MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS
HOMEWORK ASSIGNMENT

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Read this first!

- Make this assignment by yourself or together with *maximally* one fellow student that has also subscribed for this course.
- Write your name(s) and student number(s) on each sheet.
- The deadline for handing in this assignment is *Wednesday November 22 2006*. Assignments arriving after this date will be ignored.
- This assignment will be evaluated with a grade between 0 and 1. This is the bonus that will be added to your (re)examination grade in 2007. (The final grade cannot be higher than 10.)
- Provide clear arguments, and write neatly. Illegible or sloppy formulations will not be corrected. Explain conceptual steps in your proofs.

Problem 1. We define the *hyperbolic sine and cosine functions* as follows:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2} \quad (x \in \mathbb{R})\,.$$

$(\frac{1}{10})$ **a.** Prove the following identities:

$$\cosh(\xi+\eta) - \cosh(\xi-\eta) = 2\sinh\xi\,\sinh\eta\,, \tag{1}$$
$$\cosh(\xi+\eta) + \cosh(\xi-\eta) = 2\cosh\xi\,\cosh\eta\,, \tag{2}$$
$$\sinh(\xi+\eta) - \sinh(\xi-\eta) = 2\cosh\xi\,\sinh\eta\,, \tag{3}$$
$$\sinh(\xi+\eta) + \sinh(\xi-\eta) = 2\sinh\xi\,\cosh\eta\,. \tag{4}$$

These all follow straightforwardly by substitution of the defining identities for cosh and sinh:

$$\cosh(\xi+\eta) - \cosh(\xi-\eta) = \frac{e^{\xi+\eta}+e^{-\xi-\eta}}{2} - \frac{e^{\xi-\eta}+e^{-\xi+\eta}}{2} = 2\left(\frac{e^{\xi}-e^{-\xi}}{2}\right)\left(\frac{e^{\eta}-e^{-\eta}}{2}\right) = 2\sinh\xi\,\sinh\eta\,,$$
$$\cosh(\xi+\eta) + \cosh(\xi-\eta) = \frac{e^{\xi+\eta}+e^{-\xi-\eta}}{2} + \frac{e^{\xi-\eta}+e^{-\xi+\eta}}{2} = 2\left(\frac{e^{\xi}+e^{-\xi}}{2}\right)\left(\frac{e^{\eta}+e^{-\eta}}{2}\right) = 2\cosh\xi\,\cosh\eta\,,$$
$$\sinh(\xi+\eta) - \sinh(\xi-\eta) = \frac{e^{\xi+\eta}-e^{-\xi-\eta}}{2} - \frac{e^{\xi-\eta}-e^{-\xi+\eta}}{2} = 2\left(\frac{e^{\xi}+e^{-\xi}}{2}\right)\left(\frac{e^{\eta}-e^{-\eta}}{2}\right) = 2\cosh\xi\,\sinh\eta\,,$$
$$\sinh(\xi+\eta) + \sinh(\xi-\eta) = \frac{e^{\xi+\eta}-e^{-\xi-\eta}}{2} + \frac{e^{\xi-\eta}-e^{-\xi+\eta}}{2} = 2\left(\frac{e^{\xi}-e^{-\xi}}{2}\right)\left(\frac{e^{\eta}+e^{-\eta}}{2}\right) = 2\sinh\xi\,\cosh\eta\,.$$

We define the set of real-valued 2×2-matrices

$$G = \left\{ \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \,\middle|\, \xi \in \mathbb{R} \right\},$$

and endow it with a product operation in the usual way, i.e. standard matrix multiplication.

b. Show that G constitutes a group. To this end, answer the following questions, and provide proofs for your answers:

$(\frac{1}{10})$ **b1.** Is G closed under matrix multiplication? In other words, does $A, B \in G$ imply $AB \in G$?

Let

$$\Xi = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}.$$

Then

$$\Xi H = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} = \begin{pmatrix} \cosh\xi\cosh\eta + \sinh\xi\sinh\eta & \sinh\xi\cosh\eta + \cosh\xi\sinh\eta \\ \cosh\xi\sinh\eta + \sinh\xi\cosh\eta & \sinh\xi\sinh\eta + \cosh\xi\cosh\eta \end{pmatrix}.$$

Using a we can rewrite this as

$$\Xi H = \begin{pmatrix} \cosh(\xi+\eta) & \sinh(\xi+\eta) \\ \sinh(\xi+\eta) & \cosh(\xi+\eta) \end{pmatrix},$$

from which it becomes obvious that $\Xi H \in G$.

$(\frac{1}{10})$ **b2.** Show *for general $n\times n$-matrices* that matrix multiplication is associative.

Let A, B, C be $n\times n$-matrices with components A_{ij}, B_{ij}, respectively C_{ij}, $i,j=1,\ldots,n$. Then

$$((AB)C)_{ij} = \sum_{k=1}^{n}(AB)_{ik}C_{kj} = \sum_{k=1}^{n}\sum_{\ell=1}^{n} A_{i\ell}B_{\ell k}C_{kj} = \sum_{\ell=1}^{n} A_{i\ell}(BC)_{\ell j} = (A(BC))_{ij} \quad \text{for all } i,j=1,\ldots,n.$$

Consequently $(AB)C = A(BC)$.

$(\frac{1}{10})$ **b3.** Give the identity element of G.

The identity element for 2×2-matrix multiplication is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh 0 & \sinh 0 \\ \sinh 0 & \cosh 0 \end{pmatrix}.$$

From the last step and the definition of G it follows that $I \in G$.

$(\frac{1}{10})$ **b4.** Give the inverse element $A^{-1} \in G$ for given $A \in G$.

From b1 and b3 it follows that for arbitrarily chosen $\Xi \in G$, as specified in b1, the inverse Ξ^{-1} is given by

$$\Xi^{-1} = \begin{pmatrix} \cosh(-\xi) & \sinh(-\xi) \\ \sinh(-\xi) & \cosh(-\xi) \end{pmatrix}.$$

Proof: Obviously $\Xi^{-1} \in G$. Furthermore, if we substitute H by Ξ^{-1} in the solution of problem b1, then

$$\Xi\Xi^{-1} = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} \cosh(-\xi) & \sinh(-\xi) \\ \sinh(-\xi) & \cosh(-\xi) \end{pmatrix} \stackrel{\text{b1}}{=} \begin{pmatrix} \cosh 0 & \sinh 0 \\ \sinh 0 & \cosh 0 \end{pmatrix} \stackrel{\text{b3}}{=} I.$$

Likewise,

$$\Xi^{-1}\Xi = \begin{pmatrix} \cosh(-\xi) & \sinh(-\xi) \\ \sinh(-\xi) & \cosh(-\xi) \end{pmatrix} \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \stackrel{\text{b1}}{=} \begin{pmatrix} \cosh 0 & \sinh 0 \\ \sinh 0 & \cosh 0 \end{pmatrix} \stackrel{\text{b3}}{=} I.$$

Equality of left and right inverse also follows by observing, via b1, that the matrix product in G commutes (cf. also c):

$$\Xi^{-1}\Xi \stackrel{\text{b1}}{=} \Xi\Xi^{-1} \stackrel{*}{=} I,$$

once identity $*$ has been established as above.

$(\frac{1}{10})$ **c** Is G commutative?

Yes, cf. b1. With Ξ and H as before we have, for any $\xi, \eta \in \mathbb{R}$,

$$\Xi H \stackrel{\text{b1}}{=} \begin{pmatrix} \cosh(\xi+\eta) & \sinh(\xi+\eta) \\ \sinh(\xi+\eta) & \cosh(\xi+\eta) \end{pmatrix} = \begin{pmatrix} \cosh(\eta+\xi) & \sinh(\eta+\xi) \\ \sinh(\eta+\xi) & \cosh(\eta+\xi) \end{pmatrix} \stackrel{\text{b1}}{=} H\Xi.$$

Problem 2. In this problem G and H are two given groups. The infix product operator of G is indicated by a $\bullet$, whereas that of H is denoted by $\circ$. We construct the set F as follows

$$F = G \times H \stackrel{\text{def}}{=} \{(g,h) \mid g \in G, h \in H\} \,,$$

which is endowed with an infix product operator $\star$ as follows. If $f_1, f_2 \in F$, say $f_1 = (g_1, h_1)$ and $f_2 = (g_2, h_2)$ with $g_1, g_2 \in G$ and $h_1, h_2 \in H$, then

$$f_1 \star f_2 = (g_1 \bullet g_2, h_1 \circ h_2) \,.$$

a. Show that F constitutes a group. To this end, answer the following questions, and provide proofs for your answers:

$(\frac{1}{10})$ **a1.** Is F closed under $\star$?

Yes. Proof: Let $f_1, f_2 \in F$ be arbitrary, say $f_1 = (g_1, h_1)$ and $f_2 = (g_2, h_2)$ for some $g_1, g_2 \in G$ and $h_1, h_2 \in H$. Since G and H constitute groups relative to the operators $\bullet$, respectively $\circ$, there exist group elements $g_{12} = g_1 \bullet g_2 \in G$ (since G is closed) and $h_{12} = h_1 \circ h_2 \in H$ (since H is closed), so that

$$f_1 \star f_2 \stackrel{\text{def}}{=} (g_1 \bullet g_2, h_1 \circ h_2) = (g_{12}, h_{12}) \in G \times H = F \,.$$

$(\frac{1}{10})$ **a2.** Show that the operator $\star$ satisfies the associativity property.

Let $f_1, f_2, f_3 \in F$ be arbitrary, say $f_1 = (g_1, h_1)$, $f_2 = (g_2, h_2)$ and $f_3 = (g_3, h_3)$ for some $g_1, g_2, g_3 \in G$ and $h_1, h_2, h_3 \in H$. Then

$$\begin{aligned}(f_1 \star f_2) \star f_3 &\stackrel{\text{def}}{=} (g_1 \bullet g_2, h_1 \circ h_2) \star (g_3, h_3) \stackrel{\text{def}}{=} ((g_1 \bullet g_2) \bullet g_3, (h_1 \circ h_2) \circ h_3) \stackrel{*}{=} (g_1 \bullet (g_2 \bullet g_3), h_1 \circ (h_2 \circ h_3)) \\ &\stackrel{\text{def}}{=} (g_1, h_1) \star (g_2 \bullet g_3, h_2 \circ h_3) \stackrel{\text{def}}{=} f_1 \star (f_2 \star f_3) \,.\end{aligned}$$

In this derivation we have repeatedly exploited the definition of $\star$, and (in $*$) the fact that G and H constitute groups, implying that their respective operators $\bullet$ and $\circ$ are associative.

$(\frac{1}{10})$ **a3.** Give the identity element of F.

Since G and H are groups, they each have an identity element, say $e_G \in G$, respectively $e_H \in H$. Claim: $e_F = (e_G, e_H)$ is the identity element of F. Proof: Notice first of all that by construction $e_F \in F$. Now consider $f \in F$ arbitrary, say $f = (g, h)$ with $g \in G$ and $h \in H$. Then

$$e_F \star f \stackrel{\text{def}}{=} (e_G, e_H) \star (g, h) \stackrel{\text{def}}{=} (e_G \bullet g, e_H \circ h) \stackrel{*}{=} (g, h) \stackrel{\text{def}}{=} f \,.$$

Also

$$f \star e_F = (g, h) \star (e_G, e_H) \stackrel{\text{def}}{=} (g \bullet e_G, h \circ e_H) \stackrel{\star}{=} (g, h) \stackrel{\text{def}}{=} f \,.$$

In $*$ and $\star$ we have used the fact that e_G and e_H are both left identity elements ($*$) as well as right identity elements ($\star$) for G, respectively H. The two identities above thus indeed identify e_F as the identity element of F.

$(\frac{1}{10})$ **a4.** Give the inverse element $f^{-1} \in F$ for given $f \in F$.

Notation as in the solution of a3, with $f = (g, h) \in F$ chosen arbitrarily. Let $g^{-1} \in G$ and $h^{-1} \in H$ be the inverse elements of $g \in G$, respectively $h \in H$. Claim: $f^{-1} = (g^{-1}, h^{-1})$ is the inverse element of $f = (g, h)$. Proof: Notice first of all that $f^{-1} \in F$, by virtue of the fact that $g^{-1} \in G$ and $h^{-1} \in H$. Furthermore,

$$f^{-1} \star f = (g^{-1}, h^{-1}) \star (g, h) \stackrel{\text{def}}{=} (g^{-1} \bullet g, h^{-1} \circ h) \stackrel{\text{a3}}{=} (e_G, e_H) \stackrel{\text{a3}}{=} e_F \,.$$

Likewise,

$$f \star f^{-1} = (g, h) \star (g^{-1}, h^{-1}) \stackrel{\text{def}}{=} (g \bullet g^{-1}, h \circ h^{-1}) \stackrel{\text{a3}}{=} (e_G, e_H) \stackrel{\text{a3}}{=} e_F \,.$$

THE END