EXAMINATION: MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020.
Date: Monday January 18$^{\text{th}}$, 2010.
Time: 9h00 – 12h00.
Place: paviljoen b1

Read this first!

- Use a separate sheet of paper for each problem. Write your name and student identification number on each paper.
- The exam consists of 3 problems. The maximum credit for each item is indicated in parenthesis.
- Motivate your answers. The use of course notes and calculator is allowed. The use of the problem companion, opgaven- en tentamenbundel, is not allowed.
- You may provide your answers in Dutch or (preferably) in English.

Good Luck!

1 Linear Algebra

Consider the set $\mathcal{C}^\infty(S^1)$ of $\mathbb{R}$-valued, infinitely differentiable functions on the unit circle S^1. We equip the function space $\mathcal{C}^\infty(S^1)$ with the inner product

$$\langle f|g\rangle := \int_0^{2\pi} f(\alpha)\, g(\alpha)\, d\alpha \;, \text{ for } f, g \in \mathcal{C}^\infty(S^1).$$

The corresponding measure is given by $\|f\| := \sqrt{\langle f|f\rangle}$.

For our calculations we use the following orthogonal basis functions:

$$b_0 : \alpha \mapsto 1 \;, \quad b_{sn} : \alpha \mapsto \sin(n\,\alpha) \;, \quad b_{cn} : \alpha \mapsto \cos(n\,\alpha) \quad \text{with } n \in \mathbb{N}.$$

(5) **a)** Proof the trigonometric identity

$$2\cos(m\,\alpha)\cos(n\,\alpha) \;=\; \cos((m-n)\,\alpha) + \cos((m+n)\,\alpha)$$

for $n, m \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$.

(6) **b)** Verify that the basis functions b_0, b_{sn}, and b_{cn} are indeed orthogonal.
(Hint: You may use the trigonometric identities:

$$\begin{aligned}
2\sin(m\,\alpha)\,\sin(n\,\alpha) &= \cos((m-n)\,\alpha) - \cos((m+n)\,\alpha),\\
2\cos(m\,\alpha)\,\cos(n\,\alpha) &= \cos((m-n)\,\alpha) + \cos((m+n)\,\alpha),\\
2\sin(m\,\alpha)\,\cos(n\,\alpha) &= \sin((m-n)\,\alpha) + \sin((m+n)\,\alpha).)
\end{aligned}$$

(4) **c)** Normalize the orthogonal basis functions b_0, b_{sn}, and b_{cn}.
Call the normalized basis functions e_0, e_{sn}, and e_{cn}.

(5) **d)** Expand the Dirac point distribution δ with the property $\int_{S^1} \delta(\alpha)\, f(\alpha)\, d\alpha = f(0)$ in the orthonormal basis e_0, e_{sn}, e_{cn}. Thus, determine the coefficient d_0, d_{sn} and d_{cn} of the Dirac point distribution so that $\delta(\alpha) = d_0\, e_0(\alpha) + \sum_{n=1}^{\infty} d_{sn}\, e_{sn}(\alpha) + \sum_{n=1}^{\infty} d_{cn}\, e_{cn}(\alpha)$.

(6) **e)** Find the matrix representation D of differential operator ∂_α with respect to the orthonormal basis

$$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} = e_0\,, \quad \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} = e_{s1}\,, \quad \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} = e_{c1}\,, \quad \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} = e_{s2}\,, \quad \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = e_{c2}\,.$$

Neglect all basis functions with $n \geq 3$.

(5) **f)** Find the adjoint operator of the differential operator ∂_α. Provide the adjoint matrix representation $\mathsf{D}^\dagger$ and the adjoint differential operator?

(7) **g)** The differential operator $(1 - \varphi\,\partial_\alpha) f(\alpha)$ is equivalent to a minute shift $f(\alpha - \varphi)$ for infinitesimal φ. It is therefore called the infinitesimal generator of translation. Applying the infinitesimal generator of translation infinitely often yields the regular operator of translation $\mathcal{T}_\varphi$.

$$\lim_{m\to\infty} \left(1 - \frac{\varphi}{m}\,\partial_\alpha\right)^m = e^{-\varphi\,\partial_\alpha} = \mathcal{T}_\varphi.$$

Find the matrix representation of the translation operator $\mathcal{T}_\varphi$ in our orthonormal basis for $n \leq 2$. Utilize the definition $\mathcal{T}_\varphi = e^{-\varphi\,\partial_\alpha}$ and the Taylor-expansion of the exponential function. The latter defines the exponential of a matrix X.

$$e^{\mathsf{X}} := \sum_{k=0}^{\infty} \frac{1}{k!}\,\mathsf{X}^k.$$

(Hint: knowing the k^{th} power of matrix representation D is essential. Note that matrix D consists of sub-matrices in the diagonal, which can be dealt with one at a time.)

(5) **h)** Why is the following definition not an inner product for function space $\mathcal{C}^\infty(S^1)$?

$$\langle f|g\rangle := \int_0^{2\pi} f(\alpha)\, g(2\pi - \alpha)\, d\alpha\ .$$

2 Fourier Transformation

We adhere to the following definition of a Fourier transformation:

$$\hat{f}(\omega) \; = \; \mathcal{F}\,[f](\omega) \; := \; \int\limits_{-\infty}^{\infty} f(x)\, e^{-i\,\omega\,x}\, dx \; .$$

Inverse Fourier transformation:

$$f(x) \; = \; \mathcal{F}^{-1}\left[\hat{f}\right](x) \; := \; \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\,\omega\,x}\, d\omega \; .$$

(4) **a)** Show that the Fourier transformation is a linear transformation.

(6) **b)** Assume that function h is the complex conjugate of function f, thus, $h = f^*$.
Proof, that $\hat{h}(\omega) = \left(\hat{f}(-\omega)\right)^*$.

(5) **c)** Show that the Fourier transform of function $q : x \mapsto e^{-x^4}$ is $\mathbb{R}$-valued.

(6) **d)** Determine the Fourier transform of $f : x \mapsto \sin^2(x)$. (Recall that $\sin^2(x)$ stands for $(\sin(x))^2$.)

(7) **e)** Proof that the Fourier transform of $g(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ renders $\hat{g}(\omega) = e^{-\omega^2/4}$.
You may use the definite integral

$$\int\limits_{-\infty}^{\infty} e^{-(x+i\,y)^2}\, dx \; = \; \sqrt{\pi} \; .$$

(6) **f)** Derive the Fourier transform of function $p : x \mapsto \frac{1}{\sqrt{\pi}} \; \sin^2(x) \; e^{-x^2}$.

3 Distribution Theory

The Dirac point distribution, or more loosely speaking the Dirac δ-function, provides a tempered distribution with the following property.

$$T_\delta\,[\phi(x)] \; := \; \int\limits_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx \; = \; \phi(0) \; .$$

(6) **a)** Determine the result of the following distribution with $a, b \in \mathbb{R}$ and $a \neq 0$ acting on an arbitrary Schwarz function $\phi \in \mathcal{S}(\mathbb{R})$. Note, that a can be positive or negative.

$$\int_{-\infty}^{\infty} \delta(a\,x - b)\,\phi(x)\,dx\ .$$

(5) **b)** Proof for $\phi \in \mathcal{S}(\mathbb{R})$ and $T_f \in \mathcal{S}'(\mathbb{R})$ the identity $T_f''(\phi) = T_f(\phi'')$.

We consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \sin(x) & x \leq 0 \\ 0 & x > 0 \end{cases}$$

and its associated regular tempered distribution $T_f : \mathcal{S}(\mathbb{R}) \to \mathbb{R} : \phi \mapsto T_f(\phi) = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx$.

(5) **c)** Show that f satisfies the ordinary differential equation $f'' + f = 0$ *almost everywhere.* Explain what the annotation "almost everywhere" means in this case.

(7) **d)** Show that, in the distributional sense, T_f satisfies the ordinary differential equation

$$T_f'' + T_f = -T_\delta\ ,$$

in which the right hand side denotes the Dirac point distribution defined above.