\langle h|h\rangle}\, h$ .

(5) **d.** Show that $P_h \circ P_h = P_h$. The infix operator $\circ$ denotes composition.

(5) **e.** Show that $P_h^\dagger = P_h$, i.e. $\langle g|P_h f\rangle = \langle P_h g|f\rangle$ for all $f, g \in C_0^2([0,1])$.

Consider the following two functions (notice that $f(x) = f(1-x)$ and $g(x) = g(1-x)$):

$$f(x) = x^4 - 2x^3 + x^2 \quad (0 \le x \le 1) \quad \text{and} \quad g(x) = \begin{cases} -4x^3 + 3x^2 & (0 \le x \le \frac{1}{2}) \\ -4(1-x)^3 + 3(1-x)^2 & (\frac{1}{2} \le x \le 1) \end{cases}$$

($7\frac{1}{2}$) **f.** Show that $f, g \in C_0^2([0,1])$.

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(**20**) **3. Partial Differential Equations and Fourier Transformation**

Consider the following partial differential equation (p.d.e.):

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0 \qquad x \in \mathbb{R},\, t > 0\,.$$

Here $u : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R} : (x,t) \mapsto u(x,t)$ is a real valued spatial filter for each constant value of the parameter $t \in \mathbb{R}^+$.

(5) **a.** Consider, for fixed $t$, the Fourier decomposition

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{u}(\omega,t)\, e^{i\omega x}\, d\omega \qquad \text{and thus} \qquad \widehat{u}(\omega,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx\,.$$

Show that with this definition the above p.d.e. for $u(x,t)$ can be reduced to the following ordinary differential equation for $\widehat{u}(\omega,t)$, in which $\omega \in \mathbb{R}$ can be interpreted as an arbitrary parameter:

$$\frac{d^2\widehat{u}}{dt^2} - \omega^2\,\widehat{u} = 0 \qquad \omega \in \mathbb{R},\ t > 0\,.$$

(5) **b.** Show that the general solution for $\widehat{u}(\omega,t)$ is given by

$$\widehat{u}(\omega,t) = A\, e^{-t|\omega|} + B\, e^{t|\omega|}\,.$$

Here, $A$ and $B$ are two integration constants yet to be determined.
(*Hint:* Stipulate a solution of type $\widehat{u}(t) = e^{\lambda t}$ and determine the possible values of $\lambda \in \mathbb{C}$ in terms of $\omega$.)

**c.** Determine the constants $A$ en $B$ based on the following assumptions:

($2\frac{1}{2}$) **c1.** $\lim_{t\to\infty} \widehat{u}(\omega,t) = 0$ for all $\omega \neq 0$.

($2\frac{1}{2}$) **c2.** $\int_{-\infty}^{\infty} u(x,t)\, dx = 1$ for all $t > 0$.
(*Hint:* What does this normalization mean for $\widehat{u}(\omega,t)$?)

(5) **d.** Take $(A,B) = (1,0)$, so $\widehat{u}(\omega,t) = e^{-t|\omega|}$. Determine $u(x,t)$.

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(**20**) **4.** Distribution Theory

We consider the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$ given by

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$$

and its associated regular tempered distribution $T_f : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto T_f(\phi) = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx$.

(10) **a.** Show that $f$ satisfies the o.d.e. (ordinary differential equation) $u' + u = 0$ *almost everywhere*, and explain what the annotation "almost everywhere" means in this case.

(10) **b.** Show that, in distributional sense, $T_f$ satisfies the o.d.e. $u' + u = \delta$, in which the right hand side denotes the Dirac point distribution.
(*Hint:* What does it mean for $u' + u - \delta$ to be a distribution rather than a regular function?)

**THE END**