REEXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Wednesday April 11, 2012. Time: 09h00–12h00. Place: MA 1.46.

Read this first!

- Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of problem companion ("opgaven- en tentamenbundel"), calculator, laptop, or any other equipment, is *not* allowed.
- You may provide your answers in Dutch or English.
- Feel free to ask questions on linguistic matters or if you suspect an erroneous problem formulation.

Good luck!

(**30**) **1.** Inner Product Space

Consider the set

$$V = \{ f : \mathbb{R} \to \mathbb{C} \mid f \in C(\mathbb{R}) \text{ and } f(x) = \langle k_x | f \rangle \}$$

in which $k_x \in V$ is a particular element of V for every $x \in \mathbb{R}$ and $\langle\,|\,\rangle : C(\mathbb{R}) \times C(\mathbb{R}) \to \mathbb{C}$ is a complex inner product on $C(\mathbb{R})$. We take it for granted that $C(\mathbb{R})$ is a complex inner product space given the usual definitions of function addition and complex scalar multiplication.

a. Show that the function k_x has the following properties:

($2\frac{1}{2}$) **a1.** $k_x(y) = \langle k_y | k_x \rangle$;

Since $k_x \in V$ we have, by definition of V, $k_x(y) = \langle k_y | k_x \rangle$.

($2\frac{1}{2}$) **a2.** $k_x(y) = \overline{k_y(x)}$ (in which $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$);

Using the definition of V (first and last step) and the definition of a complex inner product (middle step) we find $k_x(y) = \langle k_y | k_x \rangle = \overline{\langle k_x | k_y \rangle} = \overline{k_y(x)}$.

($2\frac{1}{2}$) **a3.** $k_x(x) \geq 0$ for all $x \in \mathbb{R}$.

By virtue of non-degeneracy of an inner product we have, for all $x \in \mathbb{R}$, $k_x(x) = \langle k_x | k_x \rangle \geq 0$.

The following diagrams are abstract representations for $k_x(y)$ and $\overline{k_y(x)}$:

($2\frac{1}{2}$) **a4.** Explain what it means to say that these diagrams are mutually consistent.

Assuming that the orientation of the graphics should not matter you can interpret the second graph as a 180°-rotated copy of the first graph, with free labels x and y interchanged, i.e. as $k_x(y)$. But according to a2 this is identical to $\overline{k_y(x)}$, indeed the interpretation given to the second graph.

(10) **b.** Show that V is a complex vector space.
☞ HINT: USE THE LINEAR SUBSPACE THEOREM.

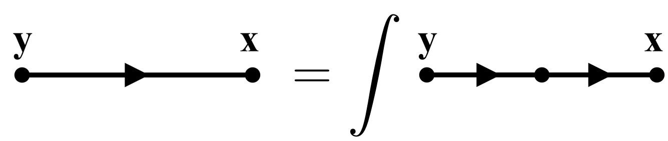
It is given that $C(\mathbb{R})$ is a complex vector space, with $V \subset C(\mathbb{R})$. So we need to prove only closure. Let $f, g \in V$ and $\lambda, \mu \in \mathbb{C}$, so in particular $f(x) = \langle k_x | f \rangle$ and $g(x) = \langle k_x | g \rangle$ for all $x \in \mathbb{R}$. We have $\langle k_x | \lambda f + \mu g \rangle \stackrel{*}{=} \lambda \langle k_x | f \rangle + \mu \langle k_x | g \rangle \stackrel{\star}{=} \lambda f(x) + \mu g(x) \stackrel{\circ}{=} (\lambda f + \mu g)(x)$, so $\lambda f + \mu g \in V$. Here we have used the linearity property of a complex inner product ($*$), the definition of V ($\star$), and the usual definition of linear function superposition ($\circ$).

Below we take $\langle \, | \, \rangle : C(\mathbb{R}) \times C(\mathbb{R}) \to \mathbb{C}$ to be the *standard* complex inner product on $C(\mathbb{R})$.

(5) **c.** Explain what this means by giving the explicit formula for $\langle f | g \rangle$.

The standard complex inner product for one-dimensional functions is given by $\langle f | g \rangle = \int_{\mathbb{R}} \overline{f(x)}\, g(x)\, dx$.

(5) **d.** Explain and prove the following diagrammatic equality:



☞ HINT: THE UNLABELED CENTRAL DOT ON THE R.H.S. REPRESENTS AN INTEGRATION DUMMY.

The r.h.s. diagram symbolizes $k_y(x) = \displaystyle\int_{\mathbb{R}} \overline{k_x(z)}\, k_y(z)\, dz$. To see that this identity is correct, consider the l.h.s. diagram. By definition this diagram equals

$$k_y(x) \stackrel{a1}{=} \langle k_x | k_y \rangle \stackrel{c}{=} \int \overline{k_x(z)}\, k_y(z)\, dz \stackrel{a1}{=} \int \overline{\langle k_z | k_x \rangle}\, \langle k_z | k_y \rangle\, dz\,.$$

The integrand on the r.h.s. consists of two factors, which can be diagrammatically represented as two oriented line elements with a common (dummy) label z, once with an incoming arrow and once with an outgoing arrow. By graphically "contracting" this dummy into a single inner node one obtains the diagram on the r.h.s.

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(35) **2.** ALGEBRA (EXAM MARCH 8, 2005, PROBLEM 1)

In this problem we consider the set $\mathcal{G} \stackrel{\text{def}}{=} \mathbb{R}^2$ endowed with certain internal and external operators. We identify an element $\theta \in \mathcal{G}$ with its column representation in $\mathbb{R}^2$:

$$\theta \stackrel{\text{def}}{=} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \quad \text{in which } \theta_1, \theta_2 \in \mathbb{R} \text{ (the "components of } \theta\text{")}.$$

To begin with we interpret $\mathcal{G}$ as the linear space over $\mathbb{R}$ by introducing vector addition and scalar multiplication, in the usual way. The vector sum of $\eta, \theta \in \mathcal{G}$ is written as $\eta + \theta$, and the scalar multiple of $\theta \in \mathcal{G}$ and $\lambda \in \mathbb{R}$ as $\lambda\,\theta$.

(5) **a.** Explain what is meant by "the usual way" by indicating explicitly how $\eta + \theta$ and $\lambda\,\theta$ are defined in terms of their components.

For arbitrary $\lambda, \mu \in \mathbb{R}$ and $\eta, \theta \in \mathcal{G}$ we define

$$\lambda\,\eta + \mu\,\theta \stackrel{\text{def}}{=} \begin{pmatrix} \lambda\,\eta_1 + \mu\,\theta_1 \\ \lambda\,\eta_2 + \mu\,\theta_2 \end{pmatrix}.$$

We furthermore introduce an algebraic operation, which we shall refer to as "multiplication". The "product" of $\eta, \theta \in \mathcal{G}$ is simply written as $\eta\,\theta$, for which we agree that, in terms of components,

$$\eta\,\theta \stackrel{\text{def}}{=} \begin{pmatrix} \eta_1\,\theta_1 \\ \eta_1\,\theta_2 + \eta_2\,\theta_1 \end{pmatrix} \in \mathcal{G}.$$

(5) **b.** Prove that $\mathcal{G}$, endowed with the aforementioned multiplication operation, constitutes an algebra. Proceed as follows (without proof we take it for granted that $\mathcal{G}$ is a linear space, cf. part a):

b1. Prove that $\forall\,\eta, \theta, \gamma \in \mathcal{G} \quad (\eta\,\theta)\,\gamma = \eta\,(\theta\,\gamma)$.

Expanding in terms of components we get

$$(\eta\,\theta)\,\gamma \stackrel{\text{def}}{=} \begin{pmatrix} \eta_1\,\theta_1 \\ \eta_1\,\theta_2 + \eta_2\,\theta_1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} (\eta_1\,\theta_1)\,\gamma_1 \\ (\eta_1\,\theta_1)\,\gamma_2 + (\eta_1\,\theta_2 + \eta_2\,\theta_1)\,\gamma_1 \end{pmatrix} \stackrel{\star}{=} \begin{pmatrix} \eta_1\,(\theta_1\,\gamma_1) \\ \eta_1\,(\theta_1\,\gamma_2 + \theta_2\,\gamma_1) + \eta_2\,(\theta_1\,\gamma_1) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \begin{pmatrix} \theta_1\,\gamma_1 \\ \theta_1\,\gamma_2 + \theta_2\,\gamma_1 \end{pmatrix} \stackrel{\text{def}}{=} \eta\,(\theta\,\gamma).$$

The triviality marked with $\star$ exploits associativity of ordinary multiplication on $\mathbb{R}$.

b2. Prove that $\forall\,\eta, \theta, \gamma \in \mathcal{G} \quad \eta\,(\theta + \gamma) = (\eta\,\theta) + (\eta\,\gamma)$.

$$\eta\,(\theta + \gamma) = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \begin{pmatrix} \theta_1 + \gamma_1 \\ \theta_2 + \gamma_2 \end{pmatrix} = \begin{pmatrix} \eta_1\,(\theta_1 + \gamma_1) \\ \eta_1\,(\theta_2 + \gamma_2) + \eta_2\,(\theta_1 + \gamma_1) \end{pmatrix} \stackrel{\star}{=} \begin{pmatrix} \eta_1\,\theta_1 \\ \eta_1\,\theta_2 + \eta_2\,\theta_1 \end{pmatrix} + \begin{pmatrix} \eta_1\,\gamma_1 \\ \eta_1\,\gamma_2 + \eta_2\,\gamma_1 \end{pmatrix} = (\eta\,\theta) + (\eta\,\gamma).$$

In $\star$ distributivity of ordinary multiplication on $\mathbb{R}$ has been used to eliminate parentheses in the components and also of the definition of vector addition to split the column vector into two terms.

b3. Prove that $\forall\,\eta, \theta, \gamma \in \mathcal{G} \quad (\eta + \theta)\,\gamma = (\eta\,\gamma) + (\theta\,\gamma)$.

You can construct the proof in terms of components analogous to the solution for b2. Alternatively one can make use of commutativity ($\star$), which will be proven under d below, and of the distributivity property in part b2 ($*$):

$$(\eta + \theta)\,\gamma \stackrel{\star}{=} \gamma\,(\eta + \theta) \stackrel{*}{=} (\gamma\,\eta) + (\gamma\,\theta) \stackrel{\star}{=} (\eta\,\gamma) + (\theta\,\gamma).$$

b4. Prove that $\forall\,\eta, \theta \in \mathcal{G}, \lambda \in \mathbb{R} \quad \lambda(\eta\,\theta) = (\lambda\,\eta)\,\theta = \eta\,(\lambda\,\theta)$.

We first prove the first equality:

$$\lambda(\eta\,\theta) \stackrel{\text{def}}{=} \lambda\begin{pmatrix} \eta_1\,\theta_1 \\ \eta_1\,\theta_2 + \eta_2\,\theta_1 \end{pmatrix} \stackrel{\star}{=} \begin{pmatrix} \lambda\,(\eta_1\,\theta_1) \\ \lambda(\eta_1\,\theta_2 + \eta_2\,\theta_1) \end{pmatrix} \stackrel{*}{=} \begin{pmatrix} (\lambda\,\eta_1)\,\theta_1 \\ (\lambda\,\eta_1)\,\theta_2 + (\lambda\,\eta_2)\,\theta_1 \end{pmatrix} \stackrel{\star}{=} \begin{pmatrix} (\lambda\,\eta)_1\,\theta_1 \\ (\lambda\,\eta)_1\,\theta_2 + (\lambda\,\eta)_2\,\theta_1 \end{pmatrix} \stackrel{\text{def}}{=} (\lambda\,\eta)\,\theta\,.$$

In $\star$ we have used the definition of scalar multiplication, in $*$ we have used associativity of multiplication on $\mathbb{R}$ to shift parentheses. In the first and last step the definition of multiplication on $\mathcal{G}$ has been used. The second equality can be proven in a similar fashion, or with the help of commutativity (part d).

(5) **c.** Show that, moreover, there exists a unit element $1 \in \mathcal{G}$ (not to be confused with the number $1 \in \mathbb{R}$), and give its column representation in $\mathbb{R}^2$.

Call the components of $1 \in \mathcal{G}$ $e_1 \in \mathbb{R}$ respectively $e_2 \in \mathbb{R}$. Let $x \in \mathcal{G}$ be arbitrary, and suppose

$$1\,x = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e_1\,x_1 \\ e_1\,x_2 + e_2\,x_1 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

for all $x_1, x_2 \in \mathbb{R}$, in which (in the final step) the definition of the identity element has been used, then it follows by necessity that $e_1 = 1$ and $e_2 = 0$. Conclusion:

$$1 \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

One still needs to verify whether $x\,1 = x$ for all $x \in \mathcal{G}$. We could do this by componentwise analysis as previously, or by exploiting once more commutativity, with a modest amount of foresight, cf. part d ($\star$):

$$x\,1 \stackrel{\star}{=} 1\,x = x \quad \text{for all } x \in \mathcal{G}\,.$$

(5) **d.** Is multiplication on $\mathcal{G}$ commutative? If so, prove, if not, give a counter example.

Suppose $x, y \in \mathcal{G}$, then

$$x\,y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1\,y_1 \\ x_1\,y_2 + x_2\,y_1 \end{pmatrix} = \begin{pmatrix} y_1\,x_1 \\ y_1\,x_2 + y_2\,x_1 \end{pmatrix} = y\,x\,.$$

We now consider the subset $\mathcal{G}_0 \subset \mathcal{G}$, defined by $\mathcal{G}_0 = \{\theta \in \mathcal{G} \mid \theta^2 = 0\}$. (With θ^2 we mean $\theta\,\theta$.)

(5) **e.** Give an explicit characterization of $\mathcal{G}_0$ by indicating what the column representation in $\mathbb{R}^2$ of an arbitrary element $\theta \in \mathcal{G}_0$ looks like.

In terms of components we have

$$\theta^2 = \begin{pmatrix} \theta_1^2 \\ 2\theta_1\theta_2 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In the last step we have used the fact that $\theta \in \mathcal{G}_0$. This is equivalent to the condition $\theta_1 = 0$, i.e. an arbitrary element $\theta \in \mathcal{G}_0$ has the form

$$\theta = \begin{pmatrix} 0 \\ \theta_2 \end{pmatrix}$$

with $\theta_2 \in \mathbb{R}$ arbitrary.

Finally we introduce on $\mathcal{G}$ a *degenerate*, non-negative, symmetric, real valued, bilinear form. For $\eta, \theta \in \mathcal{G}$ this is indicated by $\langle \eta | \theta \rangle \in \mathbb{R}$. In terms of the components of η and θ we define this as follows:

$$\langle \eta | \theta \rangle = \eta_1\,\theta_1\,.$$

Caveat: The adjective "degenerate" indicates that $\langle\;|\;\rangle$ does *not* define an inner product.

(5) **f.** Explain the adjective "degenerate" by explaining why $\langle\ |\ \rangle$ does not define an inner product.

For a (non-degenerate) inner product we have the condition that $\langle\theta|\theta\rangle = 0$ iff $\theta = 0 \in \mathcal{G}$. In the case of the definition above, however, we have $\langle\theta|\theta\rangle = 0$ iff $\theta_1 = 0$, i.e. irrespective of the value of θ_2. Thus there are non-trivial elements $\theta \in \mathcal{G}$ (viz. all elements with $\theta_1 = 0$ and $\theta_2 \neq 0$) for which $\langle\theta|\theta\rangle = 0$ ("degeneracy").

We now consider the subset $\mathcal{G}_1 \subset \mathcal{G}$, defined by $\mathcal{G}_1 = \{\theta \in \mathcal{G} \mid \langle\theta|\theta\rangle = 1\}$.

(5) **g.** Prove that $\mathcal{G}_1$ constitutes a group with respect to multiplication. Proceed as follows:

g1. Show that, if $\eta, \theta \in \mathcal{G}_1$ then $\eta\,\theta \in \mathcal{G}_1$ ("closure").

You need to show that the subset $\mathcal{G}_1 \subset \mathcal{G}$ is closed under multiplication. Suppose $\eta, \theta \in \mathcal{G}_1$, then

$$\eta\,\theta = \begin{pmatrix} \eta_1\,\theta_1 \\ \eta_1\,\theta_2 + \eta_2\,\theta_1 \end{pmatrix} \quad \text{so} \quad \langle\eta\,\theta|\eta\,\theta\rangle \stackrel{*}{=} (\eta\,\theta)_1^2 \stackrel{\star}{=} (\eta_1\,\theta_1)^2 = \eta_1^2\,\theta_1^2 = \langle\eta|\eta\rangle\,\langle\theta|\theta\rangle = 1\,.$$

In $*$ the definition of the bilinear form $\langle\eta|\theta\rangle \in \mathbb{R}$ has been used, in $\star$ that of the (first component of) the algebraic product $\eta\,\theta \in \mathcal{G}$. In the last step the fact that $\eta, \theta \in \mathcal{G}_1$ has been used. Since $\langle\eta\,\theta|\eta\,\theta\rangle = 1$ it follows that $\eta\,\theta \in \mathcal{G}_1$.

g2. Show that $\forall\,\eta, \theta, \gamma \in \mathcal{G}_1 \quad (\eta\,\theta)\,\gamma = \eta\,(\theta\,\gamma)$ ("associativity").

Multiplication on $\mathcal{G}$ is associative (cf. b1). Since $\mathcal{G}_1 \subset \mathcal{G}$ it is also associative within $\mathcal{G}_1$.

g3. Show that the unit element of part c satisfies $1 \in \mathcal{G}_1$.

$\theta \in \mathcal{G}_1$ iff $\theta \in \mathcal{G}$ and $\langle\theta|\theta\rangle = 1$. For the identity element it has already been shown in c that $1 \in \mathcal{G}$. Using the components of the identity element and the definition of the bilinear form it immediately follows that

$$\langle 1|1\rangle = 1\,.$$

Conclusion: $1 \in \mathcal{G}_1$.

g4. Show that, given $\theta \in \mathcal{G}_1$, there exists an inverse $\theta^{-1} \in \mathcal{G}_1$, such that $\theta\,\theta^{-1} = \theta^{-1}\,\theta = 1 \in \mathcal{G}_1$. Give the column representation of θ^{-1} in $\mathbb{R}^2$ in terms of the components of θ.

Given $\theta \in \mathcal{G}_1$ arbitrary. Write $\theta^{-1} = \eta$ for notational convenience. We must have $\eta\,\theta = 1 \in \mathcal{G}_1$. (If this holds then $\theta\,\eta = 1$ holds automatically by virtue of commutativity, recall part d.) In other words,

$$\begin{pmatrix} \eta_1\,\theta_1 \\ \eta_1\,\theta_2 + \eta_2\,\theta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which can also be written as

$$\begin{pmatrix} \theta_1 & 0 \\ \theta_2 & \theta_1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Inversion yields

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}^{-1} \stackrel{\text{def}}{=} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \theta_1 & 0 \\ \theta_2 & \theta_1 \end{pmatrix}^{\text{inv}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\theta_1^2} \begin{pmatrix} \theta_1 & 0 \\ -\theta_2 & \theta_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\theta_1} \\ -\dfrac{\theta_2}{\theta_1^2} \end{pmatrix} \stackrel{*}{=} \begin{pmatrix} \theta_1 \\ -\theta_2 \end{pmatrix}.$$

Note that this is well defined, since $\theta_1 \neq 0$ for all $\theta \in \mathcal{G}_1$. The last step, indicated with a $*$, exploits the fact that $\theta_1^2 = \langle\theta|\theta\rangle = 1$ for all $\theta \in \mathcal{G}_1$.

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(**35**) **3.** FOURIER TRANSFORMATION AND DISTRIBUTION THEORY

Consider the generalized function f defining a tempered distribution $T_f \in \mathscr{S}'(\mathbb{R}^2)$, given by

$$f(x,y) = u(x,y)\,\delta(y-mx)\,,$$

in which δ denotes the *one-dimensional* Dirac function, $u : \mathbb{R}^2 \to \mathbb{C}$ is a given function with well-defined Fourier transform $\hat{u} : \mathbb{R}^2 \to \mathbb{C}$, and $m \in \mathbb{R}$ is a parameter. That is, for $\phi \in \mathscr{S}(\mathbb{R}^2)$,

$$T_f(\phi) = \iint_{\mathbb{R}^2} f(x,y)\,\phi(x,y)\,dxdy\,.$$

Note that the support of f (the part of the (x,y)-domain where $f(x,y)$ may not vanish) is effectively the line given by $\ell : y = mx$. For this reason we define the function $u_m : \mathbb{R} \to \mathbb{C}$ by

$$u_m(x) = u(x, mx)\,.$$

In this problem the two-dimensional Fourier transform is defined as

$$\hat{f}(\omega,\nu) = \iint_{\mathbb{R}^2} e^{-i\omega x - i\nu y} f(x,y)\,dxdy\,.$$

(10) **a.** Show that $\hat{f}(\omega,\nu) = \hat{u}_m(\omega + m\nu)$.

Working out the definitions we have

$$\hat{f}(\omega,\nu) = \iint_{\mathbb{R}^2} e^{-i\omega x - i\nu y} f(x,y)\,dxdy = \iint_{\mathbb{R}^2} e^{-i\omega x - i\nu y} u(x,y)\,\delta(y-mx)\,dxdy = \int_{\mathbb{R}} e^{-i(\omega+m\nu)x}\, u_m(x)\;dx = \hat{u}_m(\omega+m\nu)\,.$$

($2\frac{1}{2}$) **b1.** Sketch the graph of ℓ in the (x,y)-plane.

The graph of ℓ in the (x,y)-plane is a straight line through the origin with an inclination angle α relative to the x-axis, such that $m = \tan\alpha$.

($2\frac{1}{2}$) **b2.** Express the angle α by which ℓ intersects the x-axis in terms of the parameter m.

Recall b1: $\alpha = \arctan m$.

($2\frac{1}{2}$) **b3.** Sketch the family of lines in the (ω,ν)-plane on which $\hat{f}(\omega,\nu)$ assumes constant values.

For constant $c \in \mathbb{C}$ and generic function $\hat{f}$ we have $\hat{f}(\omega,\nu) = c$, i.e. $\hat{u}_m(\omega + m\nu) = c$, along lines in the (ω,ν)-plane for which $\omega + m\nu = k$ for any constant $k \in \mathbb{R}$.

($2\frac{1}{2}$) **b4.** Under which angle does the normal vector to this family intersect the ω-axis?

The common normal of the family of lines given in b3 is, up to an arbitrary non-zero constant, equal to $(1, m)$, so that it makes the same angle $\alpha = \arctan m$ with the ω-axis as the line ℓ does with the x-axis.

Below we consider the case

$$u(x,y) = Ae^{-(x^2+y^2)}\,,$$

for some amplitude $A > 0$. In the following problem you may use the following standard integral, valid for all $\xi, \eta \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} e^{-(\xi+i\eta)^2} d\xi = \sqrt{\pi}\,.$$

(10) **c.** Compute $\hat{u}_m(\omega)$.

Plugging in the definition of the function u_m we obtain

$$\hat{u}_m(\omega) = \int_{\mathbb{R}} e^{-i\omega x}\, u_m(x)\, dx = A \int_{\mathbb{R}} e^{-i\omega x}\, e^{-(1+m^2)x^2}\, dx\,.$$

The r.h.s. can be rewritten as

$$A\, e^{-\frac{\omega^2}{4(1+m^2)}} \int_{\mathbb{R}} e^{-(x\sqrt{1+m^2}+\frac{i\omega}{2\sqrt{1+m^2}})^2}\, dx \overset{*}{=} \frac{A}{\sqrt{1+m^2}}\, e^{-\frac{\omega^2}{4(1+m^2)}} \int_{\mathbb{R}} e^{-(\xi+\frac{i\omega}{2\sqrt{1+m^2}})^2}\, d\xi \overset{\star}{=} \frac{A\sqrt{\pi}}{\sqrt{1+m^2}}\, e^{-\frac{\omega^2}{4(1+m^2)}}\,.$$

In $*$ we have substituted $x\sqrt{1+m^2} = \xi$, in $\star$ we have applied the given standard integral.

(5) **d.** Suppose the function u_m is normalized such that $\int_{-\infty}^{\infty} u_m(x)\, dx = 1$. Determine A.

This means that $\hat{u}_m(0) = 1$, so $A = \sqrt{1+m^2}/\sqrt{\pi}$.

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