EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Wednesday April 10, 2013. Time: 09h00–12h00. Place: MA 1.44

Read this first!

- Write your name and student ID on each paper.
- The exam consists of 4 problems. Maximum credits are indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of any additional material or equipment, including the problem companion ("opgaven- en tentamenbundel"), is *not* allowed.
- You may provide your answers in Dutch or English.
- Do not hesitate to ask questions on linguistic matters or if you suspect an erroneous problem formulation.

Good luck!

(35) 1. Vector Space

A real sequence s is an infinitely long array of the form $s = (s_1, s_2, s_3, \ldots)$, with $s_i \in \mathbb{R}$ for all $i \in \mathbb{N}$. It is not difficult to show that the set S of all real sequences constitutes a real vector space under entry-wise addition and scalar multiplication. (You may take this for granted.)

($2\frac{1}{2}$) **a1.** Explain by formulas the meaning of "entry-wise addition and scalar multiplication" on S.

Let $s = (s_1, s_2, s_3, \ldots) \in \mathrm{S}$, $t = (t_1, t_2, t_3, \ldots) \in \mathrm{S}$, $\lambda \in \mathbb{R}$, then we define $s + t = (s_1 + t_1, s_2 + t_2, s_3 + t_3, \ldots) \in \mathrm{S}$ and $\lambda s = (\lambda s_1, \lambda s_2, \lambda s_3, \ldots)$.

($2\frac{1}{2}$) **a2.** What is the neutral element of S? What is the inverse element of $s = (s_1, s_2, s_3, \ldots) \in \mathrm{S}$?

The neutral element is $n = (0, 0, 0, \ldots) \in \mathrm{S}$. The inverse of $s = (s_1, s_2, s_3, \ldots) \in \mathrm{S}$ is $(-s) \stackrel{\text{def}}{=} (-s_1, -s_2, -s_3, \ldots) \in \mathrm{S}$.

An *arithmetic sequence* a is a sequence of the form $a = (a_1, a_2, a_3, \ldots)$ such that subsequent terms have a common difference, i.e. for each such a sequence a there exists a constant $c \in \mathbb{R}$ such that for all $i \in \mathbb{N}$

$$a_{i+1} = a_i + c\,.$$

By A we denote the set of all real-valued arithmetic sequences.

(5) **b.** Prove that $\mathrm{A} \subset \mathrm{S}$ is a vector space.

Since S is a vector space we only need to prove closure. Suppose $a = (a_1, a_2, a_3, \ldots) \in \mathrm{A}$, $b = (b_1, b_2, b_3, \ldots) \in \mathrm{A}$, and $\lambda, \mu \in \mathrm{R}$. By definition there exist constants $c, d \in \mathbb{R}$ such that $a_{i+1} = a_i + c$ and $b_{i+1} = b_i + d$. Let $s \stackrel{\text{def}}{=} \lambda a + \mu b \in \mathrm{A}$ be an arbitrary superposition, i.e. $s_i = \lambda a_i + \mu b_i$, then it follows that $s_{i+1} = \lambda a_{i+1} + \mu b_{i+1} \stackrel{*}{=} \lambda(a_i + c) + \mu(b_i + d) = \lambda a_i + \mu b_i + \lambda c + \mu d = s_i + e$ for all $i \in \mathbb{N}$, in which $e \stackrel{\text{def}}{=} \lambda c + \mu d \in \mathrm{R}$, so that we may conclude that, by definition, $s \in \mathrm{A}$. In $*$ we have likewise used the definition of A.

An n-dimensional basis of A is a set $\{e_1, \dots, e_n\}$ of n linearly independent sequences $e_i \in \text{A}$, $i = 1, \dots, n$, such that every element $a \in \text{A}$ can be written as a linear superposition of the form $a = \lambda_1 e_1 + \dots + \lambda_n e_n$ for certain coefficients $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

($2\frac{1}{2}$) **c1.** State in terms of an explicit formula what "linear independence" of the basis elements in $\{e_1, \dots, e_n\}$ means.

$\lambda_1 e_1 + \dots + \lambda_n e_n = n = (0, 0, 0, \dots) \in \text{A}$ iff $\lambda_1 = \dots = \lambda_n = 0$.

($2\frac{1}{2}$) **c2.** Show that, if such a basis exists, then, for any given $a \in \text{A}$, the coefficients λ_i in the linear superposition $a = \lambda_1 e_1 + \dots + \lambda_n e_n$ are unique.

☞ HINT: SUPPOSE $a = \lambda_1 e_1 + \dots + \lambda_n e_n$ AND $a = \mu_1 e_1 + \dots + \mu_n e_n$, CONSIDER THE DIFFERENCE.

We have $(0, 0, 0, \dots) = n = a - a = (\lambda_1 - \mu_1)e_1 + \dots + (\lambda_n - \mu_n)e_n$. By c1 this is equivalent to $\lambda_1 = \mu_1, \dots, \lambda_n = \mu_n$.

(5) **d.** Show that A does indeed have a finite-dimensional basis, and provide one explicitly. What is the dimension?

Note that an arbitrary arithmetic sequence $a \in \text{A}$ can be written as $a = (a_1, a_1 + c, a_1 + 2c, a_1 + 3c, \dots)$ for some $a_1, c \in \mathbb{R}$. This can be written as $a = a_1 e_1 + c e_2$, with $e_1 = (1, 1, 1, \dots)$, $e_2 = (0, 1, 2, \dots)$. Thus the linear space of arithmetic sequences is 2-dimensional.

A *geometric sequence* g is a sequence of the form $g = (g_1, g_2, g_3, \dots)$ such that subsequent terms have a common ratio, i.e. for each such a sequence g there exists a nonzero constant $r \in \mathbb{R}\backslash\{0\}$ such that for all $i \in \mathbb{N}$

$$g_{i+1} = r g_i \, .$$

By G we denote the set of all real-valued geometric sequences.

(5) **e.** Prove that $\text{G} \subset \text{S}$ is *not* a vector space.

G fails to be closed. For suppose $g \in \text{G}$, $h \in \text{G}$, such that, for any $i \in \mathbb{N}$, $g_{i+1} = r g_i$ and $h_{i+1} = s h_i$ for some constants $r, s \in \mathbb{R}\backslash\{0\}$. Then we observe that $g_{i+1} + h_{i+1} = r g_i + s h_i$. In general the right hand side cannot be written as $t(g_i + h_i)$ for some constant $t \in \mathbb{R}\backslash\{0\}$.

We restrict ourselves henceforth to the set of all *positive* geometric sequences, defined as $\text{G}^+ = \{g = (g_1, g_2, g_3, \dots) \in \text{G} \mid g_i > 0 \text{ for all } i = 1, 2, 3, \dots\}$. On this set we introduce an alternative definition for addition and scalar multiplication, according to the following rules. If $g = (g_1, g_2, g_3, \dots) \in \text{G}^+$, $h = (h_1, h_2, h_3, \dots) \in \text{G}^+$, $\lambda \in \mathbb{R}$, then

$$g \oplus h = (g_1 h_1, g_2 h_2, g_3 h_3, \dots) \quad \text{and} \quad \lambda \otimes g = (g_1^\lambda, g_2^\lambda, g_3^\lambda, \dots) \, .$$

(10) **f.** Show that G^+ is closed under the actions of $\oplus$ and $\otimes$, and subsequently show that it is a vector space.

For closure we must show that $g \oplus h$ and $\lambda \otimes g$ as defined above are geometric sequences. We have $(g \oplus h)_{i+1} \stackrel{\text{def}}{=} g_{i+1} h_{i+1} \stackrel{\text{def}}{=} r g_i s h_i = r s g_i h_i \stackrel{\text{def}}{=} (rs)(g \oplus h)_i$ for some common ratios $r, s \neq 0$ (whence the effective common ratio equals $rs \neq 0$) and all $i \in \mathbb{N}$. Similarly, $(\lambda \otimes g)_{i+1} \stackrel{\text{def}}{=} g_{i+1}^\lambda \stackrel{\text{def}}{=} (r g_i)^\lambda = r^\lambda g_i^\lambda \stackrel{\text{def}}{=} r^\lambda (\lambda \otimes g)_i$. Note that the effective common ratio $r^\lambda \neq 0$ if

$r \neq 0$. Now we cannot use the subspace theorem, since $\mathrm{G}^+ \not\subset \mathrm{G}$ in the sense of a vector space inclusion, for the spaces G^+ and G have different vector operations.

♣

(35) **2.** Group Theory

In this problem we consider a finite group G consisting of 2 elements, to which we will refer as $a, b \in G$. Without loss of generality we may identify $a = \mathrm{id}$, i.e. the identity element of G.

(5) **a.** Show that either $a = b = \mathrm{id}$, or, if $a \neq b$, that $b = b^{-1}$.

The option $a = b = \mathrm{id}$ produces the trivial 1-element group $G = \{\mathrm{id}\}$. Suppose $a \neq b$, then, since a serves as the identity element, we have $ba = ab = b \neq a$, from which it follows that a cannot be the inverse of b. By closure there is no other option than that b equals its own inverse: $b^{-1} = b$, i.e. $b^2 = a$.

We henceforth assume that $a \neq b$.

(5) **b.** Provide the 2×2 group multiplication table of G, cf. the template below. With $x_1 = a, x_2 = b$, the (i, j)-th element in this table indicates $x_i \circ x_j$.

Inspection of the result in a readily provides the full multiplication table:

$\circ$	a	b
a	a	b
b	b	a

Let $T : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2 \backslash \{(0,0)\} : (x, y) \mapsto T(x, y)$ be given by $T(x, y) = \left(\dfrac{x}{x^2 + y^2}, \dfrac{y}{x^2 + y^2} \right)$.

For $k \in \mathbb{N}$ we indicate k-fold concatenation of T by $T^k \stackrel{\text{def}}{=} T \circ \ldots \leftarrow k\text{-fold} \rightarrow \ldots \circ T$. Moreover, $T^{-k} \stackrel{\text{def}}{=} (T^{\text{inv}})^k$, in which T^{inv} is the inverse function of T. The identity element is identified with $T^0 : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2 \backslash \{(0,0)\} : (x, y) \mapsto \mathrm{id}(x, y) = (x, y)$.

(5) **c.** Show that $T^{\text{inv}} = T$.
☞ Hint: Set $(x', y') = T(x, y)$, and consider the identity $T^{\text{inv}}(x', y') = (x, y)$.

Solving the following system for (x, y) in terms of (x', y')

$$\begin{array}{rcl} x' & = & \frac{x}{x^2+y^2} \\ y' & = & \frac{y}{x^2+y^2} \end{array}$$

yields

$$\begin{array}{rcl} x & = & \frac{x'}{(x')^2+(y')^2} \\ y & = & \frac{y'}{(x')^2+(y')^2} \end{array} ,$$

so $T^{\rm inv}(x',y')=T(x',y')$, i.e. $T^{\rm inv}$ has the same functional form as T (suppress irrelevant primes attached to arguments).

Consider the set $\Theta = \left\{ T^k : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}^2\backslash\{(0,0)\} \,\middle|\, k\in\mathbb{Z} \right\}$.

We say that two groups, G and H say, are *isomorphic*, notation $G\sim H$, if there is a one-to-one correspondence $\phi : G \to H : g \mapsto h = \phi(g)$ between their respective elements that preserves the group structure, i.e. $\phi(g_1)\circ_H \phi(g_2) = \phi(g_1 \circ_G g_2)$, in which $\circ_G$ and $\circ_H$ are the infix group operators on G, respectively H.

(5) **d.** Show that the set Θ, furnished with the concatenation operator $\circ$, constitutes a group isomorphic to the 2-element group G of problem b.

Obviously $T\neq T^0=\text{id}$. Since $T^{\rm inv}=T$ according to c it follows from the definition of T^k that $T^k=T$ if k is odd, and $T^k=T^0=\text{id}$ if k is even, i.e. $\Theta=\{T^0,T\}$ is a 2-element group. We have seen in problems a and b that any such 2-parameter group must have a multiplication table as given in b, with in the case at hand the formal substitutions $a\to T^0=\text{id}$ and $b=T$. In other words, $\Theta\sim G$ (as defined in a and b).

Next, consider the class of symmetric smooth functions of rapid decay,

$$\mathscr{S}_{\rm sym}(\mathbb{R}) \stackrel{\rm def}{=} \{\phi\in\mathscr{S}(\mathbb{R}) \mid \phi(x)=\phi(-x)\}\ .$$

We take it for granted that $\mathscr{S}(\mathbb{R})$ is closed under Fourier transformation, defined in this problem with the following convention:

$$\mathscr{F}(\phi)(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)\,e^{-i\omega x}\,dx \quad\text{whence}\quad \mathscr{F}^{\rm inv}(\psi)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(\omega)\,e^{i\omega x}\,d\omega\,.$$

(5) **e.** Show that $\mathscr{S}_{\rm sym}(\mathbb{R})$ is closed under Fourier transformation. ☞ HINT: $\mathscr{S}_{\rm sym}(\mathbb{R})\subset\mathscr{S}(\mathbb{R})$.

Since $\mathscr{S}_{\rm sym}(\mathbb{R})\subset\mathscr{S}(\mathbb{R})$, Fourier transform of $\phi\in\mathscr{S}_{\rm sym}(\mathbb{R})$ is well defined. We need to show that if $\phi(x)=\phi(-x)$ for all $x\in\mathbb{R}$, i.e. $\phi\in\mathscr{S}_{\rm sym}(\mathbb{R})$, then also $\mathscr{F}(\phi)(\omega)=\mathscr{F}(\phi)(-\omega)$, i.e. $\mathscr{F}(\phi)\in\mathscr{S}_{\rm sym}(\mathbb{R})$. Indeed we have

$$\mathscr{F}(\phi)(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)\,e^{-i\omega x}\,dx \stackrel{*}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(-y)\,e^{i\omega y}\,dy \stackrel{\star}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(y)\,e^{i\omega y}\,dy = \mathscr{F}(\phi)(-\omega)\,,$$

in which we have made use of a change of variables, $y=-x$, in $*$, and of the symmetry property $\phi(y)=\phi(-y)$ in $\star$.

We now consider the set $\Phi = \left\{ \mathscr{F}^k : \mathscr{S}_{\rm sym}(\mathbb{R}) \to \mathscr{S}_{\rm sym}(\mathbb{R}) \,\middle|\, k\in\mathbb{Z} \right\}$.

(5) **f.** Show that this set, furnished with the concatenation operator $\circ$, constitutes a group that is likewise isomorphic to the 2-element group G of problem b, but that this is *not* the case if we replace $\mathscr{S}_{\rm sym}(\mathbb{R})$ by $\mathscr{S}(\mathbb{R})$ in the definition of Φ.

From the solution of problem e we may conclude that $\mathscr{F}(\phi)=\mathscr{F}^{\rm inv}(\phi)$ for all $\phi\in\mathscr{S}_{\rm sym}(\mathbb{R})$ by virtue of step $\star$ and the particular definition of the Fourier transform in the case at hand. In other words, $\mathscr{F}=\mathscr{F}^{\rm inv}$ as elements of Φ. Furthermore we have

$$\begin{aligned}\mathscr{F}^2(\phi)(\xi) \;&=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x)e^{-i\omega x}dx e^{-i\omega\xi}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i\omega(x+\xi)}d\omega\phi(x)dx = \\ &\frac{1}{2\pi}\int_{-\infty}^{\infty}2\pi\delta(x+\xi)\phi(x)dx = \phi(-\xi)=\phi(\xi)\,,\end{aligned}$$

for all $\phi \in \mathscr{S}_{\rm sym}(\mathbb{R})$ and $\xi \in \mathbb{R}$. In other words, $\mathscr{F}^2 = \mathscr{F}^0 = \mathrm{id}$ as elements of Φ. By the same token as in problem d we may conclude that $\Phi \sim G$, for the 2-element group G of problem a and b.

Finally, we consider the 2-element matrix group under matrix multiplication

$$M = \left\{ I \stackrel{\rm def}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \, A \stackrel{\rm def}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} ,$$

and use it to construct the 2-dimensional linear space $V_M = \{\alpha\, I + \beta\, A \mid \alpha, \beta \in \mathbb{R}\}$

(5) **g.** Show that V_M is a semigroup, but *not* a group under matrix multiplication.

Closure is obvious, since any product of matrices in V_M will be a linear superposition of I and A as a result of the group structure of M (and the definition of matrix superposition on V_M). Associativity likewise holds trivially, since it is a general property of the matrix product. The group identity element of V_M is clearly I, since $I\,(\alpha\, I{+}\beta\, A) = (\alpha\, I{+}\beta\, A)\, I = \alpha\, I{+}\beta\, A$ for any $\alpha, \beta \in \mathbb{R}$. That V_M is not a group follows, e.g., from the fact that the vector neutral element, the null matrix, does not have an inverse. Another example which fails to have an inverse is $I + A$, for suppose $\alpha, \beta \in \mathbb{R}$ are such that $I = (I + A)(\alpha\, I + \beta\, A) = (\alpha + \beta)\, I + (\alpha + \beta)\, A$, then $\alpha + \beta = 1$ and $\alpha + \beta = 0$, which is a contradiction.

♣

(**15**) **3.** FOURIER TRANSFORMATION

In this problem we consider a general parametrization of the various one-dimensional Fourier definitions one encounters in the literature:

$$\mathscr{F}_{(a,b)}(u)(\omega) = b\int_{-\infty}^{\infty} u(x)\, e^{-ia\omega x}\, dx \quad \text{whence} \quad \mathscr{F}^{\text{inv}}_{(a,b)}(\hat{u})(x) = \frac{|a|}{2\pi\, b}\int_{-\infty}^{\infty} \hat{u}(\omega)\, e^{ia\omega x}\, d\omega\,.$$

The parameter space is $\mathrm{P} \stackrel{\text{def}}{=} \left\{(a,b)\in\mathbb{R}^2 \,|\, a\neq 0\,,\; b>0\right\}$. Consider the following reparametrization:

$$\mathrm{T}:\mathrm{P}\to\mathrm{P}:(a,b)\mapsto(a',b')=\mathrm{T}(a,b) \quad \text{with} \quad \left\{\begin{array}{rcl} a' &=& -a \\ b' &=& \dfrac{|a|}{2\pi\, b}\end{array}\right.$$

($2\frac{1}{2}$) **a1.** Show that this is a good definition, in the sense that P is indeed closed under T as stipulated by the prototype "$\mathrm{T}:\mathrm{P}\to\mathrm{P}$", i.e. $(a',b')\in\mathrm{P}$ if $(a,b)\in\mathrm{P}$.

If $a\neq 0$ then $a'=-a\neq 0$ and if $b>0$ then $b'=|a|/(2\pi b)>0$, so $(a',b')\in\mathrm{P}$.

($2\frac{1}{2}$) **a2.** Show that T is invertible, and that $\mathrm{T}^{\text{inv}}=\mathrm{T}$.
☞ HINT: SOLVE $(a,b)=\mathrm{T}^{\text{inv}}(a',b')$.

We have $\mathrm{T}^2(a,b)=T(-a,|a|/(2\pi b))=(a,b)$, from which it follows that $\mathrm{T}^2=\mathrm{id}_{\mathrm{P}}$, i.e. $\mathrm{T}^{\text{inv}}=\mathrm{T}$.

Without proof we state that the normed space $L^2(\mathbb{R})$ of square-integrable, complex-valued functions with domain $\mathbb{R}$, is closed under Fourier transformation. The norm of a function $u\in L^2(\mathbb{R})$ will be denoted by $\|u\|$. Recall that

$$\|u\|^2 = \int_{-\infty}^{\infty} u(x)\, u^*(x)\, dx\,.$$

In the problems below you may, moreover, use the following lemma: $\displaystyle\int_{-\infty}^{\infty} e^{\pm i\,a\,y\,z}\, dz = \frac{2\pi}{|a|}\,\delta(y)$.

Let $\mathrm{Q}\subset\mathrm{P}$ be the set of parameters for which $\|\mathscr{F}_{(a,b)}(u)(\omega)\|^2=\|u\|^2$ for all $u\in L^2(\mathbb{R})$ (*unitarity*).

(10) **b.** Determine Q, and show that the convention that was used in problem 1 provides an example of a unitary Fourier transform, i.e. show that $(a,b)=(1,1/\sqrt{2\pi})\in\mathrm{Q}$.

Insert

$$u(x)=\frac{|a|}{2\pi\, b}\int_{-\infty}^{\infty}\hat{u}(\omega)\, e^{ia\omega x}\, d\omega \quad \text{and} \quad u^*(x)=\frac{|a|}{2\pi\, b}\int_{-\infty}^{\infty}\hat{u}^*(\omega')\, e^{-ia\omega' x}\, d\omega'$$

into

$$\|u\|^2 = \int_{-\infty}^{\infty} u(x)\, u^*(x)\, dx\,.$$

The result is

$$\begin{aligned}\|u\|^2 &= \left[\frac{|a|}{2\pi\, b}\right]^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{u}(\omega)\,\hat{u}^*(\omega')\int_{-\infty}^{\infty} e^{ia(\omega-\omega')x}\, dx\, d\omega\, d\omega' = \left[\frac{|a|}{2\pi\, b}\right]^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{u}(\omega)\,\hat{u}^*(\omega')\,\frac{2\pi}{|a|}\delta(\omega-\omega')\, d\omega\, d\omega' \\ &= \frac{|a|}{2\pi b^2}\int_{-\infty}^{\infty}\hat{u}^*(\omega)\,\hat{u}^*(\omega)\, d\omega = \frac{|a|}{2\pi b^2}\|\hat{u}\|^2\,.\end{aligned}$$

Unitarity requires $|a| = 2\pi b^2$. Thus $\mathrm{Q} \stackrel{\text{def}}{=} \left\{(a,b) \in \mathrm{P} \,|\, |a| = 2\pi b^2\right\}$. In particular we observe that $(a,b) = (1, 1/\sqrt{2\pi}) \in \mathrm{Q}$.

♣

(**15**) **4.** Distribution Theory (Exam June 14, 2005, Problem 4)

Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto f(x)$, in which $m > 0$ is a constant, defined as

$$f(x) = \begin{cases} 0 & \text{als } x \leq 0 \\ m\,x & \text{als } 0 < x < \frac{1}{m} \\ 1 & \text{als } x \geq \frac{1}{m} \end{cases}$$

(5) **a.** Determine the (classical) derivative f' of f. Clearly indicate the domain of definition of f'.
☞ Hint: Sketch the graph of f.

Het domein van f' is $\mathrm{Dom}\, f' = \mathbb{R} \backslash \{0, \frac{1}{m}\}$. Het functievoorschrift is:

$$f'(x) = \begin{cases} 0 & \text{als } x < 0 \\ m & \text{als } 0 < x < \frac{1}{m} \\ 0 & \text{als } x > \frac{1}{m} \end{cases}$$

De functie is niet gedefinieerd in de aansluitpunten $x = 0$ en $x = \frac{1}{m}$.

By $T_f \in \mathcal{S}'(\mathbb{R})$ we denote the regular tempered distribution corresponding to the function f:

$$T_f : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{R} : \phi \mapsto T_f[\phi] \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx\,.$$

Derivatives of regular tempered distributions are defined as usual: $T_f^{(k)}[\phi] \stackrel{\text{def}}{=} (-1)^k\, T_f[\phi^{(k)}]$. The superscript $k \in \mathbb{N}$ indicates order of differentiation.

(5) **b.** Show that $T'_f[\phi] = m \displaystyle\int_0^{\frac{1}{m}} \phi(x)\,dx$, i.e. that $T'_f = T_g$, with $g : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto g(x)$ given by

$$g(x) = m\,\chi_{[0,\frac{1}{m}]}(x)\,.$$

Here, χ_I is the indicator function on the set $I \subset \mathbb{R}$, i.e. $\chi_I(x) = 1$ if $x \in I$, $\chi_I(x) = 0$ if $x \notin I$.

Er geldt

$$\begin{aligned} T'_f[\phi] &\stackrel{\text{def}}{=} -T_f[\phi'] \stackrel{\text{def}}{=} -\int_{-\infty}^{\infty} f(x)\,\phi'(x)\,dx \stackrel{\text{def}}{=} -m \int_0^{\frac{1}{m}} x\,\phi'(x)\,dx - \int_{\frac{1}{m}}^{\infty} \phi'(x)\,dx \\ &\stackrel{*}{=} -m\,\left[x\,\phi(x)\right]_0^{\frac{1}{m}} + m \int_0^{\frac{1}{m}} \phi(x)\,dx - [\phi(x)]_{\frac{1}{m}}^{\infty} \stackrel{\star}{=} m \int_0^{\frac{1}{m}} \phi(x)\,dx \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} g(x)\,\phi(x)\,dx \stackrel{\text{def}}{=} T_g[\phi]\,, \end{aligned}$$

waarin g (respectievelijk T_g) de functie (respectievelijk reguliere getemperde distributie) is zoals hierboven gedefinieerd. Bij $*$ is gebruik gemaakt van partiële integratie, bij $\star$ zijn de randvoorwaarden gebruikt, met i.h.b. de eigenschap dat $\lim_{x\to\infty} \phi(x) = 0$ voor elke testfunctie $\phi \in \mathcal{S}(\mathbb{R})$. Aangezien dit resultaat geldt voor alle $\phi \in \mathcal{S}(\mathbb{R})$ volgt dat de distributies in linker- en rechterlid gelijk zijn: $T'_f = T_g$.

(5) **c.** Prove: $\lim_{m\to\infty} T'_f = \delta$, in which δ is the Dirac distribution, $\delta : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{R} : \phi \mapsto \delta[\phi] \stackrel{\text{def}}{=} \phi(0)$

☞ HINT: SUBSTITUTE $\xi = mx$ IN THE INTEGRAL EXPRESSION FOR $T'_f[\phi]$ BEFORE TAKING THE LIMIT.

Volg de hint en gebruik het resultaat bij onderdeel b:

$$\lim_{m\to\infty} T'_f[\phi] \stackrel{\text{b}}{=} \lim_{m\to\infty} m \int_0^{\frac{1}{m}} \phi(x)\,dx \stackrel{*}{=} \lim_{m\to\infty} \int_0^1 \phi(\frac{\xi}{m})\,d\xi \stackrel{\star}{=} \int_0^1 \phi(0)\,d\xi = \phi(0) \stackrel{\text{def}}{=} \delta[\phi]\,.$$

Bij $*$ is de genoemde substitutie van variabelen uitgevoerd, bij $\star$ zijn limiet- en integraaloperaties omgewisseld en in de laatste stap is de definitie van de Dirac distributie gebruikt. Aangezien dit resultaat geldt voor alle $\phi \in \mathcal{S}(\mathbb{R})$ volgt dat de distributies in linker- en rechterlid gelijk zijn: $\lim_{m\to\infty} T'_f = \delta$.

THE END