EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Wednesday April 10, 2013. Time: 09h00–12h00. Place: MA 1.44

Read this first!

- Write your name and student ID on each paper.
- The exam consists of 4 problems. Maximum credits are indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of any additional material or equipment, including the problem companion ("opgaven- en tentamenbundel"), is *not* allowed.
- You may provide your answers in Dutch or English.
- Do not hesitate to ask questions on linguistic matters or if you suspect an erroneous problem formulation.

Good luck!

(**35**) **1.** VECTOR SPACE

A real sequence s is an infinitely long array of the form $s = (s_1, s_2, s_3, \ldots)$, with $s_i \in \mathbb{R}$ for all $i \in \mathbb{N}$. It is not difficult to show that the set S of all real sequences constitutes a real vector space under entry-wise addition and scalar multiplication. (You may take this for granted.)

($2\frac{1}{2}$) **a1.** Explain by formulas the meaning of "entry-wise addition and scalar multiplication" on S.

($2\frac{1}{2}$) **a2.** What is the neutral element of S? What is the inverse element of $s = (s_1, s_2, s_3, \ldots) \in$ S?

An *arithmetic sequence* a is a sequence of the form $a = (a_1, a_2, a_3, \ldots)$ such that subsequent terms have a common difference, i.e. for each such a sequence a there exists a constant $c \in \mathbb{R}$ such that for all $i \in \mathbb{N}$

$$a_{i+1} = a_i + c\,.$$

By A we denote the set of all real-valued arithmetic sequences.

(5) **b.** Prove that A $\subset$ S is a vector space.

An n-dimensional basis of A is a set $\{e_1, \ldots, e_n\}$ of n linearly independent sequences $e_i \in$ A, $i = 1, \ldots, n$, such that every element $a \in$ A can be written as a linear superposition of the form $a = \lambda_1 e_1 + \ldots + \lambda_n e_n$ for certain coefficients $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

($2\frac{1}{2}$) **c1.** State in terms of an explicit formula what "linear independence" of the basis elements in $\{e_1, \ldots, e_n\}$ means.

($2\frac{1}{2}$) **c2.** Show that, if such a basis exists, then, for any given $a \in$ A, the coefficients λ_i in the linear superposition $a = \lambda_1 e_1 + \ldots + \lambda_n e_n$ are unique.

☞ HINT: SUPPOSE $a = \lambda_1 e_1 + \ldots + \lambda_n e_n$ AND $a = \mu_1 e_1 + \ldots + \mu_n e_n$, CONSIDER THE DIFFERENCE.

(5) **d.** Show that A does indeed have a finite-dimensional basis, and provide one explicitly. What is the dimension?

A *geometric sequence* g is a sequence of the form $g = (g_1, g_2, g_3, \ldots)$ such that subsequent terms have a common ratio, i.e. for each such a sequence g there exists a nonzero constant $r \in \mathbb{R}\backslash\{0\}$ such that for all $i \in \mathbb{N}$

$$g_{i+1} = r g_i \,.$$

By G we denote the set of all real-valued geometric sequences.

(5) **e.** Prove that $\mathrm{G} \subset \mathrm{S}$ is *not* a vector space.

We restrict ourselves henceforth to the set of all *positive* geometric sequences, defined as $\mathrm{G}^+ = \{g = (g_1, g_2, g_3, \ldots) \in \mathrm{G} \mid g_i > 0 \text{ for all } i = 1, 2, 3, \ldots\}$. On this set we introduce an alternative definition for addition and scalar multiplication, according to the following rules. If $g = (g_1, g_2, g_3, \ldots) \in \mathrm{G}^+$, $h = (h_1, h_2, h_3, \ldots) \in \mathrm{G}^+$, $\lambda \in \mathbb{R}$, then

$$g \oplus h = (g_1 h_1, g_2 h_2, g_3 h_3, \ldots) \quad \text{and} \quad \lambda \otimes g = (g_1^\lambda, g_2^\lambda, g_3^\lambda, \ldots)\,.$$

(10) **f.** Show that G^+ is closed under the actions of $\oplus$ and $\otimes$, and subsequently show that it is a vector space.

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(**35**) **2.** Group Theory

In this problem we consider a finite group G consisting of 2 elements, to which we will refer as $a, b \in G$. Without loss of generality we may identify $a = \mathrm{id}$, i.e. the identity element of G.

(5) **a.** Show that either $a = b = \mathrm{id}$, or, if $a \neq b$, that $b = b^{-1}$.

We henceforth assume that $a \neq b$.

(5) **b.** Provide the 2×2 group multiplication table of G, cf. the template below. With $x_1 = a$, $x_2 = b$, the (i, j)-th element in this table indicates $x_i \circ x_j$.

$\circ$	a	b
a		
b		

Let $T : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}^2\backslash\{(0,0)\} : (x, y) \mapsto T(x, y)$ be given by $T(x, y) = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\right)$.

For $k \in \mathbb{N}$ we indicate k-fold concatenation of T by $T^k \stackrel{\text{def}}{=} T \circ \ldots \leftarrow k\text{-fold} \rightarrow \ldots \circ T$. Moreover, $T^{-k} \stackrel{\text{def}}{=} (T^{\text{inv}})^k$, in which T^{inv} is the inverse function of T. The identity element is identified with $T^0 : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2 \backslash \{(0,0)\} : (x,y) \mapsto \text{id}(x,y) = (x,y)$.

(5) **c.** Show that $T^{\text{inv}} = T$.

☞ Hint: Set $(x', y') = T(x,y)$, and consider the identity $T^{\text{inv}}(x', y') = (x,y)$.

Consider the set $\Theta = \left\{ T^k : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2 \backslash \{(0,0)\} \;\middle|\; k \in \mathbb{Z} \right\}$.

We say that two groups, G and H say, are *isomorphic*, notation $G \sim H$, if there is a one-to-one correspondence $\phi : G \to H : g \mapsto h = \phi(g)$ between their respective elements that preserves the group structure, i.e. $\phi(g_1) \circ_H \phi(g_2) = \phi(g_1 \circ_G g_2)$, in which $\circ_G$ and $\circ_H$ are the infix group operators on G, respectively H.

(5) **d.** Show that the set Θ, furnished with the concatenation operator $\circ$, constitutes a group isomorphic to the 2-element group G of problem b.

Next, consider the class of symmetric smooth functions of rapid decay,

$$\mathscr{S}_{\text{sym}}(\mathbb{R}) \stackrel{\text{def}}{=} \{\phi \in \mathscr{S}(\mathbb{R}) \mid \phi(x) = \phi(-x)\} \,.$$

We take it for granted that $\mathscr{S}(\mathbb{R})$ is closed under Fourier transformation, defined in this problem with the following convention:

$$\mathscr{F}(\phi)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x)\, e^{-i\omega x}\, dx \quad \text{whence} \quad \mathscr{F}^{\text{inv}}(\psi)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(\omega)\, e^{i\omega x}\, d\omega \,.$$

(5) **e.** Show that $\mathscr{S}_{\text{sym}}(\mathbb{R})$ is closed under Fourier transformation. ☞ Hint: $\mathscr{S}_{\text{sym}}(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$.

We now consider the set $\Phi = \left\{ \mathscr{F}^k : \mathscr{S}_{\text{sym}}(\mathbb{R}) \to \mathscr{S}_{\text{sym}}(\mathbb{R}) \;\middle|\; k \in \mathbb{Z} \right\}$.

(5) **f.** Show that this set, furnished with the concatenation operator $\circ$, constitutes a group that is likewise isomorphic to the 2-element group G of problem b, but that this is *not* the case if we replace $\mathscr{S}_{\text{sym}}(\mathbb{R})$ by $\mathscr{S}(\mathbb{R})$ in the definition of Φ.

Finally, we consider the 2-element matrix group under matrix multiplication

$$M = \left\{ I \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,\; A \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} ,$$

and use it to construct the 2-dimensional linear space $V_M = \{\alpha\, I + \beta\, A \mid \alpha, \beta \in \mathbb{R}\}$

(5) **g.** Show that V_M is a semigroup, but *not* a group under matrix multiplication.

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(**15**) **3.** FOURIER TRANSFORMATION

In this problem we consider a general parametrization of the various one-dimensional Fourier definitions one encounters in the literature:

$$\mathscr{F}_{(a,b)}(u)(\omega) = b \int_{-\infty}^{\infty} u(x)\, e^{-ia\omega x}\, dx \quad \text{whence} \quad \mathscr{F}^{\text{inv}}_{(a,b)}(\hat{u})(x) = \frac{|a|}{2\pi\, b} \int_{-\infty}^{\infty} \hat{u}(\omega)\, e^{ia\omega x}\, d\omega\,.$$

The parameter space is $\mathrm{P} \stackrel{\text{def}}{=} \left\{(a,b) \in \mathbb{R}^2 \mid a \neq 0\,,\; b > 0\right\}$. Consider the following reparametrization:

$$\mathrm{T} : \mathrm{P} \to \mathrm{P} : (a,b) \mapsto (a',b') = \mathrm{T}(a,b) \quad \text{with} \quad \begin{cases} a' &= -a \\ b' &= \dfrac{|a|}{2\pi\, b} \end{cases}$$

($2\frac{1}{2}$) **a1.** Show that this is a good definition, in the sense that P is indeed closed under T as stipulated by the prototype "$\mathrm{T} : \mathrm{P} \to \mathrm{P}$", i.e. $(a',b') \in \mathrm{P}$ if $(a,b) \in \mathrm{P}$.

($2\frac{1}{2}$) **a2.** Show that T is invertible, and that $\mathrm{T}^{\text{inv}} = \mathrm{T}$.
☞ HINT: SOLVE $(a,b) = \mathrm{T}^{\text{inv}}(a',b')$.

Without proof we state that the normed space $L^2(\mathbb{R})$ of square-integrable, complex-valued functions with domain $\mathbb{R}$, is closed under Fourier transformation. The norm of a function $u \in L^2(\mathbb{R})$ will be denoted by $\|u\|$. Recall that

$$\|u\|^2 = \int_{-\infty}^{\infty} u(x)\, u^*(x)\, dx\,.$$

In the problems below you may, moreover, use the following lemma: $\displaystyle\int_{-\infty}^{\infty} e^{\pm i\,a\,y\,z}\, dz = \frac{2\pi}{|a|}\,\delta(y)$.

Let $\mathrm{Q} \subset \mathrm{P}$ be the set of parameters for which $\|\mathscr{F}_{(a,b)}(u)(\omega)\|^2 = \|u\|^2$ for all $u \in L^2(\mathbb{R})$ (*unitarity*).

(10) **b.** Determine Q, and show that the convention that was used in problem 1 provides an example of a unitary Fourier transform, i.e. show that $(a,b) = (1, 1/\sqrt{2\pi}) \in \mathrm{Q}$.

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(**15**) **4.** DISTRIBUTION THEORY (EXAM JUNE 14, 2005, PROBLEM 4)

Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto f(x)$, in which $m > 0$ is a constant, defined as

$$f(x) = \begin{cases} 0 & \text{als } x \leq 0 \\ m\,x & \text{als } 0 < x < \frac{1}{m} \\ 1 & \text{als } x \geq \frac{1}{m} \end{cases}$$

(5) **a.** Determine the (classical) derivative f' of f. Clearly indicate the domain of definition of f'.
☞ HINT: SKETCH THE GRAPH OF f.

By $T_f \in \mathcal{S}'(\mathbb{R})$ we denote the regular tempered distribution corresponding to the function f:

$$T_f : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{R} : \phi \mapsto T_f[\phi] \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx\,.$$

Derivatives of regular tempered distributions are defined as usual: $T_f^{(k)}[\phi] \stackrel{\text{def}}{=} (-1)^k\, T_f[\phi^{(k)}]$. The superscript $k \in \mathbb{N}$ indicates order of differentiation.

(5) **b.** Show that $T_f'[\phi] = m \displaystyle\int_0^{\frac{1}{m}} \phi(x)\,dx$, i.e. that $T_f' = T_g$, with $g : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto g(x)$ given by

$$g(x) = m\,\chi_{[0,\frac{1}{m}]}(x)\,.$$

Here, χ_I is the indicator function on the set $I \subset \mathbb{R}$, i.e. $\chi_I(x) = 1$ if $x \in I$, $\chi_I(x) = 0$ if $x \notin I$.

(5) **c.** Prove: $\lim\limits_{m\to\infty} T_f' = \delta$, in which δ is the Dirac distribution, $\delta : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{R} : \phi \mapsto \delta[\phi] \stackrel{\text{def}}{=} \phi(0)$

☞ Hint: Substitute $\xi = mx$ in the integral expression for $T_f'[\phi]$ before taking the limit.

THE END