MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS
HOMEWORK ASSIGNMENT

Course code: 8D020. Teacher: Dr L.M.J. Florack, WH 3.108 (secretariat WH 2.106), **E** L.M.J.Florack@tue.nl, **T** 040 2475377, **F** 040 2472740, **W** www.bmi2.bmt.tue.nl/image-analysis/people/lflorack

Read this first!

- Make this assignment by yourself or together with *maximally* one fellow student that has also subscribed for this course.
- Write your name(s) and student number(s) on each sheet.
- The deadline for handing in this assignment is *Wednesday December 13 2006.* Assignments arriving after this date will be ignored.
- This assignment will be evaluated with a grade between 0 and 1. This is the bonus that will be added to your (re)examination grade in 2007. (The final grade cannot be higher than 10.)
- Provide clear arguments, and write neatly. Illegible or sloppy formulations will not be corrected. Explain conceptual steps in your proofs.

Problem 1. In this problem V is a vector space over $\mathbb{R}$ equipped with a real inner product $\langle\_|\_\rangle : V \times V \to \mathbb{R}$. Furthermore, $a \in V$ is a fixed unit vector: $\langle a|a\rangle = 1$.

($\frac{1}{10}$) **a.** Show that the subset $V_a \subset V$ generated by a and defined as

$$V_a = \{v \in V \mid \langle a|v\rangle = 0\} \, ,$$

constitutes a linear subspace of V.

b. The vector a, moreover, induces a mapping $\phi_a : V \to V$, as follows:

$$\phi_a(v) = v - \langle a|v\rangle \, a \, .$$

($\frac{1}{10}$) **b1.** Prove that ϕ_a is a linear map.

($\frac{1}{10}$) **b2.** Prove that $\phi_a(v) \in V_a$ for all $v \in V$.

($\frac{1}{10}$) **b3.** Prove that $\phi_a(\phi_a(v)) = \phi_a(v)$ for all $v \in V$.

($\frac{1}{10}$) **b4.** Prove that $\langle \phi_a(v)|w\rangle = \langle v|\phi_a(w)\rangle$ for all $v, w \in V$.

($\frac{1}{10}$) **b5.** Suppose $w \in V$ is such that $\langle \phi_a(v)|w\rangle = 0$ for all $v \in V$. Show that $w = \lambda \, a$ for some $\lambda \in \mathbb{R}$ and determine the value of λ in terms of a en w.
(*Hint:* Use the previous part and the defining properties of the inner product.)

Problem 2. We define the set of functions $C_0^\infty(\mathbb{R})$ as follows:

$$C_0^\infty(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) \mid f^{(n)}(0) = 0 \text{ voor alle } n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \right\} .$$

In this definition $f^{(n)}(x)$ stands for the n-th order derivative of f evaluated at x. The set $C^\infty(\mathbb{R})$ is the collection of all smooth real-valued functions with domain $\mathbb{R}$, endowed with the usual definitions of vector addition and scalar multiplication. You may take it for granted that $C^\infty(\mathbb{R})$ constitutes a linear space.

($\frac{1}{10}$) **a.** Provide (an) unambiguous formula(s) for the "usual definitions" alluded to above.

($\frac{1}{10}$) **b.** Prove that $C_0^\infty(\mathbb{R}) \subset C^\infty(\mathbb{R})$ constitutes a linear subspace.

($\frac{1}{10}$) **c.** Suppose $f \in C^\omega(\mathbb{R}) \cap C_0^\infty(\mathbb{R})$, i.e. f is an *analytical* function within the class $C_0^\infty(\mathbb{R})$. Show that $f = 0$, i.e. the null function of $C_0^\infty(\mathbb{R})$.
(*Hint:* Analyticity implies that f is equal to its Taylor series.)

($\frac{1}{10}$) **d.** Show by means of an explicit example that $C_0^\infty(\mathbb{R})$ contains nontrivial elements $f \neq 0$.
(*Hint:* Stipulate a function of type $f(x) = e^{g(x)}$ and deduce what properties the function g should have, then find a concrete instance.)

THE END