EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Monday June 15, 2009. Time: 14h00–17h00. Place: HG 10.01 C.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student identification number on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes and calculator is allowed. The use of the problem companion, "opgaven- en tentamenbundel", is *not* allowed.
- You may provide your answers in Dutch or (preferably) in English.

GOOD LUCK!

(35) **1.** LINEAR ALGEBRA & GROUP THEORY

Definition. Let V be a vector space over $\mathbb{R}$. A *real inner product* is a nondegenerate positive definite symmetric bilinear mapping $\langle\,|\,\rangle : V \times V \longrightarrow \mathbb{R}$ satisfying the following properties. For all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$ we have

- $\langle \lambda u + \mu v | w \rangle = \lambda \langle u | w \rangle + \mu \langle v | w \rangle$,
- $\langle u | \lambda v + \mu w \rangle = \lambda \langle u | v \rangle + \mu \langle u | w \rangle$,
- $\langle u | v \rangle = \langle v | u \rangle$,
- $\langle u | u \rangle > 0$ for all $u \neq 0$.

($2\frac{1}{2}$) **a.** Show that either the first or the second criterion is redundant.

The second property follows from the first and third properties: $\langle u | \lambda v + \mu w \rangle \stackrel{3}{=} \langle \lambda v + \mu w | u \rangle \stackrel{1}{=} \lambda \langle v | u \rangle + \mu \langle w | u \rangle \stackrel{3}{=} \lambda \langle u | v \rangle + \mu \langle u | w \rangle$.

Below we consider the following binary map:

$$\langle\,|\,\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} : (v, w) \mapsto \langle v | w \rangle \stackrel{\text{def}}{=} v_1 w_1 - v_2 w_2 \,. \qquad (*)$$

($7\frac{1}{2}$) **b.** Verify whether $\langle\,|\,\rangle$ defines a real inner product. To this end, indicate explicitly which of the relevant criteria are satisfied, respectively violated. Support your claims by proofs.

The first three properties hold, the last one does not. Note that $(\lambda v + \mu w)_i = \lambda v_i + \mu w_i$ for each component $i = 1, 2$. Linearity w.r.t. left hand side (first property): $\langle u | \lambda v + \mu w \rangle \stackrel{\text{def}}{=} u_1 (\lambda v + \mu w)_1 - u_2 (\lambda v + \mu w)_2 = \lambda (u_1 v_1 - u_2 v_2) + \mu (u_1 w_1 - u_2 w_2) \stackrel{\text{def}}{=} \lambda \langle u | w \rangle + \mu \langle v | w \rangle$. Linearity w.r.t. right hand side (second property) may be proven in a similar fashion, but according to **a** does not require a proof if first and third properties hold. Symmetry (third property): $\langle u|v\rangle \stackrel{\text{def}}{=} u_1v_1 - u_2v_2 = v_1u_1 - v_2u_2 \stackrel{\text{def}}{=} \langle v|u\rangle$. Positivity (fourth property) does not hold, for $\langle u|u\rangle \stackrel{\text{def}}{=} u_1^2 - u_2^2 < 0$ if $|u_1| > |u_2|$.

The functions $\cosh, \sinh : \mathbb{R} \to \mathbb{R}$ are defined as follows:

$$\cosh t \stackrel{\text{def}}{=} \frac{e^t + e^{-t}}{2} \quad \text{respectively} \quad \sinh t \stackrel{\text{def}}{=} \frac{e^t - e^{-t}}{2}.$$

The notation $\cosh^2 t$ and $\sinh^2 t$ is equivalent to $(\cosh t)^2$, resp. $(\sinh t)^2$.

($2\frac{1}{2}$) **c.** Show that $\cosh^2 t - \sinh^2 t = 1$ for all $t \in \mathbb{R}$.

Using the definition and straightforward algebraic simplifications we obtain $\cosh^2 t - \sinh^2 t \stackrel{\text{def}}{=} \left(\frac{e^t+e^{-t}}{2}\right)^2 - \left(\frac{e^t-e^{-t}}{2}\right)^2 = \frac{1}{4}\left(e^{2t} + 2 + e^{-2t} - e^{2t} + 2 - e^{-2t}\right) = 1$.

Definition. An *abelian group* is a collection G together with an internal operation

$$\circ : G \times G \longrightarrow G : (x, y) \mapsto x \circ y,$$

such that

- the operation is associative, i.e. $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in G$,
- there exists an identity element $e \in G$ such that $x \circ e = e \circ x = x$ for all $x \in G$,
- for each $x \in G$ there exists an inverse element $x^{-1} \in G$ such that $x^{-1} \circ x = x \circ x^{-1} = e$,
- for all $x, y \in G$ we have $x \circ y = y \circ x$.

Consider the 1-parameter linear mapping $A_t : \mathbb{R}^2 \to \mathbb{R}^2 : v \mapsto A_t(v) = \mathbf{A}(t)v$, with matrix representation

$$\mathbf{A}(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

($7\frac{1}{2}$) **d.** Show that the set $G = \{A_t \mid t \in \mathbb{R}\}$ constitutes an abelian group under operator composition. (*Hint:* First show that $A_s \circ A_t = A_{s+t}$.)

The hint provides the key to prove this conjecture. Let us assume that $A_s \circ A_t = A_{s+t}$ does indeed hold for all $s, t \in \mathbb{R}$. Associativity (first property) then follows from the observation that $(A_s \circ A_t) \circ A_u = A_{s+t} \circ A_u = A_{(s+t)+u} \stackrel{*}{=} A_{s+(t+u)} = A_s \circ A_{t+u} = A_s \circ (A_t \circ A_u)$, where in $*$ we have used associativity of ordinary addition on $\mathbb{R}$. The identity element is clearly $e \stackrel{\text{def}}{=} A_0$, for $A_0(v) = \mathbf{A}(0)v = \mathbf{I}v = v$ for all $v \in \mathbb{R}^2$. Given $s \in \mathbb{R}$, we have $A_s \circ A_{-s} = A_{s-s} = A_0 = A_{-s+s} = A_{-s} \circ A_s$, so A_{-s} is the inverse element of A_s. Commutativity is also inherited from that of ordinary addition (identity $*$), since $A_s \circ A_t = A_{s+t} \stackrel{*}{=} A_{t+s} = A_t \circ A_s$. Conclusion: G constitutes an abelian group.

It remains to prove the conjecture given in the hint: $(A_s \circ A_t)(v) \stackrel{\text{def}}{=} A_s(A_t(v)) \stackrel{\text{def}}{=} \mathbf{A}(s)(\mathbf{A}(t)v) \stackrel{*}{=} (\mathbf{A}(s)\mathbf{A}(t))v \stackrel{\star}{=} \mathbf{A}(s+t))v$. Here the identity $*$ follows from associativity of matrix multiplication, and $\star$ from algebraic simplification of the matrix product $\mathbf{A}(s)\mathbf{A}(t)$,

$$\mathbf{A}(s)\mathbf{A}(t) = \begin{pmatrix} \cosh s \cosh t + \sinh s \sinh t & \cosh s \sinh t + \sinh s \cosh t \\ \cosh s \sinh t + \sinh s \cosh t & \cosh s \cosh t + \sinh s \sinh t \end{pmatrix},$$

using the observations that $\cosh s \cosh t + \sinh s \sinh t = \cosh(s+t)$ and $\cosh s \sinh t + \sinh s \cosh t = \sinh(s+t)$.

(5) **e.** Show that $\langle A_t(v)|A_t(w)\rangle = \langle v|w\rangle$ for all $v, w \in \mathbb{R}^2$.

We have

$$\begin{aligned}
\langle A_t(v)|A_t(w)\rangle &= \langle \mathbf{A}(t)v|\mathbf{A}(t)w\rangle = \langle \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} | \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rangle \\
&= (\cosh t v_1 + \sinh t v_2)(\cosh t w_1 + \sinh t w_2) - (\sinh t v_1 + \cosh t v_2)(\sinh t w_1 + \cosh t w_2) \\
&= (\cosh^2 t - \sinh^2 t) v_1 w_1 + (\sinh^2 t - \cosh^2 t) v_2 w_2 \stackrel{c}{=} v_1 w_1 - v_2 w_2 = \langle v|w\rangle
\end{aligned}$$

for all $v, w \in \mathbb{R}^2$ and $t \in \mathbb{R}$.

Definition. A *norm* is a nondegenerate positive definite mapping $\|\ \| : V \longrightarrow \mathbb{R}$ such that for all $v, w \in V$, $\lambda \in \mathbb{R}$,

- $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$,
- $\|\lambda\, v\| = |\lambda|\, \|v\|$,
- $\|v + w\| \leq \|v\| + \|w\|$.

Below we introduce the unary operator $\|\ \| : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\|v\|^2 \stackrel{\text{def}}{=} \langle v|v\rangle \quad \text{for all } v \in \mathbb{R}^2.$$

Here the bracket operator $\langle\ |\ \rangle$ is the one defined in the equation marked by $(*)$ above.

($7\frac{1}{2}$) **f.** Verify whether $\|\ \|$ defines a norm. To this end, indicate explicitly which of the three criteria are satisfied, respectively violated. Support your claims by proofs.

First and last properties are violated, since $\|v\|^2 \stackrel{\text{def}}{=} \langle v|v\rangle = v_1^2 - v_2^2 < 0$ if $|v_1| < |v_2|$, in which case $\|v\| \notin \mathbb{R}$, rendering the inequalities meaningless. The second property seems to hold for all $v \in \mathbb{R}^2$, since if $\lambda \in \mathbb{R}$, $\|\lambda v\|^2 = \langle \lambda v|\lambda v\rangle \stackrel{*}{=} \lambda^2 \langle v|v\rangle = \lambda^2 \|v\|^2$, whence $\|\lambda v\| = |\lambda| \|v\|$. However, this is only the case if we admit cases in which $\|v\|$ is imaginary, which is precluded by the prescribed prototype $\|\ \| : \mathbb{R}^2 \longrightarrow \underline{\mathbb{R}}$. Identity $*$ makes use of bilinearity of the bracket operator, recall **b**.

Consider the subsets

$$\mathbb{R}^2_- \stackrel{\text{def}}{=} \left\{ v \in \mathbb{R}^2 \mid \|v\|^2 < 0 \right\}, \quad \mathbb{R}^2_0 \stackrel{\text{def}}{=} \left\{ v \in \mathbb{R}^2 \mid \|v\|^2 = 0 \right\}, \quad \mathbb{R}^2_+ \stackrel{\text{def}}{=} \left\{ v \in \mathbb{R}^2 \mid \|v\|^2 > 0 \right\}.$$

($2\frac{1}{2}$) **g.** Show that the subsets $\mathbb{R}^2_-$, $\mathbb{R}^2_0$ and $\mathbb{R}^2_+$ are invariant under A_t, i.e. if

$$A_t(\mathbb{R}^2_{\pm,0}) \stackrel{\text{def}}{=} \left\{ A_t(v) \mid v \in \mathbb{R}^2_{\pm,0} \right\},$$

show that $A_t(\mathbb{R}^2_{\pm,0}) = \mathbb{R}^2_{\pm,0}$ for all $t \in \mathbb{R}$.

We have already proven that $\langle A_t(v)|A_t(w)\rangle = \langle v|w\rangle$ in **e**. By taking $v = w$ this implies that $\|A_t(v)\|^2 = \|v\|^2$ for all $v \in \mathbb{R}^2$ and $t \in \mathbb{R}$, whence the result follows.

♣

(**25**) **2.** Algebra

Definition. An algebra $\mathcal{A}$ over the field $\mathbb{R}$ is a linear space enriched with a multiplication operator. Denoting the infix multiplication operator by $\circ$, we have, for all $a, b, c \in \mathcal{A}$:

$$\begin{aligned} (a \circ b) \circ c &\stackrel{1}{=} a \circ (b \circ c)\,, \\ a \circ (b + c) &\stackrel{2}{=} a \circ b + a \circ c\,, \\ (a + b) \circ c &\stackrel{3}{=} a \circ c + b \circ c\,. \end{aligned}$$

Moreover, scalar multiplication must be such that for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{R}$,

$$\lambda(a \circ b) \stackrel{4}{=} (\lambda a) \circ b \stackrel{4}{=} a \circ (\lambda b)\,.$$

If, in addition,

$$a \circ b \stackrel{5}{=} b \circ a\,,$$

for all $a, b \in \mathcal{A}$, then $\mathcal{A}$ is called a commutative algebra. If, in addition to properties 1–4, there exists an identity element $e \in \mathcal{A}$ such that

$$e \circ a \stackrel{6}{=} a \circ e \stackrel{6}{=} a\,,$$

for all $a \in \mathcal{A}$, then $\mathcal{A}$ is called an algebra with identity. If, in addition to properties 1–4 and 6, every nonzero element $a \in \mathcal{A}$ has an inverse $a^{-1} \in \mathcal{A}$ such that

$$a \circ a^{-1} \stackrel{7}{=} a^{-1} \circ a \stackrel{7}{=} e\,,$$

then $\mathcal{A}$ is called a regular algebra. A singular algebra is one in which we cannot invert all nonzero elements.

We now consider the 2-dimensional linear space

$$\mathbb{D} = \text{span}\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

equipped with the usual operators for matrix addition and scalar multiplication, and extend it with the usual matrix multiplication operator.

($2\frac{1}{2}$) **a.** Show that $\mathbb{D}$ is closed with respect to matrix multiplication.

It suffices to compute the following products to show closure:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

all of which are within $\mathbb{D}$. Alternatively we may multiply two arbitrary elements from $\mathbb{D}$ to verify closure:

$$\begin{pmatrix} b & a \\ 0 & b \end{pmatrix}\begin{pmatrix} d & c \\ 0 & d \end{pmatrix} = \begin{pmatrix} bd & bc + ad \\ 0 & bd \end{pmatrix} \in \mathbb{D}\,.$$

($12\frac{1}{2}$) **b.** Show that $\mathbb{D}$ is a commutative algebra over the field $\mathbb{R}$ (i.e. satisfies identities labeled 1–5).

Closure has been proven in **a**.

- Property 1 (associativity): matrix multiplication in general is associative, which thus carries over to $\mathbb{D}$. A formal proof: let A, B, C be square matrices with components a_{ij}, b_{ij}, respectively c_{ij}, then we have for all $i, j = 1, 2, 3$,

$$((AB)C)_{ij} = \sum_{k=1}^{n} (AB)_{ik} C_{kj} = \sum_{k=1}^{n} \sum_{\ell=1}^{n} (A_{i\ell} B_{\ell k}) C_{kj} = \sum_{\ell=1}^{n} A_{i\ell} \sum_{k=1}^{n} (B_{\ell k} C_{kj}) = \sum_{\ell=1}^{n} A_{i\ell} (BC)_{\ell j} = (A(BC))_{ij} .$$

Conclusion: $(AB)C = A(BC)$ for all square matrices A, B, C and therefore for all $A, B, C \in \mathbb{D}$.

- Properties 2–3 (distributivity): matrix multiplication in general is distributive relative to matrix addition, which thus carries over to $\mathbb{D}$. A formal proof: let A, B, C be square matrices with components a_{ij}, b_{ij}, respectively c_{ij}, then we have for all $i, j = 1, 2, 3$,

$$(A(B+C))_{ij} = \sum_{k=1}^{n} A_{ik}(B+C)_{kj} = \sum_{k=1}^{n} A_{ik}(B_{kj}+C_{kj}) = \sum_{k=1}^{n} A_{ik} B_{kj} + \sum_{k=1}^{n} A_{ik} C_{kj} = (AB)_{ij} + (AC)_{ij} = (AB + AC)_{ij} .$$

Conclusion: $A(B+C) = AB + AC$ for all square matrices A, B, C and therefore for all $A, B, C \in \mathbb{D}$. The proof of property 3 is analogous.

- Property 4 (distributivity): matrix multiplication in general is distributive relative to scalar multiplication, which thus carries over to $\mathbb{D}$. A formal proof: let A, B be square matrices with components a_{ij}, b_{ij}, respectively, and $\lambda \in \mathbb{R}$, then

$$(\lambda(AB))_{ij} \stackrel{*}{=} \lambda(AB)_{ij} \stackrel{\star}{=} \lambda \sum_{k=1}^{n} A_{ik} B_{kj} = \sum_{k=1}^{n} \lambda A_{ik} B_{kj} \stackrel{*}{=} \sum_{k=1}^{n} (\lambda A)_{ik} B_{kj} \stackrel{\star}{=} ((\lambda A)B)_{ij} ,$$

for all $i, j = 1, \ldots, n$, whence $\lambda(AB) = (\lambda A)B$. Here $*$ and $\star$ pertain to the definition of scalar multiplication of a matrix with a scalar, and that of the matrix product, respectively. The proof of $\lambda(AB) = A(\lambda B)$ is similar.

- For general $n \times n$ matrices we do *not* have $AB = BA$, but in this particular case of subset $\mathbb{D}$ we do. Proof: Consider all possible mutual (ordered) products of the two basis matrices. The identity matrix commutes with any matrix, and any matrix obviously commutes with itself, whence if $E_a \in \mathbb{D}$ denotes one of the given basis matrices ($a = 1, 2$), then $E_a E_b = E_b E_a$ for all $a, b = 1, 2$. This commutativity of basis carries over to all linear combinations, i.e. to all of $\mathbb{D}$. Alternatively, recalling a result from **a.**, the product

$$\begin{pmatrix} b & a \\ 0 & b \end{pmatrix} \begin{pmatrix} d & c \\ 0 & d \end{pmatrix} = \begin{pmatrix} bd & bc + ad \\ 0 & bd \end{pmatrix} = \begin{pmatrix} d & c \\ 0 & d \end{pmatrix} \begin{pmatrix} b & a \\ 0 & b \end{pmatrix} ,$$

in which the last equality follows from the observation that the product is symmetric w.r.t. interchange $b \leftrightarrow d$ and $a \leftrightarrow c$.

($2\frac{1}{2}$) **c.** Show that $\mathbb{D}$ has an identity element (identity 6).

It is given that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{D}$.

($2\frac{1}{2}$) **d.** Show that $\mathbb{D}$ is a singular algebra, and identify those elements which cannot be inverted.

Since $\det \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$ there apparently exists at least one nontrivial non-invertible element in $\mathbb{D}$. In general we have

$$\begin{pmatrix} b & a \\ 0 & b \end{pmatrix}^{-1} = \frac{1}{b^2} \begin{pmatrix} b & -a \\ 0 & b \end{pmatrix} \in \mathbb{D}$$

exists iff $b \neq 0$.

(5) **e.** Show that for $a, b, c, d \in \mathbb{R}$, $c \neq 0$, division must be defined on $\mathbb{D}$ as follows:

$$\frac{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}}{\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}} = \begin{pmatrix} \dfrac{a}{c} & \dfrac{bc - ad}{c^2} \\ 0 & \dfrac{a}{c} \end{pmatrix} .$$

(*Hint:* How should one define "division" in terms of multiplication?)

First of all, since multiplication is commutative, we have $AB^{-1} = B^{-1}A$ for all $A, B \in \mathbb{D}$, so we may define

$$\frac{A}{B} \stackrel{\text{def}}{=} AB^{-1} \quad \text{or} \quad \frac{A}{B} \stackrel{\text{def}}{=} B^{-1}A$$

as we please, without risk of confusion. Let

$$A \stackrel{\text{def}}{=} \begin{pmatrix} a & b \\ 0 & b \end{pmatrix} \quad \text{and} \quad B \stackrel{\text{def}}{=} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix}.$$

For B to be invertible we must require $\det B = c^2 \neq 0$, so $c \neq 0$. In that case we have

$$\frac{A}{B} \stackrel{\text{def}}{=} B^{-1}A = \frac{1}{c^2}\begin{pmatrix} c & bc-ad \\ 0 & c \end{pmatrix}\begin{pmatrix} a & b \\ 0 & b \end{pmatrix} = \begin{pmatrix} \dfrac{a}{c} & \dfrac{cb-ad}{c^2} \\ 0 & \dfrac{a}{c} \end{pmatrix}.$$

♣

(**20**) **3.** Distribution Theory

We consider the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$ given by

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$$

and its associated regular tempered distribution $T_f : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto T_f(\phi) = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx$.

(10) **a.** Show that f satisfies the o.d.e. (ordinary differential equation) $u' + u = 0$ *almost everywhere*, and explain what the annotation "almost everywhere" means in this case.

For $x < 0$ it is clear that f is differentiable (with $f(x) = f'(x) = 0$) and trivially satisfies the o.d.e. For $x > 0$ f is likewise differentiable, and we have $f'(x) = -e^{-x} = -f(x)$, which shows that also on this subdomain f satisfies the o.d.e. $u' + u = 0$. However, at $x = 0$ f is not differentiable, so this point needs to be excluded. This explains what is meant by the statement that f satisfies the o.d.e. "almost everywhere".

(10) **b.** Show that, in distributional sense, T_f satisfies the o.d.e. $u' + u = \delta$, in which the right hand side denotes the Dirac point distribution.

We have, respectively,

$$T_f(\phi) = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx = \int_0^{\infty} e^{-x}\,\phi(x)\,dx\,,$$

and

$$T_f'(\phi) \stackrel{*}{=} -T_f(\phi') = -\int_{-\infty}^{\infty} f(x)\,\phi'(x)\,dx = -\int_0^{\infty} e^{-x}\,\phi'(x)\,dx \stackrel{\star}{=} -e^{-x}\,\phi(x)\,\Big|_0^{\infty} - \int_0^{\infty} e^{-x}\,\phi(x)\,dx = \phi(0) - T_f(\phi)\,.$$

The equality marked by $*$ holds by definition of distributional differentiation, the one marked by $\star$ follows by partial integration. Using the definition of the Dirac point distribution, $\delta(\phi) = \phi(0)$, we may rewrite the result as

$$T_f'(\phi) = \delta(\phi) - T_f(\phi)\,,$$

which shows that T_f satisfies the inhomogeneous o.d.e. $u' + u = \delta$ in distributional sense. Notice that no restrictions on the domain of definition need to be imposed, and that the result is consistent with the "classical" result under a, since $\delta(x) = 0$ for $x \neq 0$.

♣

(**20**) **4.** FOURIER ANALYSIS

For each $n \in \mathbb{N}$ we define the function $f_n : \mathbb{R} \to \mathbb{R}$ as follows:

$$f_n(x) \stackrel{\text{def}}{=} \frac{1}{x^n}\,.$$

We employ the following Fourier convention:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \quad \text{with, as a result,} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{i\omega x}\, d\omega\,.$$

Without proof we state the Fourier transform of the function f_1, viz. $\widehat{f_1}(\omega) = -i\,\pi\,\mathrm{sgn}\,(\omega)$. Here, $\mathrm{sgn}\,(\omega) = -1$ for $\omega < 0$, $\mathrm{sgn}\,(0) = 0$, and $\mathrm{sgn}\,(\omega) = +1$ for $\omega > 0$.

The convolution product of two functions f and g is defined as

$$(f * g)(x) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy\,,$$

provided the integral on the right hand side exists. If this is not the case, but the functions f and g do permit Fourier transformation, we employ the following *implicit definition* for the convolution product ($\mathcal{F}(u)$ is here synonymous for $\widehat{u}$):

$$\mathcal{F}(f * g) = \mathcal{F}(f)\,\mathcal{F}(g)\,.$$

(5) **a.** Show that the function $\widehat{f_n}$ is purely imaginary for odd $n \in \mathbb{N}$, and real for even $n \in \mathbb{N}$. (*Hint:* Use the (anti-)symmetry property $f_n(x) = (-1)^n f_n(-x)$ for all $x \in \mathbb{R}$.)

If $z = a + bi \in \mathbb{C}$ we write the complex conjugate as $z^* = a - bi$, $a, b \in \mathbb{R}$. For $\omega \in \mathbb{R}$ arbitrary we have

$$\begin{aligned}\widehat{f}_n(\omega) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f_n(x)\, e^{-i\omega x}\, dx \stackrel{\text{hint}}{=} (-1)^n \int_{-\infty}^{\infty} f_n(-x)\, e^{-i\omega x}\, dx \stackrel{*}{=} (-1)^n \int_{-\infty}^{\infty} f_n(y)\, e^{i\omega y}\, dy \stackrel{\star}{=} (-1)^n \left(\int_{-\infty}^{\infty} f_n(y)\, e^{-i\omega y}\, dy\right)^* \\ &= (-1)^n\, \widehat{f}_n^*(\omega)\,.\end{aligned}$$

In $*$ substitution of variables, $x = -y$, has been used. In $\star$ the fact that $f_n(y) \in \mathbb{R}$ for all $y \in \mathbb{R}$ has been used, as well as the fact that $\int_\Omega f^*(x)\, dx = \left(\int_\Omega f(x)\, dx\right)^*$ for any integration domein $\Omega \subset \mathbb{R}$. Conclusion: For even n we have $\widehat{f}_n(\omega) = \widehat{f}_n^*(\omega)$, i.e. $\widehat{f}_n(\omega) \in \mathbb{R}$. For odd n we have $\widehat{f}_n(\omega) = -\widehat{f}_n^*(\omega)$, i.e. $\widehat{f}_n(\omega) \in i\mathbb{R}$, i.e. purely imaginary.

b. Prove the following recursions for the functions f_n, respectively $\widehat{f_n}$:

($2\frac{1}{2}$) **b1.** $f_{n+1}(x) = -\dfrac{1}{n}\, f_n'(x)$, $n \in \mathbb{N}$.

Straightforward differentiation yields $f_n'(x) \stackrel{\text{def}}{=} \left[x^{-n}\right]' = -n\, x^{-n-1} \stackrel{\text{def}}{=} -n\, f_{n+1}(x)$, from which the conjecture follows.

(2$\frac{1}{2}$) **b2.** $\widehat{f}_{n+1}(\omega) = -\frac{1}{n}\, i\omega\, \widehat{f}_n(\omega)$, $n \in \mathbb{N}$.

We have $\mathcal{F}(f_{n+1})(\omega) \stackrel{*}{=} -\frac{1}{n}\,\mathcal{F}(f_n')(\omega) \stackrel{\star}{=} -\frac{1}{n}\, i\omega\, \mathcal{F}(f_n)(\omega)$. In $*$ problem b1 has been used together with linearity of Fourier transformation. In $\star$ the following property has been used: $\mathcal{F}(f')(\omega) = i\omega\mathcal{F}(f)(\omega)$.

(5) **c.** Determine $\widehat{f}_n(\omega)$ for each $n \in \mathbb{N}$, given that $\widehat{f}_1(\omega) = -i\,\pi\,\mathrm{sgn}\,(\omega)$.

Claim (induction hypothesis): $\widehat{f}_n(\omega) = \frac{\pi}{i}\,\frac{(-i\omega)^{n-1}}{(n-1)!}\,\mathrm{sgn}(\omega)$. Proof by induction: For $n = 1$ this result agrees with the one given. Furthermore, $\widehat{f}_{n+1}(\omega) \stackrel{\mathrm{b2}}{=} -\frac{1}{n}\, i\omega\, \widehat{f}_n(\omega) \stackrel{*}{=} -\frac{1}{n}\, i\omega\, \frac{\pi}{i}\,\frac{(-i\omega)^{n-1}}{(n-1)!}\,\mathrm{sgn}(\omega) = \frac{\pi}{i}\,\frac{(-i\omega)^n}{n!}\,\mathrm{sgn}(\omega)$. In $*$ the induction hypothesis has been invoked for $\widehat{f}_n(\omega)$.

(5) **d.** Prove: $\widehat{f}_n * \widehat{f}_m = 2\pi\, \widehat{f}_{n+m}$ for all $n, m \in \mathbb{N}$.

It is evident that $f_n\, f_m = f_{n+m}$ ($\star$), as for all $x \in \mathbb{R}$ we have $f_n(x)\, f_m(x) = x^{-n}\, x^{-m} = x^{-(n+m)} = f_{n+m}(x)$. Consequently: $\widehat{f}_n * \widehat{f}_m = \mathcal{F}(f_n) * \mathcal{F}(f_m) \stackrel{*}{=} 2\pi\mathcal{F}(f_n\, f_m) \stackrel{\star}{=} 2\pi\,\mathcal{F}(f_{n+m}) = 2\pi\,\widehat{f}_{n+m}$. In $*$ we have used the fact that for two functions u_1 en u_2 we have, provided left and right hand sides exist, $\mathcal{F}(u_1\, u_2) = \frac{1}{2\pi}\,\mathcal{F}(u_1) * \mathcal{F}(u_2)$. In $\star$ we have used the first observation above.

THE END