EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Thursday April 07, 2011. Time: 14h00–17h00. Place:

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of problem companion ("opgaven- en tentamenbundel"), calculator, laptop, or other equipment, is *not* allowed.
- You may provide your answers in Dutch or English.

GOOD LUCK!

(**30**) **1.** CLIFFORD ALGEBRA

Attention: For each numeric symbol (matrix entry, scalar multiplier, et cetera) that you use in your arguments below please state explicitly whether it is real or complex.

Let σ be any 2×2 matrix with complex entries $\sigma_{ij} \in \mathbb{C}$ in i-th row and j-th column. The set of all complex 2×2 matrices constitutes a vector space over the *real* numbers (i.e. scalar multiplication pertains to *real* scalars), which we indicate here by $\mathbb{M}_{2\times 2}$. For simplicity of notation we write the identity matrix as $1 \in \mathbb{M}_{2\times 2}$. The complex conjugate of a complex number $z = a + b\,i$, $a, b \in \mathbb{R}$, is indicated by $z^* = a - b\,i$.

The *trace* operator $\mathrm{tr} : \mathbb{M}_{2\times 2} \to \mathbb{C}$ is defined by $\mathrm{tr}\,\sigma = \sigma_{11} + \sigma_{22}$.

The *hermitian conjugate* operator $^\dagger : \mathbb{M}_{2\times 2} \to \mathbb{M}_{2\times 2}$ is defined by $\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}^\dagger = \begin{pmatrix} \sigma_{11}^* & \sigma_{21}^* \\ \sigma_{12}^* & \sigma_{22}^* \end{pmatrix}$.

(10) **a.** Show that $\mathrm{tr} : \mathbb{M}_{2\times 2} \to \mathbb{C} : \sigma \mapsto \mathrm{tr}\,\sigma$ and $^\dagger : \mathbb{M}_{2\times 2} \to \mathbb{M}_{2\times 2} : \sigma \mapsto \sigma^\dagger$ are linear operators.

For $\lambda, \mu \in \mathbb{R}$, $\sigma, \tau \in \mathbb{M}_{2\times 2}$, we have $\mathrm{tr}\,(\lambda\,\sigma + \mu\,\tau) = (\lambda\,\sigma + \mu\tau)_{11} + (\lambda\,\sigma + \mu\tau)_{22} = \lambda\,\sigma_{11} + \mu\,\tau_{11} + \lambda\,\sigma_{22} + \mu\,\tau_{22} = \lambda\,(\sigma_{11} + \sigma_{22}) + \mu\,(\tau_{11} + \tau_{22}) = \lambda\,\mathrm{tr}\,\sigma + \mu\,\mathrm{tr}\,\tau$. Furthermore, $(\lambda\,\sigma + \mu\,\tau)^\dagger_{ij} = (\lambda\,\sigma + \mu\,\tau)^*_{ji} = \lambda\,\sigma^*_{ji} + \mu\,\tau^*_{ji} = \lambda\,\sigma^\dagger_{ij} + \mu\,\tau^\dagger_{ij}$, for all $i, j = 1, 2$, so $(\lambda\,\sigma + \mu\,\tau)^\dagger = \lambda\,\sigma^\dagger + \mu\,\tau^\dagger$.

The three so-called Pauli matrices are defined as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Together they define a basis for a 3-dimensional vector space $V \subset \mathbb{M}_{2\times 2}$ over the real numbers.

(5) **b.** What is the general form of an element $\sigma \in V$?

If $\sigma \in V$ then it can be written as a real linear combination of Pauli matrices, so

$$\sigma = \lambda_1\sigma_1 + \lambda_2\sigma_2 + \lambda_3\sigma_3 = \begin{pmatrix} \lambda_3 & \lambda_1 - \lambda_2\, i \\ \lambda_1 + \lambda_2\, i & -\lambda_3 \end{pmatrix} ,$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, in other words a traceless hermitian 2×2 matrix.

(5) **c.** Show that $\operatorname{tr} \sigma = 0$ and $\sigma^\dagger = \sigma$ for all $\sigma \in V$.

By linearity it suffices to show this for the Pauli matrices, for which it is obvious from b.

We subsequently enrich the vector space V with a multiplication operator, viz. standard matrix multiplication.

(5) **d.** Compute all nine products of the form $\sigma_k\,\sigma_\ell$ for $k, \ell = 1, 2, 3$.

Straightforward computation yields:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 ,$$

and

$$\begin{aligned} \sigma_1\,\sigma_2 = -\sigma_2\,\sigma_1 &= i\sigma_3 \\ \sigma_3\,\sigma_1 = -\sigma_1\,\sigma_3 &= i\sigma_2 \\ \sigma_2\,\sigma_3 = -\sigma_3\,\sigma_2 &= i\sigma_1 . \end{aligned}$$

This can be condensed as follows:

$$\sigma_k\,\sigma_\ell = \delta_{k\ell}\,1 + i \sum_{m=1}^{3} \epsilon_{k\ell m}\,\sigma_m ,$$

in which $\delta_{k\ell} = 1$ if $k = \ell$ and 0 otherwise, and in which the symbol $\epsilon_{k\ell m}$ is defined as $+1$ if (k, ℓ, m) is an even permutation of $(1, 2, 3)$, -1 if (k, ℓ, m) is an odd permutation of $(1, 2, 3)$, and 0 otherwise.

We now consider the set A consisting of all real linear combinations of *all possible* products (i.e. with an arbitrary number of factors) of Pauli matrices. In this construct we allow for the "empty" product, and define it to produce the identity matrix $1 \in \mathbb{M}_{2\times2}$.

(5) **e.** Interpreted as a vector space over the real numbers, show that A has dimension 8 by providing an explicit set of 8 basis vectors.
(*Hint:* $\dim \mathbb{M}_{2\times2} = 8$.)

A basis is given by $\{1, \sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2 = i\sigma_3, \sigma_3\sigma_1 = i\sigma_2, \sigma_2\sigma_3 = i\sigma_1, \sigma_1\sigma_2\sigma_3 = i\}$. (With i we mean $i\,1$, i.e. i times identity matrix.) Other choices are of course possible. Note that σ_i and $i\sigma_i$, $i = 1, 2, 3$, are independent vectors, since they cannot be related by real valued linear combinations, i.e. there exists no $\lambda, \mu \in \mathbb{R}$ other than zero such that $\lambda\,\sigma_i + \mu\,i\sigma_i = 0$. Likewise for the vectors 1 and i (the latter of which is obtained as $\sigma_1\sigma_2\sigma_3$). In fact, since $A \subset \mathbb{M}_{2\times2}$ has the same dimension as $\mathbb{M}_{2\times2}$ itself we must have $A = \mathbb{M}_{2\times2}$. Clearly other products than those considered in this basis are therefore redundant.

♣

(**25**) 2. STAIRCASE FUNCTION

We introduce the staircase function $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$ given by $f(x) = \lfloor x \rfloor$, i.e. the so-called *entier* of $x \in \mathbb{R}$, which is defined as the largest integer $k \in \mathbb{Z}$ such that $k \leq x$.

(5) **a.** Sketch the graph of $y = f(x)$ on the interval $[-3, 3]$, clearly illustrating its discontinuities.

See Figure 1. Black (part of graph) and red (not part of graph) bullets indicate discontinuities.

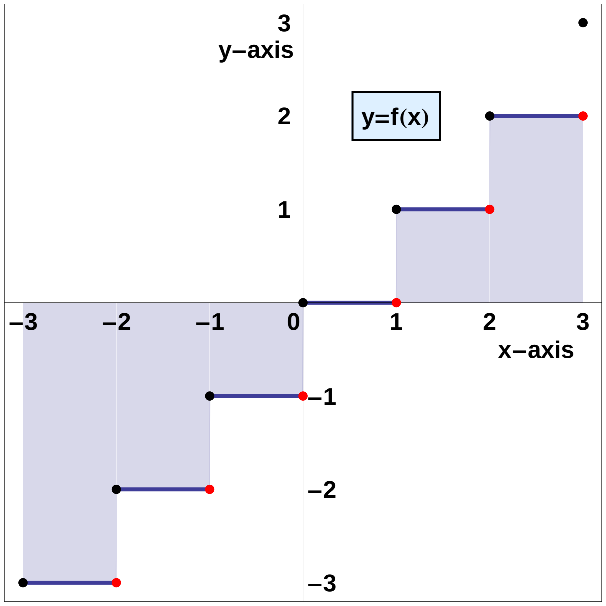

Figure 1: Graph of $y = f(x) = \lfloor x \rfloor$.

(5) **b.** Give the formula for the classical derivative $f'(x)$, together with its domain of definition.

We have discontinuities at each integer, and constant values in-between:

$$f'(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Z} \\ \text{undefined} & \text{if } x \in \mathbb{Z}\,. \end{cases}$$

The regular tempered distribution $T_g : \mathscr{S}(\mathbb{R}) \to \mathbb{R}$ associated with a function $g : \mathbb{R} \to \mathbb{R}$ is the tempered distribution given by

$$T_g(\phi) = \int_{-\infty}^{\infty} g(x)\, \phi(x)\, dx\,,$$

for any $\phi \in \mathscr{S}(\mathbb{R})$.

(5) **c.** Show that, in distributional sense, $T_f' \neq T_{f'}$ for the staircase function f defined above.

The function f' is zero almost everywhere, so $T_{f'}(\phi) = \int_{-\infty}^{\infty} f'(x)\, \phi(x)\, dx = 0$. However, $T_f'(\phi) = -T(\phi') = \int_{-\infty}^{\infty} f(x)\, \phi'(x)\, dx$, which does not vanish in general, depending on the choice of $\phi \in \mathscr{S}(\mathbb{R})$.

For any $a \in \mathbb{R}$ we furthermore define the (shifted) Dirac distribution $\delta_a : \mathscr{S}(\mathbb{R}) \to \mathbb{R}$ by

$$\delta_a(\phi) = \phi(a)\,,$$

for any $\phi \in \mathscr{S}(\mathbb{R})$.

(10) **d.** Prove that $T_f' = \sum_{k\in\mathbb{Z}} \delta_k$.

For $f(x) = \lfloor x \rfloor$ we get $T_f'(\phi) = -T_f(\phi') = -\int_{\mathbb{R}} \lfloor x \rfloor \phi(x)dx = -\sum_{k\in\mathbb{Z}} \int_k^{k+1} k\phi'(x)dx = -\sum_{k\in\mathbb{Z}} k(\phi(k+1) - \phi(k)) = \sum_{k\in\mathbb{Z}} \phi(k) = \sum_{k\in\mathbb{Z}} \delta_k(\phi)$ for any $\phi \in \mathscr{S}(\mathbb{R})$, from which the result follows.

♣

(**20**) **3.** GROUP THEORY

A discrete group G_n with n distinct elements $x_i \in G_n$, $i = 1, \ldots, n$, can be represented by means of a multiplication table. The term multiplication refers to the group operation, which will be denoted by the infix operator $\circ : G \times G \to G : (x_i, x_j) \mapsto x_i \circ x_j$. The element on i-th row and j-th column in the table specifies the product $x_i \circ x_j \in G$:

$\circ$	x_1	$\cdots$	x_j	$\cdots$	x_n
x_1	$x_1 \circ x_1$	$\cdots$	$x_1 \circ x_j$	$\cdots$	$x_1 \circ x_n$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
x_i	$x_i \circ x_1$	$\cdots$	$x_i \circ x_j$	$\cdots$	$x_i \circ x_n$
$\vdots$	$\vdots$		$\vdots$		$\vdots$
x_n	$x_n \circ x_1$	$\cdots$	$x_n \circ x_j$	$\cdots$	$x_n \circ x_n$

Of course, one obtains full knowledge about the group if all entries in this table are provided. However, due to the specific structure of a group it is not necessary to provide all entries in order to uniquely specify the group. An incomplete table may be completed or partially completed with the help of the defining group properties.

(10) **a.** Prove that elements in any given row are all distinct. Likewise for elements in any given column.

Fixing the row label i, let us assume that $x_i \circ x_j = x_i \circ x_k$, i.e. the j-th and k-th entries on the i-th row coincide. Applying x_i^{inv}, i.e. the inverse of x_i, to this identity by multiplication from the left yields $x_j = e \circ x_j = (x_i^{\text{inv}} \circ x_i) \circ x_j = x_i^{\text{inv}} \circ (x_i \circ x_j) = x_i^{\text{inv}} \circ (x_i \circ x_k) = (x_i^{\text{inv}} \circ x_i) \circ x_k = e \circ x_k = x_k$, which is a contradiction unless $j = k$. Similarly, by multiplying from the right by x_i one obtains the proof of the second claim.

We now consider the specific case of $G_4 = \{E, A, B, C\}$. This group has the following properties:

- E is the identity element,
- B equals its own inverse,
- $A \circ A = C \circ C \neq E$.

(10) **b.** Complete the multiplication table for G_4, and explain how you obtained your result:

$\circ$	E	A	B	C
E				
A				
B				
C				

Blue entries are given by the first two properties in the list. Suppose $A \circ A = C \circ C = X$ for some $X \in G_4$, then $X \neq A$ and $X \neq C$ by virtue of the uniqueness property proven in problem a. Hence either $X = E$ or $X = B$. But the former has been explicitly excluded, so $X = B$. Furthermore, $A \circ B \neq E$, for if $A \circ B = E$, then $A = B^{\text{inv}} = B$, which is a contradiction. So $A \circ B = C$. The table can now be completed in a unique way by exploiting the uniqueness property proven in problem a.

$\circ$	$\boldsymbol{E}$	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{C}$
$\boldsymbol{E}$	$\boldsymbol{E}$	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{C}$
$\boldsymbol{A}$	$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{E}$
$\boldsymbol{B}$	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{E}$	$\boldsymbol{A}$
$\boldsymbol{C}$	$\boldsymbol{C}$	$\boldsymbol{E}$	$\boldsymbol{A}$	$\boldsymbol{B}$

In fact we may observe that in this case all elements are powers (group autoproducts) of A, viz. $E = A^0$, $A = A^1$, $B = A^2 = A \circ A$, $C = A^3 = A \circ A \circ A$, under the periodicity assumption that $A^4 = A \circ A \circ A \circ A = E$. You may therefore visualize A as a planar rotation over $\pm\pi/2$, and the group product as a concatenation of such rotations.

♣

(25) **4.** FOURIER THEORY[1]

Consider the following initial value problem for the function $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} & = & \left(\Delta - m^2\right) u(x,t) & \text{for } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(x,0) & = & f(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

Here $f : \mathbb{R}^n \to \mathbb{R}$ is a real valued image and $m > 0$ a positive constant. Boundary and initial conditions are such that this initial value problem has a unique, sufficiently nice solution.

In this problem the following Fourier convention applies. For (sufficiently nice) $u : \mathbb{R}^n \to \mathbb{C}$ we define the function $\widehat{u} = \mathcal{F}(u) : \mathbb{R}^n \to \mathbb{C}$ as follows:

$$\widehat{u}(\omega) = \int_{\mathbb{R}^n} e^{-i\omega\cdot x}\, u(x)\, dx \qquad \text{or, equivalently,} \qquad u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\omega\cdot x}\, \widehat{u}(\omega)\, d\omega\,,$$

in which $\omega \cdot x$ denotes $\omega_1 x_1 + \ldots + \omega_n x_n$. For the sake of brevity we furthermore write $\|x\|^2 = x \cdot x$, respectively $\|\omega\|^2 = \omega \cdot \omega$.

In the problems below you may use the standard integral

$$\int_{-\infty}^{\infty} e^{-(x+iy)^2}\, dx = \sqrt{\pi} \qquad \text{irrespective of the value of } y \in \mathbb{R}\,.$$

(5) **a.** Show that the initial value problem above is equivalent to the following initial value problem in the Fourier domain:

$$\begin{cases} \dfrac{d\widehat{u}(\omega,t)}{dt} & = & -\left(\|\omega\|^2 + m^2\right) \widehat{u}(\omega,t) & \text{voor } (\omega,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ \widehat{u}(\omega,0) & = & \widehat{f}(\omega) & \text{for } \omega \in \mathbb{R}^n \end{cases}$$

(Note that we may interpret this as an *ordinary* differential equation with initial condition, whence the alternative notation for the t-derivative on the left hand side.)

This follows by using linearity of Fourier transformation and the appropriate formal identity

$$\mathcal{F}(\frac{\partial}{\partial x^i}) = i\omega_i \quad \text{for } i = 1,\ldots,n, \text{ so in particular } \mathcal{F}(\Delta) = -\|\omega\|^2\,.$$

Note that no Fourier transform is applied with respect to the t-coordinate.

(5) **b.** Find the solution $\widehat{u}(\omega,t)$.
(*Hint:* Stipulate a solution of type $\widehat{u}(\omega,t) = Ae^{Bt}$ and determine the (ω-dependent) parameters A, B.)

By subsituting $\widehat{u}(\omega,t) = Ae^{Bt}$ into the differential equation we find $B = -(\|\omega^2\| + m^2)$. By imposing the initial condition we find $A = \widehat{f}(\omega)$. The solution is therefore

$$\widehat{u}(\omega,t) = \widehat{f}(\omega)\, e^{-(\|\omega^2\|+m^2)\,t}\,.$$

(5) **c.** Show that for fixed $t \in \mathbb{R}^+$ the spatial solution is given by the convolution product

[1] Exam June 14, 2005, problem 3.

$$u(x,t) = (\phi_t * f)(x)$$

for some convolution filter $\phi_t : \mathbb{R}^n \to \mathbb{R}$.

From b it follows that

$$\widehat{u}(\omega,t) = \widehat{f}(\omega)\,\widehat{\phi}_t(\omega) \quad \text{waarin } \widehat{\phi}_t(\omega) \stackrel{\text{def}}{=} e^{-(\|\omega^2\|+m^2)t}\,.$$

A function product Fourier space corresponds to a convolution product in the spatial domain, so

$$u(x,t) = (f * \phi_t)(x)\,.$$

(Caveat: t is considered as a constant parameter and therefore plays no role in the convolution integral.) Here we have defined $\phi_t \stackrel{\text{def}}{=} \mathcal{F}^{\text{inv}}(\widehat{\phi}_t)$, i.e. the spatial convolution filter with Fourier representation $\widehat{\phi}_t(\omega)$.

(5) **d.** Determine the form $\phi_t(x)$ of this convolution filter.

Fourier inversion yields:

$$\phi_t(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\omega\cdot x}\,\widehat{\phi}_t(\omega)\,d\omega = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\omega\cdot x-(\|\omega^2\|+m^2)t}\,d\omega = e^{-m^2t}\,\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\omega\cdot x-\|\omega^2\|t}\,d\omega\,.$$

Now consider the following integral:

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\omega\cdot x-\|\omega^2\|t}\,d\omega \stackrel{*}{=} \frac{1}{(2\pi)^n}e^{-\frac{\|x\|^2}{4t}}\int_{\mathbb{R}^n} e^{-\|\omega\sqrt{t}-\frac{ix}{2\sqrt{t}}\|^2}\,d\omega \stackrel{\star}{=} \frac{1}{(2\pi)^n}e^{-\frac{\|x\|^2}{4t}}\,\frac{1}{\sqrt{t}^{\,n}}\int_{\mathbb{R}^n} e^{-\|\omega'-\frac{ix}{2\sqrt{t}}\|^2}\,d\omega'\,.$$

The equality marked by $*$ exploits the identity $i\omega\cdot x - \|\omega^2\|t = -\|\omega\sqrt{t} - \dfrac{ix}{2\sqrt{t}}\|^2 - \dfrac{\|x\|^2}{4t}$, while $\star$ uses change of variables: $\omega\sqrt{t} = \omega' \in \mathbb{R}^n$ (note the Jacobian!). Finally, using the standard integral given, it follows that

$$\int_{\mathbb{R}^n} e^{-\|\omega'-\frac{ix}{2\sqrt{t}}\|^2}\,d\omega' = \int_{-\infty}^{\infty} e^{-(\omega_1'-\frac{ix_1}{2\sqrt{t}})^2}\,d\omega_1' \ldots \int_{-\infty}^{\infty} e^{-(\omega_n'-\frac{ix_n}{2\sqrt{t}})^2}\,d\omega_n' = \sqrt{\pi}^{\,n}\,.$$

All in all:

$$\phi_t(x) = \frac{1}{\sqrt{4\pi t}^{\,n}}\,e^{-\frac{\|x\|^2}{4t}-m^2t}\,.$$

(5) **e.** Prove: $\displaystyle\int_{\mathbb{R}^n} \phi_t(x)\,dx = e^{-m^2t}$.

(*Hint:* Consider $\widehat{\phi}_t(\omega)$.)

Cf. b: $\widehat{\phi}_t(\omega) \stackrel{\text{def}}{=} e^{-(\|\omega^2\|+m^2)\,t}$, whence $\displaystyle\int_{\mathbb{R}^n}\phi_t(x)\,dx = \int_{\mathbb{R}^n} e^{-i\omega\cdot x}\phi_t(x)\,dx\Big|_{\omega\,=\,0} = \widehat{\phi}_t(0) = e^{-m^2t}$.

THE END