MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS
HOMEWORK ASSIGNMENT

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Read this first!

- Make this assignment by yourself or together with *maximally* one fellow student that has also subscribed for this course.
- Write your name(s) and student number(s) on each sheet.
- The deadline for handing in this assignment is *Wednesday December 13 2006*. Assignments arriving after this date will be ignored.
- This assignment will be evaluated with a grade between 0 and 1. This is the bonus that will be added to your (re)examination grade in 2007. (The final grade cannot be higher than 10.)
- Provide clear arguments, and write neatly. Illegible or sloppy formulations will not be corrected. Explain conceptual steps in your proofs.

Problem 1. In this problem V is a vector space over $\mathbb{R}$ equipped with a real inner product $\langle \_|\_ \rangle : V \times V \to \mathbb{R}$. Furthermore, $a \in V$ is a fixed unit vector: $\langle a|a \rangle = 1$.

$(\frac{1}{10})$ **a.** Show that the subset $V_a \subset V$ generated by a and defined as

$$V_a = \{v \in V \mid \langle a|v \rangle = 0\} \,,$$

constitutes a linear subspace of V.

Choose $v, w \in V_a$ and $\lambda, \mu \in \mathbb{R}$ arbitrarily. Then, using the defining properties of an inner product, $\langle a|\lambda v + \mu w \rangle = \lambda \langle a|v \rangle + \mu \langle a|w \rangle = 0$. The last equality follows from the definition of V_a. Thus $\lambda v + \mu w \in V_a$ (closure), whence it follows that V_a is a linear subspace of V.

b. The vector a, moreover, induces a mapping $\phi_a : V \to V$, as follows:

$$\phi_a(v) = v - \langle a|v \rangle \, a \,.$$

$(\frac{1}{10})$ **b1.** Prove that ϕ_a is a linear map.

Choose $v, w \in V$ and $\lambda, \mu \in \mathbb{R}$ arbitrarily. Consider

$$\phi_a(\lambda v + \mu w) \stackrel{\text{def}}{=} \lambda v + \mu w - \langle a|\lambda v + \mu w \rangle \, a \stackrel{*}{=} \lambda v + \mu w - \lambda \langle a|v \rangle \, a - \mu \langle a|w \rangle \, a = \lambda(v - \langle a|v \rangle \, a) + \mu(w - \langle a|w \rangle \, a) \stackrel{\text{def}}{=} \lambda \phi_a(v) + \mu \phi_a(w) \,.$$

In $*$ linearity of the inner product has been used.

$(\frac{1}{10})$ **b2.** Prove that $\phi_a(v) \in V_a$ for all $v \in V$.

Consider

$$\langle a|\phi_a(v) \rangle \stackrel{\text{def}}{=} \langle a|v - \langle a|v \rangle \, a \rangle \stackrel{*}{=} \langle a|v \rangle - \langle a|v \rangle \, \langle a|a \rangle \stackrel{\star}{=} 0 \,.$$

In $*$ linearity of the inner product has been used, in $\star$ the fact that a is a unit vector.

($\frac{1}{10}$) **b3.** Prove that $\phi_a(\phi_a(v)) = \phi_a(v)$ for all $v \in V$.

Substitution yields:

$$\phi_a(\phi_a(v)) \stackrel{\text{def}}{=} \phi_a(v) - \langle a|\phi_a(v)\rangle\, a = \phi_a(v)\,.$$

In the last step we have used the fact that $\phi_a(v) \in V_a$ according to b2.

($\frac{1}{10}$) **b4.** Prove that $\langle \phi_a(v)|w\rangle = \langle v|\phi_a(w)\rangle$ for all $v, w \in V$.

Substitution yields:

$$\langle \phi_a(v)|w\rangle \stackrel{\text{def}}{=} \langle v - \langle a|v\rangle\, a|w\rangle \stackrel{*}{=} \langle v|w\rangle - \langle a|v\rangle\,\langle a|w\rangle \stackrel{*}{=} \langle v|w\rangle - \langle\langle a|w\rangle\, a|v\rangle \stackrel{\star}{=} \langle v|w\rangle - \langle v|\langle a|w\rangle\, a\rangle \stackrel{*}{=} \langle v|w - \langle a|w\rangle\, a\rangle \stackrel{\text{def}}{=} \langle v|\phi_a(w)\rangle\,.$$

In $*$ we have used linearity, in $\star$ symmetry of the (real) inner product.

($\frac{1}{10}$) **b5.** Suppose $w \in V$ is such that $\langle \phi_a(v)|w\rangle = 0$ for all $v \in V$. Show that $w = \lambda\, a$ for some $\lambda \in \mathbb{R}$ and determine the value of λ in terms of a en w.
(*Hint:* Use the previous part and the defining properties of the inner product.)

From the previous result it follows that for any $a, v \in V$ $\langle \phi_a(v)|w\rangle = \langle v|\phi_a(w)\rangle$. Assume therefore that $\langle v|\phi_a(w)\rangle = 0$ for some $w \in V$. Since this must hold for all $v \in V$ it follows, by virtue of the non-negativity and non-degeneracy of the inner product, that $\phi_a(w) = w - \langle a|w\rangle\, a = 0$, in other words, that $w = \lambda\, a$ with $\lambda = \langle a|w\rangle$. Vice versa, if $w = \lambda\, a$ for some $\lambda \in \mathbb{R}$, then

$$\langle \phi_a(v)|w\rangle \stackrel{\circ}{=} \langle \phi_a(v)|\lambda\, a\rangle \stackrel{*}{=} \lambda\,\langle \phi_a(v)|a\rangle \stackrel{\text{def}}{=} \langle v - \langle a|v\rangle\, a|a\rangle \stackrel{*}{=} \langle v|a\rangle - \langle a|v\rangle\langle a|a\rangle \stackrel{\star}{=} 0\,.$$

In $\circ$ we have used the assumption on $w \in V$. In $*$ we have used linearity of the inner product, and in $\star$ we have exploited symmetry of the inner product, and the fact that $a \in V$ is a unit vector.

Problem 2. We define the set of functions $C_0^\infty(\mathbb{R})$ as follows:

$$C_0^\infty(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) \mid f^{(n)}(0) = 0 \text{ voor alle } n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \right\}\,.$$

In this definition $f^{(n)}(x)$ stands for the n-th order derivative of f evaluated at x. The set $C^\infty(\mathbb{R})$ is the collection of all smooth real-valued functions with domain $\mathbb{R}$, endowed with the usual definitions of vector addition and scalar multiplication. You may take it for granted that $C^\infty(\mathbb{R})$ constitutes a linear space.

($\frac{1}{10}$) **a.** Provide (an) unambiguous formula(s) for the "usual definitions" alluded to above.

The "usual definitions" of vector addition and scalar multiplication pertain to the following way to define linear combinations: Let $f, g \in C_0^\infty(\mathbb{R})$ and $\lambda, \mu \in \mathbb{R}$ be arbitrarily chosen, then

$$(\lambda\, f + \mu\, g)\,(x) \stackrel{\text{def}}{=} \lambda\, f(x) + \mu\, g(x) \quad \text{for all } x \in \mathbb{R}.$$

($\frac{1}{10}$) **b.** Prove that $C_0^\infty(\mathbb{R}) \subset C^\infty(\mathbb{R})$ constitutes a linear subspace.

Since $C^\infty(\mathbb{R})$ is a linear space it suffices to prove closure of $C_0^\infty(\mathbb{R}) \subset C^\infty(\mathbb{R})$ under linear combination. Let $f, g \in C_0^\infty(\mathbb{R})$ and $\lambda, \mu \in \mathbb{R}$ be arbitrarily chosen, then $\lambda\, f + \mu\, g$ is the function defined by $(\lambda\, f + \mu\, g)\,(x) \stackrel{\text{def}}{=} \lambda\, f(x) + \mu\, g(x)$ for all $x \in \mathbb{R}$. Since differentiation is a linear operation we have

$$(\lambda\, f + \mu\, g)^{(n)}\,(x) = \lambda\, f^{(n)}(x) + \mu\, g^{(n)}(x)\,,$$

for any order $n \in \mathbb{N}_0$, so that in particular

$$(\lambda\, f + \mu\, g)^{(n)}\,(0) = \lambda\, f^{(n)}(0) + \mu\, g^{(n)}(0) = 0$$

for all $f, g \in C_0^\infty(\mathbb{R})$. Therefore $\lambda f + \mu g \in C_0^\infty(\mathbb{R})$.

$(\frac{1}{10})$ **c.** Suppose $f \in C^\omega(\mathbb{R}) \cap C_0^\infty(\mathbb{R})$, i.e. f is an *analytical* function within the class $C_0^\infty(\mathbb{R})$. Show that $f = 0$, i.e. the null function of $C_0^\infty(\mathbb{R})$.
(*Hint:* Analyticity implies that f is equal to its Taylor series.)

Since $f \in C^\omega(\mathbb{R})$, $f(x)$ equals its convergent Taylor series, i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n \stackrel{*}{=} 0 \quad \text{for all } x \in \mathbb{R}.$$

In $*$ we have used the fact that $f \in C_0^\infty(\mathbb{R})$. Conclusion: $f = 0 \in C_0^\infty(\mathbb{R})$.

$(\frac{1}{10})$ **d.** Show by means of an explicit example that $C_0^\infty(\mathbb{R})$ contains nontrivial elements $f \neq 0$.
(*Hint:* Stipulate a function of type $f(x) = e^{g(x)}$ and deduce what properties the function g should have, then find a concrete instance.)

Following the hint, let us take $g(x) = -x^{-2}$, and define $f(x) = e^{g(x)}$ whenever $x \neq 0$, and $f(0) = 0 = \lim_{x\to 0} f(x)$. Then $f'(x) = g'(x)\, e^{g(x)} = 2x^{-3}\, e^{-x^{-2}}$ for $x \neq 0$, and $f'(0) = 0 = \lim_{x\to 0} f'(x)$. (That is, the derivative is well-defined and continuous by virtue of identical left and right limits.) In fact, for any order $n \in \mathbb{N}_0$ we have

$$f^{(n)}(x) = p_n(\frac{1}{x})\, f(x)\,,$$

for some polynomial p_n (which depends on order n). Proof: The conjecture is apparently true for $n = 0$ (take $p_0(\frac{1}{x}) = 1$). If the conjecture is true for some $n \in \mathbb{N}_0$, then

$$f^{(n+1)}(x) = \frac{d}{dx} f^{(n)}(x) = \frac{d}{dx}\left(p_n(\frac{1}{x})\, f(x)\right) \stackrel{*}{=} -\frac{1}{x^2}\, p_n'(\frac{1}{x})\, f(x) + p_n(\frac{1}{x}) f'(x) = \left(-\frac{1}{x^2}\, p_n'(\frac{1}{x}) + \frac{2}{x^3}\, p_n(\frac{1}{x})\right) f(x)\,,$$

which is indeed of the stipulated form $f^{(n+1)}(x) = p_{n+1}(\frac{1}{x})\, f(x)$ for some polynomial p_{n+1}. This proves the conjecture. In particular we have that

$$f^{(n)}(0) = 0 = \lim_{x\to 0} f^{(n)}(x)\,,$$

by virtue of the fact that

$$\lim_{x\to 0} x^{-m}\, e^{-x^{-2}} = 0 \quad \text{for any } m \in \mathbb{N}_0.$$

THE END