EXAMINATION: MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020.
Date: Monday January 18^{th}, 2010.
Time: 9h00 – 12h00.
Place: paviljoen b1

1 Linear Algebra

Consider the set $\mathcal{C}^\infty(S^1)$ of $\mathbb{R}$-valued, infinitely differentiable functions on the unit circle S^1. We equip the function space $\mathcal{C}^\infty(S^1)$ with the inner product

$$\langle f|g\rangle := \int_0^{2\pi} f(\alpha)\, g(\alpha)\, d\alpha \ , \ \text{for } f, g \in \mathcal{C}^\infty(S^1).$$

The corresponding measure is given by $\|f\| := \sqrt{\langle f|f\rangle}$.

For our calculations we use the following orthogonal basis functions:

$$b_0 : \alpha \mapsto 1 \ , \quad b_{sn} : \alpha \mapsto \sin(n\,\alpha) \ , \quad b_{cn} : \alpha \mapsto \cos(n\,\alpha) \quad \text{with } n \in \mathbb{N}.$$

(5) **a)** Proof the trigonometric identity

$$2\,\cos(m\,\alpha)\,\cos(n\,\alpha) \ = \ \cos((m-n)\,\alpha) \ + \ \cos((m+n)\,\alpha)$$

for $n, m \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$.

Solution:

$$\begin{aligned}
2\,\cos(m\,\alpha)\,\cos(n\,\alpha) &= \left(e^{i\,m\,\alpha} + e^{-i\,m\,\alpha}\right)\left(e^{i\,n\,\alpha} + e^{-i\,n\,\alpha}\right) \\
&= e^{i\,(m+n)\,\alpha} + e^{i\,(m-n)\,\alpha} + e^{i\,(-m+n)\,\alpha} + e^{i\,(-m-n)\,\alpha} \\
&= e^{i\,(m+n)\,\alpha} + e^{-i\,(m+n)\,\alpha} + e^{i\,(m-n)\,\alpha} + e^{-i\,(m-n)\,\alpha} \\
&= \cos((m+n)\,\alpha)\,\cos((m-n)\,\alpha)\ .
\end{aligned}$$

(6) **b)** Verify that the basis functions b_0, b_{sn}, and b_{cn} are indeed orthogonal.
(Hint: You may use the trigonometric identities:
$2\,\sin(m\,\alpha)\,\sin(n\,\alpha) \ = \ \cos((m-n)\,\alpha) \ - \ \cos((m+n)\,\alpha)$,
$2\,\cos(m\,\alpha)\,\cos(n\,\alpha) \ = \ \cos((m-n)\,\alpha) \ + \ \cos((m+n)\,\alpha)$,

$2\sin(m\,\alpha)\,\cos(n\,\alpha) \;=\; \sin((m-n)\,\alpha) \;+\; \sin((m+n)\,\alpha).)$

Solution Assuming $m \neq n$, $m, n \geq 1$, and $m, n \in \mathbb{N}$.

$$
\begin{aligned}
\langle b_0 | b_{\mathrm{s}n} \rangle &= \int_0^{2\pi} \sin(n\,\alpha)\, d\alpha \;=\; -\frac{\cos(n\,\alpha)}{n}\bigg|_0^{2\pi} \;=\; -\frac{1-1}{n} \;=\; 0, \\
\langle b_0 | b_{\mathrm{c}n} \rangle &= \int_0^{2\pi} \cos(n\,\alpha)\, d\alpha \;=\; \frac{\sin(n\,\alpha)}{n}\bigg|_0^{2\pi} \;=\; \frac{0-0}{n} \;=\; 0, \\
\langle b_{\mathrm{s}m} | b_{\mathrm{c}n} \rangle &= \int_0^{2\pi} \sin(m\,\alpha)\,\cos(n\,\alpha)\, d\alpha \;=\; \int_0^{2\pi} \frac{1}{2}\left(\sin((m-n)\,\alpha) + \sin((m+n)\,\alpha)\right) d\alpha \\
&= -\frac{\cos((m-n)\,\alpha)}{2(m-n)}\bigg|_0^{2\pi} - \frac{\cos((m+n)\,\alpha)}{2(m+n)}\bigg|_0^{2\pi} \;=\; -\frac{1-1}{2(m-n)} - \frac{1-1}{2(m+n)} \;=\; 0, \\
\langle b_{\mathrm{s}m} | b_{\mathrm{s}n} \rangle &= \int_0^{2\pi} \sin(m\,\alpha)\,\sin(n\,\alpha)\, d\alpha \;=\; \int_0^{2\pi} \frac{1}{2}\left(\cos((m-n)\,\alpha) \;-\; \cos((m+n)\,\alpha)\right) d\alpha \\
&= \frac{\sin((m-n)\,\alpha)}{2(m-n)}\bigg|_0^{2\pi} - \frac{\sin((m+n)\,\alpha)}{2(m+n)}\bigg|_0^{2\pi} \;=\; -\frac{0-0}{2(m-n)} - \frac{0-0}{2(m+n)} \;=\; 0, \\
\langle b_{\mathrm{c}m} | b_{\mathrm{c}n} \rangle &= \int_0^{2\pi} \cos(m\,\alpha)\,\cos(n\,\alpha)\, d\alpha \;=\; \int_0^{2\pi} \frac{1}{2}\left(\cos((m-n)\,\alpha) \;+\; \cos((m+n)\,\alpha)\right) d\alpha \\
&= \frac{\sin((m-n)\,\alpha)}{2(m-n)}\bigg|_0^{2\pi} + \frac{\sin((m+n)\,\alpha)}{2(m+n)}\bigg|_0^{2\pi} \;=\; -\frac{0-0}{2(m-n)} - \frac{0-0}{2(m+n)} \;=\; 0,
\end{aligned}
$$

(4) **c)** Normalize the orthogonal basis functions b_0, $b_{\mathrm{s}n}$, and $b_{\mathrm{c}n}$.
Call the normalized basis functions e_0, $e_{\mathrm{s}n}$, and $e_{\mathrm{c}n}$.

Solution:

$$
\begin{aligned}
\|b_0\|^2 &= \int_0^{2\pi} d\alpha = 2\pi \text{ , so that } e_0 : \alpha \mapsto \frac{b_0(\alpha)}{\|b_0\|} = \frac{1}{\sqrt{2\pi}}, \\
\|b_{\mathrm{s}n}\|^2 &= \int_0^{2\pi} \sin(n\,\alpha)\,\sin(n\,\alpha)\, d\alpha = \frac{1}{2}\int_0^{2\pi} \left(\cos(0\,\alpha) - \cos(2n\,\alpha)\right) d\alpha = \frac{1}{2}\left(\alpha - \frac{\sin 2n\,\alpha}{2n}\right)\bigg|_0^{2\pi} \\
&= \frac{1}{2}\left(2\pi - 0 - \frac{1}{2n} + \frac{1}{2n}\right) \;=\; \pi \text{ , so that } e_{\mathrm{s}n} : \alpha \mapsto \frac{b_{\mathrm{s}n}(\alpha)}{\|b_{\mathrm{s}n}\|} = \frac{\sin(n\,\alpha)}{\sqrt{\pi}}, \\
\|b_{\mathrm{c}n}\|^2 &= \int_0^{2\pi} \cos(n\,\alpha)\,\cos(n\,\alpha)\, d\alpha = \frac{1}{2}\int_0^{2\pi} \left(\cos(0\,\alpha) + \cos(2n\,\alpha)\right) d\alpha = \frac{1}{2}\left(\alpha + \frac{\sin 2n\,\alpha}{2n}\right)\bigg|_0^{2\pi} \\
&= \frac{1}{2}\left(2\pi - 0 + \frac{1}{2n} - \frac{1}{2n}\right) \;=\; \pi \text{ , so that } e_{\mathrm{c}n} = \frac{b_{\mathrm{c}n}(\alpha)}{\|b_{\mathrm{c}n}\|} = \frac{\cos(n\,\alpha)}{\sqrt{\pi}}.
\end{aligned}
$$

(5) **d)** Expand the Dirac point distribution δ with the property $\int_{S^1} \delta(\alpha)\, f(\alpha)\, d\alpha = f(0)$ in the orthonormal basis e_0, e_{sn}, e_{cn}. Thus, determine the coefficient d_0, d_{sn} and d_{cn} of the Dirac point distribution so that $\delta(\alpha) = d_0\, e_0(\alpha) \;+\; \sum_{n=1}^{\infty} d_{sn}\, e_{sn}(\alpha) \;+\; \sum_{n=1}^{\infty} d_{cn}\, e_{cn}(\alpha)$.

Solution: The projection of δ onto the basis is given by

$$\delta(\alpha) \;=\; \langle \delta | e_0 \rangle \; e_0(\alpha) \;+\; \sum_{n=1}^{\infty} \langle \delta | e_{sn} \rangle \; e_{sn}(\alpha) \;+\; \sum_{n=1}^{\infty} \langle \delta | e_{cn} \rangle \; e_{cn}(\alpha)$$

The inner products $\langle \delta | e_0 \rangle$, $\langle \delta | e_{sn} \rangle$ and $\langle \delta | e_{cn} \rangle$ are easily determined.

$$\begin{aligned}
\langle \delta | e_0 \rangle &= \int_0^{2\pi} \delta(\alpha)\, \frac{1}{\sqrt{2\pi}}\, d\alpha \;=\; \frac{1}{\sqrt{2\pi}}, \\
\langle \delta | e_{sn} \rangle &= \int_0^{2\pi} \delta(\alpha)\, \frac{\sin(n\,\alpha)}{\sqrt{\pi}}\, d\alpha \;=\; \frac{\sin(n\,0)}{\sqrt{\pi}} \;=\; 0, \\
\langle \delta | e_{cn} \rangle &= \int_0^{2\pi} \delta(\alpha)\, \frac{\cos(n\,\alpha)}{\sqrt{\pi}}\, d\alpha \;=\; \frac{\cos(n\,0)}{\sqrt{\pi}} \;=\; \frac{1}{\sqrt{\pi}}.
\end{aligned}$$

Hence, the expansion is

$$\delta(\alpha) \;=\; \frac{1}{\sqrt{2\pi}}\, e_0(\alpha) \;+\; \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi}}\, e_{cn}(\alpha).$$

(6) **e)** Find the matrix representation D of differential operator ∂_α with respect to the orthonormal basis

$$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} = e_0\,, \quad \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} = e_{s1}\,, \quad \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} = e_{c1}\,, \quad \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} = e_{s2}\,, \quad \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = e_{c2}\,.$$

Neglect all basis functions with $n \geq 3$.

Solution Simply apply ∂_α to the basis functions e_0, e_{sn}, and e_{cn} and read of the resulting coefficients. For example, $\partial_\alpha\, e_{sn} \;=\; \partial_\alpha \frac{\sin(n\alpha)}{\sqrt{\pi}} \;=\; n\, \frac{\cos(n\alpha)}{\sqrt{\pi}} \;=\; n\, e_{cn}$. Likewise, $\partial_\alpha\, e_0 \;=\; 0$ and $\partial_\alpha\, e_{cn} \;=\; -n\, e_{sn}$. Now, one can easily construct matrix D below. See, how each of the basis vectors are carried over via matrix multiplication into the resulting basis vector time 0, n, or $-n$.

$$\mathsf{D} \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

A more tedious way to construct matrix D is via the inner products $\mathsf{D}_{ij} = \langle e_j | \partial_\alpha \, e_j \rangle$. Example, $\mathsf{D}_{\mathrm{s2\,c2}} = \langle e_{\mathrm{s2}} | \partial_\alpha \, e_{\mathrm{c2}} \rangle = \langle e_{\mathrm{s2}} | (-2) \, e_{\mathrm{s2}} \rangle = -2 \, \langle e_{\mathrm{s2}} | e_{\mathrm{s2}} \rangle = -2$. Recall, that we operate with an orthonormal basis $\langle e_i | e_j \rangle = \delta_{ij}$.

(5) **f)** Find the adjoint operator of the differential operator ∂_α. Provide the adjoint matrix representation $\mathsf{D}^\dagger$ and the adjoint differential operator?

Solution: According to the script the adjoint partial derivative is $-\partial_\alpha$. The adjoint matrix $\mathsf{D}^\dagger$ is in the case of real-valued matrices the transposed D^T, which is $-\mathsf{D}$ in our case, thus, confirming the claim $\partial_\alpha^\dagger = -\partial_\alpha$.

(7) **g)** The differential operator $(1 - \varphi \, \partial_\alpha) f(\alpha)$ is equivalent to a minute shift $f(\alpha - \varphi)$ for infinitesimal φ. It is therefore called the infinitesimal generator of translation. Applying the infinitesimal generator of translation infinitely often yields the regular operator of translation $\mathcal{T}_\varphi$.

$$\lim_{m \to \infty} \left(1 - \frac{\varphi}{m} \, \partial_\alpha \right)^m \; = \; e^{-\varphi \, \partial_\alpha} \; = \; \mathcal{T}_\varphi .$$

Find the matrix representation of the translation operator $\mathcal{T}_\varphi$ in our orthonormal basis for $n \leq 2$. Utilize the definition $\mathcal{T}_\varphi \; = \; e^{-\varphi \, \partial_\alpha}$ and the Taylor-expansion of the exponential function. The latter defines the exponential of a matrix X.

$$e^{\mathsf{X}} \; := \; \sum_{k=0}^{\infty} \frac{1}{k!} \, \mathsf{X}^k .$$

(Hint: knowing the k^{th} power of matrix representation D is essential. Note, that matrix D consists of sub-matrices in the diagonal, which can be dealt with one at a time.)

Solution: Because of the sub-matrices in the diagonal of D, basis functions/vectors with different n do not mix, nor does b_0 mix with any other vector. Hence, $\mathrm{span}\,(b_0)$ and $\mathrm{span}\,(b_{\mathrm{s}n}, b_{\mathrm{c}n})$ are invariant subspaces under D and we can treat their sub-matrices one at a time.
First, we consider the sub-matrix in $\mathrm{span}\,(b_0)$, which is $\mathsf{D}_0 = (0)$. Hence, $(\mathsf{D}_0)^0 = (1)$ and $(\mathsf{D}_0)^k = (0)$ for any $k \leq 1$. Consequently, $e^{\mathsf{D}_0} = (1) + \sum_{k=0}^{\infty} \frac{1}{k!} (0) = (1)$.
Second, we consider the sub-matrices in $\mathrm{span}\,(b_{\mathrm{s}n}, b_{\mathrm{c}n})$, which are $\mathsf{D}_n = \begin{pmatrix} 0 & -n \\ n & 0 \end{pmatrix}$. Even powers of $\mathsf{D}_n^{2k} = (-1)^k \begin{pmatrix} n^{2k} & 0 \\ 0 & n^{2k} \end{pmatrix}$ and odd powers of $\mathsf{D}_n^{2k+1} = (-1)^k \begin{pmatrix} 0 & -n^{2k+1} \\ n^{2k+1} & 0 \end{pmatrix}$. Comparison with the Taylor expansion of cos and sin yield $e^{-\varphi \, \mathsf{D}_n} \; = \; \begin{pmatrix} \cos(n\varphi) & -\sin(n\varphi) \\ \sin(n\varphi) & \cos(n\varphi) \end{pmatrix}$. In total we get

$$e^{-\varphi \, \mathsf{D}} \; = \; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) & 0 & 0 & & \\ 0 & \sin(\varphi) & \cos(\varphi) & 0 & 0 & & \\ 0 & 0 & 0 & \cos(2\varphi) & -\sin(2\varphi) & & \\ 0 & 0 & 0 & \sin(2\varphi) & \cos(2\varphi) & & \end{pmatrix} .$$

(5) **h)** Why is the following definition not an inner product for function space $\mathcal{C}^\infty(S^1)$?

$$\langle f | g \rangle := \int_0^{2\pi} f(\alpha) \, g(2\pi - \alpha) \, d\alpha \; .$$

Solution A simple counter example that contradicts the third axiom of inner products ($\langle f|f\rangle = 0$, if and only if $f : \alpha \mapsto 0$) is

$$f : \alpha \mapsto \begin{cases} 0 & 0 \le \alpha < \pi, \\ \sin^2(\alpha) & \pi \le \alpha < 2\pi. \end{cases}$$

2 Fourier Transformation

We adhere to the following definition of a Fourier transformation:

$$\hat{f}(\omega) \;=\; \mathcal{F}\,[f](\omega) \;:=\; \int\limits_{-\infty}^{\infty} f(x)\,e^{-i\,\omega\,x}\,dx \;.$$

Inverse Fourier transformation:

$$f(x) \;=\; \mathcal{F}^{-1}\Big[\hat{f}\Big](x) \;:=\; \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \hat{f}(\omega)\,e^{i\,\omega\,x}\,d\omega \;.$$

(4) **a)** Show that the Fourier transformation is a linear transformation.

Solution: $\mathcal{F}\,[\lambda\,f \,+\, \mu\,g](\omega) \;=\; \int_{-\infty}^{\infty} (\lambda\,f(x) \,+\, \mu\,g(x))\;e^{-i\,\omega\,x}\,dx \;=\; \lambda\int_{-\infty}^{\infty} f(x)\,e^{-i\,\omega\,x}\,dx \,+\, \mu\int_{-\infty}^{\infty} g(x)\,e^{-i\,\omega\,x}\,dx \;=\; \lambda\,\mathcal{F}\,[f](\omega) \,+\, \mu\,\mathcal{F}\,[g](\omega)$. q.e.d.

(6) **b)** Assume that function h is the complex conjugate of function f, thus, $h = f^*$.
Proof, that $\hat{h}(\omega) = \left(\hat{f}(-\omega)\right)^*$.

Solution:

$$\begin{aligned} \hat{h}(\omega) \;&=\; \mathcal{F}\,[f^*](\omega) \;=\; \int\limits_{-\infty}^{\infty} f^*(x)\,e^{-i\,\omega\,x}\,dx \\ &=\; \int\limits_{-\infty}^{\infty} \left(f(x)\,e^{i\,\omega\,x}\right)^*\,dx \;=\; \int\limits_{-\infty}^{\infty} \left(f(x)\,e^{-i\,(-\omega)\,x}\right)^*\,dx \;=\; \left(\hat{f}(-\omega)\right)^* \;.\ \text{q.e.d.} \end{aligned}$$

(5) **c)** Show that the Fourier transform of function $q : x \mapsto e^{-x^4}$ is $\mathbb{R}$-valued.

Solution: q is an even function $q(x) = q(-x)$. Hence, the Fourier transform is real-valued.

(6) **d)** Determine the Fourier transform of $f : x \mapsto \sin^2(x)$. (Recall that $\sin^2(x)$ stands for $(\sin(x))^2$.)

Solution:

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} \frac{i}{4}\left(e^{-i\,x} - e^{i\,x}\right)^2 e^{-i\,\omega\,x}\,dx \\
&= \int_{-\infty}^{\infty} \frac{-1}{4}\left(e^{-2i\,x} - 2 + e^{2i\,x}\right) e^{-i\,\omega\,x}\,dx \\
&= \frac{-1}{4}\left(\int_{-\infty}^{\infty} e^{-i\,(\omega+2)\,x}\,dx - 2\int_{-\infty}^{\infty} e^{-i\,\omega\,x}\,dx + \int_{-\infty}^{\infty} e^{-i\,(\omega-2)\,x}\,dx\right) \\
&= \frac{-2\pi}{4}\left(\delta(\omega+2) - 2\,\delta(\omega) + \delta(\omega-2)\right).
\end{aligned}$$

(7) **e)** Proof that the Fourier transform of $g(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ renders $\hat{g}(\omega) = e^{-\omega^2/4}$. You may use the definite integral

$$\int_{-\infty}^{\infty} e^{-(x+i\,y)^2}\,dx \;=\; \sqrt{\pi}\;.$$

Solution:

$$\begin{aligned}
\hat{g}(\omega) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2 - i\,\omega\,x}\,dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2 - i\,\omega\,x + (\omega/2)^2 - (\omega/2)^2}\,dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(x+i\,\omega/2)^2 - (\omega/2)^2}\,dx \\
&= \frac{1}{\sqrt{\pi}} e^{-(\omega/2)^2} \underbrace{\int_{-\infty}^{\infty} e^{-(x+i\,\omega/2)^2}\,dx}_{\sqrt{\pi}} \;=\; e^{-\omega^2/4}.
\end{aligned}$$

(6) **f)** Derive the Fourier transform of function $p : x \mapsto \frac{1}{\sqrt{\pi}}\,\sin^2(x)\,e^{-x^2}$.

Solution Note, that $p = f\,g$. Hence

$$\begin{aligned}
\hat{p} &= \frac{\hat{f} * \hat{g}}{2\pi} \\
&= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{-2\pi}{4} \left(\delta(\nu+2) - 2\,\delta(\nu) + \delta(\nu-2)\right)\, e^{-\frac{(\omega-\nu)^2}{4}}\, d\nu \\
&= \frac{-1}{4} \left(e^{-\frac{(\omega-2)^2}{4}} - 2\,e^{-\frac{\omega^2}{4}} + e^{-\frac{(\omega+2)^2}{4}} \right) .
\end{aligned}$$

3 Distribution Theory

The Dirac point distribution, or more loosely speaking the Dirac δ-function, provides a tempered distribution with the following property.

$$T_\delta\,[\phi(x)] \;:=\; \int\limits_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx \;=\; \phi(0)\ .$$

(6) **a)** Determine the result of the following distribution with $a, b \in \mathbb{R}$ and $a \neq 0$ acting on an arbitrary Schwarz function $\phi \in \mathcal{S}(\mathbb{R})$. Note, that a can be positive or negative.

$$\int\limits_{-\infty}^{\infty} \delta(a\,x - b)\,\phi(x)\,dx\ .$$

Solution: Solve this task via substitution $\xi = a\,x - b$ with $d\xi = a\,dx$ and $x = \frac{\xi+b}{a}$.

$$\begin{aligned}
\int\limits_{-\infty}^{\infty} \delta(a\,x-b)\,\phi(x)\,dx &= \begin{cases} \int_{-\infty}^{\infty} \delta(\xi)\,\phi(\frac{\xi+b}{a})\,\frac{d\xi}{a} & a > 0 \\ \int_{\infty}^{-\infty} \delta(\xi)\,\phi(\frac{\xi+b}{a})\,\frac{d\xi}{a} & a < 0 \end{cases} \\
&= \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} \delta(\xi)\,\phi(\frac{\xi+b}{a})\,d\xi & a > 0 \\ -\frac{1}{a}\int_{-\infty}^{\infty} \delta(\xi)\,\phi(\frac{\xi+b}{a})\,d\xi & a < 0 \end{cases} \\
&= \begin{cases} \frac{1}{a}\,\phi(\frac{0+b}{a}) & a > 0 \\ -\frac{1}{a}\,\phi(\frac{0+b}{a}) & a < 0 \end{cases} \\
&= \frac{1}{|a|}\phi(\frac{b}{a})\ .
\end{aligned}$$

(5) **b)** Proof for $\phi \in \mathcal{S}(\mathbb{R})$ and $T_f \in \mathcal{S}'(\mathbb{R})$ the identity $T_f''(\phi) = T_f(\phi'')$.

Solution: Here we rely on the differentiation property $T_f'(\phi) = -T_f(\phi')$ of tempered distribution and obtain for two concatenated differentiations $T_f''(\phi) = -T_f'(\phi') = -(-T_f((\phi')')) = T_f(\phi'')$.

We consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \sin(x) & x \leq 0 \\ 0 & x > 0 \end{cases}$$

and its associated regular tempered distribution $T_f : \mathcal{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto T_f(\phi) = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx$.

(5) **c)** Show that f satisfies the ordinary differential equation $f'' + f = 0$ *almost everywhere.* Explain what the annotation "almost everywhere" means in this case.

Solution f is not differentiable at $x = 0$. For all other $x \in \mathbb{R}$ we obtain $f''(x) = \begin{cases} -\sin(x) & x < 0 \\ 0 & x > 0 \end{cases}$.

Hence, $f'' + f = 0$ for all points $x \in \mathbb{R}$ except for $x = 0$.

(7) **d)** Show that, in the distributional sense, T_f satisfies the ordinary differential equation

$$T_f'' + T_f = -T_\delta \ ,$$

in which the right hand side denotes the Dirac point distribution defined above.

Solution: Applying partial integration twice to determine $T_f''[\phi]$ we obtain

$$\begin{aligned} T_f''[\phi] = T_f[\phi''] &= \int_{-\infty}^{\infty} f(x)\phi''(x)\,dx = \int_{-\infty}^{0} \sin(x)\phi''(x)\,dx \\ &= \sin(x)\phi'(x)\big|_{-\infty}^{0} - \int_{-\infty}^{0} \cos(x)\phi'(x)\,dx \\ &= (0\,\phi'(0) - \sin(-\infty)\,0) - \cos(x)\phi(x)\big|_{-\infty}^{0} - \int_{-\infty}^{0} \sin(x)\phi(x)\,dx \\ &= -\phi(0) + 0 - \int_{-\infty}^{\infty} f(x)\phi(x)\,dx \\ &= -T_\delta[\phi] - T_f[\phi]. \end{aligned}$$

Note above, that the Schwartz function $\phi(x)$ and its derivative $\phi'(x)$ approach 0 for $x \longrightarrow \pm\infty$. Hence, $T_f'' + T_f = -T_\delta$.