EXAMINATION MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020. Date: Monday June 15, 2009. Time: 14h00–17h00. Place: HG 10.01 C.

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student identification number on each paper.
- The exam consists of 4 problems. The maximum credit for each item is indicated in the margin.
- Motivate your answers. The use of course notes and calculator is allowed. The use of the problem companion, "opgaven- en tentamenbundel", is *not* allowed.
- You may provide your answers in Dutch or (preferably) in English.

GOOD LUCK!

(**35**) **1.** LINEAR ALGEBRA & GROUP THEORY

Definition. Let V be a vector space over $\mathbb{R}$. A *real inner product* is a nondegenerate positive definite symmetric bilinear mapping $\langle\ |\ \rangle : V \times V \longrightarrow \mathbb{R}$ satisfying the following properties. For all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$ we have

- $\langle \lambda u + \mu v | w \rangle = \lambda \langle u|w \rangle + \mu \langle v|w \rangle$,
- $\langle u | \lambda v + \mu w \rangle = \lambda \langle u|v \rangle + \mu \langle u|w \rangle$,
- $\langle u|v \rangle = \langle v|u \rangle$,
- $\langle u|u \rangle > 0$ for all $u \neq 0$.

($2\frac{1}{2}$) **a.** Show that either the first or the second criterion is redundant.

Below we consider the following binary map:

$$\langle\ |\ \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} : (v, w) \mapsto \langle v|w \rangle \stackrel{\text{def}}{=} v_1 w_1 - v_2 w_2 \,. \qquad (*)$$

($7\frac{1}{2}$) **b.** Verify whether $\langle\ |\ \rangle$ defines a real inner product. To this end, indicate explicitly which of the relevant criteria are satisfied, respectively violated. Support your claims by proofs.

The functions $\cosh, \sinh : \mathbb{R} \to \mathbb{R}$ are defined as follows:

$$\cosh t \stackrel{\text{def}}{=} \frac{e^t + e^{-t}}{2} \quad \text{respectively} \quad \sinh t \stackrel{\text{def}}{=} \frac{e^t - e^{-t}}{2} \,.$$

The notation $\cosh^2 t$ and $\sinh^2 t$ is equivalent to $(\cosh t)^2$, resp. $(\sinh t)^2$.

($2\frac{1}{2}$) **c.** Show that $\cosh^2 t - \sinh^2 t = 1$ for all $t \in \mathbb{R}$.

Definition. An *abelian group* is a collection G together with an internal operation

$$\circ : G \times G \longrightarrow G : (x, y) \mapsto x \circ y \,,$$

such that

- the operation is associative, i.e. $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in G$,
- there exists an identity element $e \in G$ such that $x \circ e = e \circ x = x$ for all $x \in G$,
- for each $x \in G$ there exists an inverse element $x^{-1} \in G$ such that $x^{-1} \circ x = x \circ x^{-1} = e$,
- for all $x, y \in G$ we have $x \circ y = y \circ x$.

Consider the 1-parameter linear mapping $A_t : \mathbb{R}^2 \to \mathbb{R}^2 : v \mapsto A_t(v) = \mathbf{A}(t)v$, with matrix representation

$$\mathbf{A}(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} .$$

($7\frac{1}{2}$) **d.** Show that the set $G = \{A_t \mid t \in \mathbb{R}\}$ constitutes an abelian group under operator composition. (*Hint:* First show that $A_s \circ A_t = A_{s+t}$.)

(5) **e.** Show that $\langle A_t(v) | A_t(w) \rangle = \langle v | w \rangle$ for all $v, w \in \mathbb{R}^2$.

Definition. A *norm* is a nondegenerate positive definite mapping $\| \; \| : V \longrightarrow \mathbb{R}$ such that for all $v, w \in V$, $\lambda \in \mathbb{R}$,

- $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$,
- $\|\lambda \, v\| = |\lambda| \, \|v\|$,
- $\|v + w\| \leq \|v\| + \|w\|$.

Below we introduce the unary operator $\| \; \| : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\|v\|^2 \stackrel{\text{def}}{=} \langle v | v \rangle \quad \text{for all } v \in \mathbb{R}^2.$$

Here the bracket operator $\langle \; | \; \rangle$ is the one defined in the equation marked by $(*)$ above.

($7\frac{1}{2}$) **f.** Verify whether $\| \; \|$ defines a norm. To this end, indicate explicitly which of the three criteria are satisfied, respectively violated. Support your claims by proofs.

Consider the subsets

$$\mathbb{R}^2_- \stackrel{\text{def}}{=} \left\{ v \in \mathbb{R}^2 \mid \|v\|^2 < 0 \right\} \,, \quad \mathbb{R}^2_0 \stackrel{\text{def}}{=} \left\{ v \in \mathbb{R}^2 \mid \|v\|^2 = 0 \right\} \,, \quad \mathbb{R}^2_+ \stackrel{\text{def}}{=} \left\{ v \in \mathbb{R}^2 \mid \|v\|^2 > 0 \right\} \,.$$

($2\frac{1}{2}$) **g.** Show that the subsets $\mathbb{R}^2_-$, $\mathbb{R}^2_0$ and $\mathbb{R}^2_+$ are invariant under A_t, i.e. if

$$A_t(\mathbb{R}^2_{\pm,0}) \stackrel{\text{def}}{=} \left\{A_t(v) \mid v \in \mathbb{R}^2_{\pm,0}\right\} ,$$

show that $A_t(\mathbb{R}^2_{\pm,0}) = \mathbb{R}^2_{\pm,0}$ for all $t \in \mathbb{R}$.

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(**25**) **2.** Algebra

Definition. An algebra $\mathcal{A}$ over the field $\mathbb{R}$ is a linear space enriched with a multiplication operator. Denoting the infix multiplication operator by $\circ$, we have, for all $a, b, c \in \mathcal{A}$:

$$\begin{aligned} (a \circ b) \circ c &\stackrel{1}{=} a \circ (b \circ c)\,, \\ a \circ (b + c) &\stackrel{2}{=} a \circ b + a \circ c\,, \\ (a + b) \circ c &\stackrel{3}{=} a \circ c + b \circ c\,. \end{aligned}$$

Moreover, scalar multiplication must be such that for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{R}$,

$$\lambda(a \circ b) \stackrel{4}{=} (\lambda a) \circ b \stackrel{4}{=} a \circ (\lambda b)\,.$$

If, in addition,

$$a \circ b \stackrel{5}{=} b \circ a\,,$$

for all $a, b \in \mathcal{A}$, then $\mathcal{A}$ is called a commutative algebra. If, in addition to properties 1–4, there exists an identity element $e \in \mathcal{A}$ such that

$$e \circ a \stackrel{6}{=} a \circ e \stackrel{6}{=} a\,,$$

for all $a \in \mathcal{A}$, then $\mathcal{A}$ is called an algebra with identity. If, in addition to properties 1–4 and 6, every nonzero element $a \in \mathcal{A}$ has an inverse $a^{-1} \in \mathcal{A}$ such that

$$a \circ a^{-1} \stackrel{7}{=} a^{-1} \circ a \stackrel{7}{=} e\,,$$

then $\mathcal{A}$ is called a regular algebra. A singular algebra is one in which we cannot invert all nonzero elements.

We now consider the 2-dimensional linear space

$$\mathbb{D} = \text{span}\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

equipped with the usual operators for matrix addition and scalar multiplication, and extend it with the usual matrix multiplication operator.

($2\frac{1}{2}$) **a.** Show that $\mathbb{D}$ is closed with respect to matrix multiplication.

($12\frac{1}{2}$) **b.** Show that $\mathbb{D}$ is a commutative algebra over the field $\mathbb{R}$ (i.e. satisfies identities labeled 1–5).

($2\frac{1}{2}$) **c.** Show that $\mathbb{D}$ has an identity element (identity 6).

($2\frac{1}{2}$) **d.** Show that $\mathbb{D}$ is a singular algebra, and identify those elements which cannot be inverted.

(5) **e.** Show that for $a, b, c, d \in \mathbb{R}$, $c \neq 0$, division must be defined on $\mathbb{D}$ as follows:

$$\frac{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}}{\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}} = \begin{pmatrix} \dfrac{a}{c} & \dfrac{bc - ad}{c^2} \\ 0 & \dfrac{a}{c} \end{pmatrix}.$$

(*Hint:* How should one define "division" in terms of multiplication?)

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(**20**) **3.** Distribution Theory

We consider the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$ given by

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$$

and its associated regular tempered distribution $T_f : \mathscr{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto T_f(\phi) = \int_{-\infty}^{\infty} f(x)\, \phi(x)\, dx$.

(10) **a.** Show that f satisfies the o.d.e. (ordinary differential equation) $u' + u = 0$ *almost everywhere*, and explain what the annotation "almost everywhere" means in this case.

(10) **b.** Show that, in distributional sense, T_f satisfies the o.d.e. $u' + u = \delta$, in which the right hand side denotes the Dirac point distribution.

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(**20**) **4.** Fourier Analysis

For each $n \in \mathbb{N}$ we define the function $f_n : \mathbb{R} \to \mathbb{R}$ as follows:

$$f_n(x) \stackrel{\text{def}}{=} \frac{1}{x^n}.$$

We employ the following Fourier convention:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \quad \text{with, as a result,} \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{i\omega x}\, d\omega\,.$$

Without proof we state the Fourier transform of the function f_1, viz. $\widehat{f_1}(\omega) = -i\,\pi\, \mathrm{sgn}\,(\omega)$. Here, $\mathrm{sgn}\,(\omega) = -1$ for $\omega < 0$, $\mathrm{sgn}\,(0) = 0$, and $\mathrm{sgn}\,(\omega) = +1$ for $\omega > 0$.

The convolution product of two functions f and g is defined as

$$(f * g)(x) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy\,,$$

provided the integral on the right hand side exists. If this is not the case, but the functions f and g do permit Fourier transformation, we employ the following *implicit definition* for the convolution product ($\mathcal{F}(u)$ is here synonymous for $\widehat{u}$):

$$\mathcal{F}(f * g) = \mathcal{F}(f)\, \mathcal{F}(g)\,.$$

(5) **a.** Show that the function $\widehat{f_n}$ is purely imaginary for odd $n \in \mathbb{N}$, and real for even $n \in \mathbb{N}$.
(*Hint:* Use the (anti-)symmetry property $f_n(x) = (-1)^n\, f_n(-x)$ for all $x \in \mathbb{R}$.)

b. Prove the following recursions for the functions f_n, respectively $\widehat{f_n}$:

($2\frac{1}{2}$) **b1.** $f_{n+1}(x) = -\dfrac{1}{n}\, f_n'(x)$, $n \in \mathbb{N}$.

($2\frac{1}{2}$) **b2.** $\widehat{f}_{n+1}(\omega) = -\dfrac{1}{n}\, i\omega\, \widehat{f_n}(\omega)$, $n \in \mathbb{N}$.

(5) **c.** Determine $\widehat{f_n}(\omega)$ for each $n \in \mathbb{N}$, given that $\widehat{f_1}(\omega) = -i\,\pi\, \mathrm{sgn}\,(\omega)$.

(5) **d.** Prove: $\widehat{f_n} * \widehat{f_m} = 2\pi\, \widehat{f}_{n+m}$ for all $n, m \in \mathbb{N}$.

THE END