REEXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Wednesday April 11, 2012. Time: 09h00–12h00. Place: MA 1.46.

Read this first!

- Write your name and student ID on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes is allowed. The use of problem companion ("opgaven- en tentamenbundel"), calculator, laptop, or any other equipment, is *not* allowed.
- You may provide your answers in Dutch or English.
- Feel free to ask questions on linguistic matters or if you suspect an erroneous problem formulation.

Good luck!

(**30**) **1.** Inner Product Space

Consider the set

$$V = \{ f : \mathbb{R} \to \mathbb{C} \mid f \in C(\mathbb{R}) \text{ and } f(x) = \langle k_x | f \rangle \}$$

in which $k_x \in V$ is a particular element of V for every $x \in \mathbb{R}$ and $\langle \, | \, \rangle : C(\mathbb{R}) \times C(\mathbb{R}) \to \mathbb{C}$ is a complex inner product on $C(\mathbb{R})$. We take it for granted that $C(\mathbb{R})$ is a complex inner product space given the usual definitions of function addition and complex scalar multiplication.

a. Show that the function k_x has the following properties:

($2\frac{1}{2}$) **a1.** $k_x(y) = \langle k_y | k_x \rangle$;

($2\frac{1}{2}$) **a2.** $k_x(y) = \overline{k_y(x)}$ (in which $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$);

($2\frac{1}{2}$) **a3.** $k_x(x) \geq 0$ for all $x \in \mathbb{R}$.

The following diagrams are abstract representations for $k_x(y)$ and $\overline{k_y(x)}$:

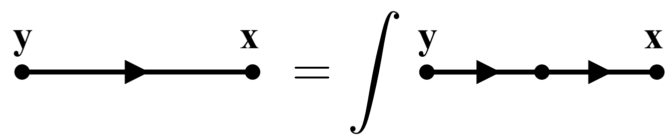
($2\frac{1}{2}$) **a4.** Explain what it means to say that these diagrams are mutually consistent.

(10) **b.** Show that V is a complex vector space.
☞ Hint: Use the linear subspace theorem.

Below we take $\langle\ |\ \rangle : C(\mathbb{R}) \times C(\mathbb{R}) \to \mathbb{C}$ to be the *standard* complex inner product on $C(\mathbb{R})$.

(5) **c.** Explain what this means by giving the explicit formula for $\langle f|g\rangle$.

(5) **d.** Explain and prove the following diagrammatic equality:



☞ HINT: THE UNLABELED CENTRAL DOT ON THE R.H.S. REPRESENTS AN INTEGRATION DUMMY.

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(35) 2. ALGEBRA (EXAM MARCH 8, 2005, PROBLEM 1)

In this problem we consider the set $\mathcal{G} \stackrel{\text{def}}{=} \mathbb{R}^2$ endowed with certain internal and external operators. We identify an element $\theta \in \mathcal{G}$ with its column representation in $\mathbb{R}^2$:

$$\theta \stackrel{\text{def}}{=} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \quad \text{in which } \theta_1, \theta_2 \in \mathbb{R} \text{ (the "components of } \theta\text{").}$$

To begin with we interpret $\mathcal{G}$ as the linear space over $\mathbb{R}$ by introducing vector addition and scalar multiplication, in the usual way. The vector sum of $\eta, \theta \in \mathcal{G}$ is written as $\eta + \theta$, and the scalar multiple of $\theta \in \mathcal{G}$ and $\lambda \in \mathbb{R}$ as $\lambda\,\theta$.

(5) **a.** Explain what is meant by "the usual way" by indicating explicitly how $\eta + \theta$ and $\lambda\,\theta$ are defined in terms of their components.

We furthermore introduce an algebraic operation, which we shall refer to as "multiplication". The "product" of $\eta, \theta \in \mathcal{G}$ is simply written as $\eta\,\theta$, for which we agree that, in terms of components,

$$\eta\,\theta \stackrel{\text{def}}{=} \begin{pmatrix} \eta_1\,\theta_1 \\ \eta_1\,\theta_2 + \eta_2\,\theta_1 \end{pmatrix} \in \mathcal{G}\,.$$

(5) **b.** Prove that $\mathcal{G}$, endowed with the aforementioned multiplication operation, constitutes an algebra. Proceed as follows (without proof we take it for granted that $\mathcal{G}$ is a linear space, cf. part a):

b1. Prove that $\forall\,\eta, \theta, \gamma \in \mathcal{G} \quad (\eta\,\theta)\,\gamma = \eta\,(\theta\,\gamma)$.

b2. Prove that $\forall\,\eta, \theta, \gamma \in \mathcal{G} \quad \eta\,(\theta + \gamma) = (\eta\,\theta) + (\eta\,\gamma)$.

b3. Prove that $\forall\,\eta, \theta, \gamma \in \mathcal{G} \quad (\eta + \theta)\,\gamma = (\eta\,\gamma) + (\theta\,\gamma)$.

b4. Prove that $\forall\,\eta, \theta \in \mathcal{G}, \lambda \in \mathbb{R} \quad \lambda(\eta\,\theta) = (\lambda\,\eta)\,\theta = \eta\,(\lambda\,\theta)$.

(5) **c.** Show that, moreover, there exists a unit element $1 \in \mathcal{G}$ (not to be confused with the number $1 \in \mathbb{R}$), and give its column representation in $\mathbb{R}^2$.

(5) **d.** Is multiplication on $\mathcal{G}$ commutative? If so, prove, if not, give a counter example.

We now consider the subset $\mathcal{G}_0 \subset \mathcal{G}$, defined by $\mathcal{G}_0 = \{\theta \in \mathcal{G} \mid \theta^2 = 0\}$. (With θ^2 we mean $\theta\,\theta$.)

(5) **e.** Give an explicit characterization of $\mathcal{G}_0$ by indicating what the column representation in $\mathbb{R}^2$ of an arbitrary element $\theta \in \mathcal{G}_0$ looks like.

Finally we introduce on $\mathcal{G}$ a *degenerate*, non-negative, symmetric, real valued, bilinear form. For $\eta, \theta \in \mathcal{G}$ this is indicated by $\langle \eta | \theta \rangle \in \mathbb{R}$. In terms of the components of η and θ we define this as follows:

$$\langle \eta | \theta \rangle = \eta_1\, \theta_1\, .$$

Caveat: The adjective "degenerate" indicates that $\langle\ |\ \rangle$ does *not* define an inner product.

(5) **f.** Explain the adjective "degenerate" by explaining why $\langle\ |\ \rangle$ does not define an inner product.

We now consider the subset $\mathcal{G}_1 \subset \mathcal{G}$, defined by $\mathcal{G}_1 = \{\theta \in \mathcal{G} \mid \langle \theta | \theta \rangle = 1\}$.

(5) **g.** Prove that $\mathcal{G}_1$ constitutes a group with respect to multiplication. Proceed as follows:

g1. Show that, if $\eta, \theta \in \mathcal{G}_1$ then $\eta\,\theta \in \mathcal{G}_1$ ("closure").

g2. Show that $\forall\, \eta, \theta, \gamma \in \mathcal{G}_1 \quad (\eta\,\theta)\,\gamma = \eta\,(\theta\,\gamma)$ ("associativity").

g3. Show that the unit element of part c satisfies $1 \in \mathcal{G}_1$.

g4. Show that, given $\theta \in \mathcal{G}_1$, there exists an inverse $\theta^{-1} \in \mathcal{G}_1$, such that $\theta\,\theta^{-1} = \theta^{-1}\,\theta = 1 \in \mathcal{G}_1$. Give the column representation of θ^{-1} in $\mathbb{R}^2$ in terms of the components of θ.

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(35) 3. Fourier Transformation and Distribution Theory

Consider the generalized function f defining a tempered distribution $T_f \in \mathscr{S}'(\mathbb{R}^2)$, given by

$$f(x, y) = u(x, y)\, \delta(y - mx)\, ,$$

in which δ denotes the *one-dimensional* Dirac function, $u : \mathbb{R}^2 \to \mathbb{C}$ is a given function with well-defined Fourier transform $\hat{u} : \mathbb{R}^2 \to \mathbb{C}$, and $m \in \mathbb{R}$ is a parameter. That is, for $\phi \in \mathscr{S}(\mathbb{R}^2)$,

$$T_f(\phi) = \iint_{\mathbb{R}^2} f(x, y)\, \phi(x, y)\, dxdy\, .$$

Note that the support of f (the part of the (x, y)-domain where $f(x, y)$ may not vanish) is effectively the line given by $\ell : y = mx$. For this reason we define the function $u_m : \mathbb{R} \to \mathbb{C}$ by

$$u_m(x) = u(x, mx)\, .$$

In this problem the two-dimensional Fourier transform is defined as

$$\hat{f}(\omega, \nu) = \iint_{\mathbb{R}^2} e^{-i\omega x - i\nu y} f(x, y)\, dxdy\,.$$

(10) **a.** Show that $\hat{f}(\omega, \nu) = \hat{u}_m(\omega + m\nu)$.

($2\frac{1}{2}$) **b1.** Sketch the graph of ℓ in the (x, y)-plane.

($2\frac{1}{2}$) **b2.** Express the angle α by which ℓ intersects the x-axis in terms of the parameter m.

($2\frac{1}{2}$) **b3.** Sketch the family of lines in the (ω, ν)-plane on which $\hat{f}(\omega, \nu)$ assumes constant values.

($2\frac{1}{2}$) **b4.** Under which angle does the normal vector to this family intersect the ω-axis?

Below we consider the case

$$u(x, y) = Ae^{-(x^2+y^2)}\,,$$

for some amplitude $A > 0$. In the following problem you may use the following standard integral, valid for all $\xi, \eta \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} e^{-(\xi+i\eta)^2} d\xi = \sqrt{\pi}\,.$$

(10) **c.** Compute $\hat{u}_m(\omega)$.

(5) **d.** Suppose the function u_m is normalized such that $\int_{-\infty}^{\infty} u_m(x)\, dx = 1$. Determine A.

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