EXAMINATION:
MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020.
Date: Thursday April 8th, 2010.
Time: 14h00 – 17h00.
Place: AUD 13

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems on 5 pages. The maximum credit for each item is indicated in parenthesis.
- Motivate your answers. The use of course notes and calculator is allowed. The use of the problem companion, opgaven- en tentamenbundel, is not allowed.
- You may provide your answers in Dutch or (preferably) in English.

Good Luck!

1 Linear Algebra

Consider the set $\mathcal{C}^\infty(H^1)$ of $\mathbb{C}$-valued, infinitely differentiable functions on a unit half-cirlce H^1. We parametrize functions $f \in \mathcal{C}^\infty(H^1)$ either by an angular coordinate $\theta \in [0, \pi]$ or by the corresponding projection onto the z-axis, being $z = \cos\theta$ (see Figure 1).
We equip the function space $\mathcal{C}^\infty(H^1)$ with the inner product

$$\langle f|g\rangle := \int_0^\pi f^*(\theta)\, g(\theta)\, d\theta \ , \text{ for } f, g \in \mathcal{C}^\infty(H^1) \ , \tag{1}$$

with f^* denoting the complex-conjugate of f.
The corresponding measure is given by $\|f\| := \sqrt{\langle f|f\rangle}$.
For our calculations we utilize the orthogonal basis functions

$$b_n : \ \theta \mapsto \cos(n\,\theta), \text{ for } n \in \{0, 1, 2, \dots\} .$$

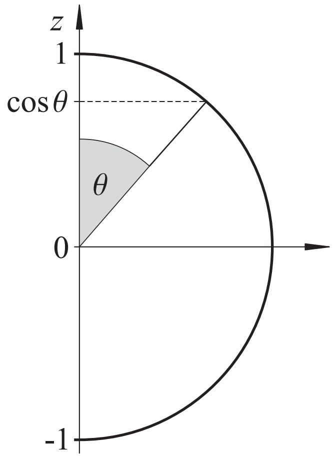

Figure 1: Half-cirlce H^1 parameterized by angle $\theta \in [0, \pi]$ or projection $z = \cos\theta \ \in [-1, 1]$.

(4) **a)** With $e^{i\,\theta} = \cos\theta + i\,\sin\theta$, proof the trigonometric identity

$$\cos^2\theta + \sin^2\theta \ = \ 1 \qquad \text{for } \theta \in \mathbb{R}.$$

Note, that $\cos^2\theta$ stands for $(\cos\theta)^2$ and $\sin^2\theta$ stands for $(\sin\theta)^2$.

Solution: Solving $e^{i\,\theta} = \cos\theta + i\,\sin\theta$ for sin and cos, we obtain
$\cos^2\theta = \frac{1}{4}\left(e^{i\,\theta} + e^{-i\,\theta}\right)^2 = \frac{1}{4}\left(e^{2i\,\theta} + 2\,e^{i\,\theta}e^{-i\,\theta} + e^{-2i\,\theta}\right) = \frac{1}{4}\left(e^{2i\,\theta} + 2 + e^{-2i\,\theta}\right)$ and
$\sin^2\theta = -\frac{1}{4}\left(e^{i\,\theta} - e^{-i\,\theta}\right)^2 = -\frac{1}{4}\left(e^{2i\,\theta} - 2\,e^{i\,\theta}e^{-i\,\theta} + e^{-2i\,\theta}\right) = -\frac{1}{4}\left(e^{2i\,\theta} - 2 + e^{-2i\,\theta}\right)$
Hence, $\cos^2\theta + \sin^2\theta = \frac{1}{4}(2+2) = 1$.

(4) **b)** Proof Moivre's formula

$$(\cos\theta + i\sin\theta)^n \ = \ \cos(n\,\theta) + i\sin(n\,\theta)$$

for $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$.

Solution: $(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{i\,n\,\theta} = \cos(n\,\theta) + i\sin(n\,\theta)$.

(6) **c)** Show, that the inner product

$$\langle f|g\rangle := \int\limits_{-1}^{1} f^*(\arccos z)\, g(\arccos z)\, \frac{dz}{\sqrt{1-z^2}} \ , \ \text{for } f, g \in \mathcal{C}^\infty(H^1) \ . \tag{2}$$

is equivalent to the inner product in equation (1). Note, that arccos is the inverse function of cos. Hence, $z = \cos\theta$, $\theta = \arccos z$, and $\theta = \arccos(\cos\theta)$ for all $\theta \in [0, \pi]$.

Solution: One has to perform in equation (1) the substitution $z = \cos\theta$ to obtain equation (2). Steps in that substitution are $dz = -\sin\theta\, d\theta$ and $-\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-z^2}$ for $\theta \in [0, \pi]$, so that $d\theta = -\frac{dz}{\sqrt{1-z^2}}$. Note, that the boundaries $\cos 0 = 1$ and $\cos\pi = -1$ of the integration need to be flipped, inducing another overall minus sign.

$$\int\limits_0^\pi f^*(\theta)\, g(\theta)\, d\theta = -\int\limits_1^{-1} f^*(\arccos z)\, g(\arccos z)\, \frac{dz}{\sqrt{1-z^2}} == \int\limits_{-1}^{1} f^*(\arccos z)\, g(\arccos z)\, \frac{dz}{\sqrt{1-z^2}}.$$

(5) **d)** Express the first three basis functions b_n with $n = 0, 1, 2$ as polynomials $T_n(z)$ of z. Remark: the polynomials $T_n(z)$ are the so-called Chebyshev polynomials of the first kind.

Solution:

$$\begin{aligned}
T_0(z) &= b_0(\arccos z) = 1, \\
T_1(z) &= b_1(\arccos z) = \cos(\arccos z) = z, \\
T_2(z) &= b_2(\arccos z) = \cos(2 \arccos z) = \cos^2(\arccos z) - \sin^2(\arccos z) \\
&= z^2 - \left(1 - \cos^2(\arccos z)\right) = 2\, z^2 - 1
\end{aligned}$$

(5) **e)** Verify the orthogonality relation for all $n = \{0, 1, 2, 3, \dots\}$.

$$\int_{-1}^{1} T_n^*(z)\, T_m(z)\, \frac{dz}{\sqrt{1-z^2}} = \begin{cases} \pi & , n = m = 0 \\ \frac{\pi}{2} & , n = m \neq 0 \\ 0 & , n \neq m \end{cases} . \tag{3}$$

Tip: Remember the relation between $T_n(z)$ and $b_n(\theta)$.

Solution: $\int_{-1}^{1} T_n^*(z)\, T_m(z)\, \frac{dz}{\sqrt{1-z^2}} = \int_0^\pi b_n^*(\theta) b_m(\theta)\, d\theta = \int_0^\pi \cos(n\,\theta)\, \cos(m\,\theta)\, d\theta$.
We now utilize the trigonometric identity (see script) $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \pm \sin(\alpha)\sin(\beta)$. Adding the two versions of this identity results in $\cos(\alpha-\beta)+\cos(\alpha+\beta) = 2\cos(\alpha)\cos(\beta)$. Substituting for $\alpha \mapsto n\,\theta$ and $\beta \mapsto m\,\theta$ we have $\cos(m\,\theta)\cos(n\,\theta) = \frac{1}{2}(\cos((n-m)\,\theta)+\cos((m+n)\,\theta))$. We return to the integral. $\int_0^\pi \cos(n\,\theta)\, \cos(m\,\theta)\, d\theta = \frac{1}{2}\left(\int_0^\pi \cos((n-m)\,\theta)\, d\theta + \int_0^\pi \cos((m+n)\,\theta)\, d\theta\right)$.
Consider the case $n = m = 0$: $\frac{1}{2}\left(\int_0^\pi 1\, d\theta + \int_0^\pi 1\,\theta)\, d\theta\right) = \pi$.
Consider the case $n = m \neq 0$: $\frac{1}{2}\left(\int_0^\pi 1\, d\theta + \int_0^\pi \cos(2n\,\theta)\, d\theta\right) = \frac{1}{2}(\pi + 0) = \frac{\pi}{2}$.
Consider the case $n \neq m$: $\frac{1}{2}\left(\frac{\sin(\pi(m-n))}{m-n} + \frac{\sin(\pi(m+n))}{m+n}\right) = \frac{1}{2}(0+0) = 0$.

(6) **f)** Derive the recursion relation

$$T_n(z) = 2\, z\, T_{n-1}(z) - T_{n-2}(z)$$

for all $n \in \{2, 3, 4, \dots\}$.
Tip: First proof the trigonometric relation $\cos(n\ \theta) = 2\ \cos(\theta)\ \cos((n-1)\ \theta) - \cos((n-2)\ \theta)$.

Solution:

$$\begin{aligned}
&2\ \cos(\theta)\ \cos((n-1)\,\theta) - \cos((n-2)\,\theta) \\
&= \left(e^{i\,\theta} + e^{-i\,\theta}\right) \frac{1}{2} \left(e^{i\,(n-1)\,\theta} + e^{-i\,(n-1)\,\theta}\right) - \frac{1}{2}\left(e^{i\,(n-2)\,\theta} + e^{-i\,(n-2)\,\theta}\right) \\
&= \frac{1}{2}\left(e^{i\,(n)\,\theta} + e^{-i\,(n)\,\theta}\right) + \frac{1}{2}\left(e^{i\,(n-2)\,\theta} + e^{-i\,(n-2)\,\theta}\right) - \frac{1}{2}\left(e^{i\,(n-2)\,\theta} + e^{-i\,(n-2)\,\theta}\right) \\
&= \frac{1}{2}\left(e^{i\,(n)\,\theta} + e^{-i\,(n)\,\theta}\right) \\
&= \cos(n\ \theta)
\end{aligned}$$

Substituting $\cos(q\,\theta)$ by $T_q(z)$ and $T_1(z)$ by z, one obtains the recursion.

(5) **g)** Determine the expansion of function $v : z \mapsto \sqrt{1-z^2}$ in the orthonormal basis

$$\begin{pmatrix}1\\0\\0\\0\\0\\\vdots\end{pmatrix} = \frac{1}{\sqrt{\pi}}\,T_0\,,\quad \begin{pmatrix}0\\1\\0\\0\\0\\\vdots\end{pmatrix} = \sqrt{\frac{2}{\pi}}\,T_1\,,\quad \begin{pmatrix}0\\0\\1\\0\\0\\\vdots\end{pmatrix} = \sqrt{\frac{2}{\pi}}\,T_2\,,\quad \begin{pmatrix}0\\0\\0\\1\\0\\\vdots\end{pmatrix} = \sqrt{\frac{2}{\pi}}\,T_3\,,\quad \begin{pmatrix}0\\0\\0\\0\\1\\\vdots\end{pmatrix} = \sqrt{\frac{2}{\pi}}\,T_4\,,\ldots$$

Neglect all basis functions with $n \geq 5$.

Solution: All inner products below are integrals of simple polynomials.

$$v \;=\; \begin{pmatrix} \left\langle \sqrt{1-z^2} \middle| \frac{1}{\sqrt{\pi}}\,T_0 \right\rangle \\ \left\langle \sqrt{1-z^2} \middle| \sqrt{\frac{2}{\pi}}\,T_1 \right\rangle \\ \left\langle \sqrt{1-z^2} \middle| \sqrt{\frac{2}{\pi}}\,T_2 \right\rangle \\ \left\langle \sqrt{1-z^2} \middle| \sqrt{\frac{2}{\pi}}\,T_3 \right\rangle \\ \left\langle \sqrt{1-z^2} \middle| \sqrt{\frac{2}{\pi}}\,T_4 \right\rangle \\ \vdots \end{pmatrix} \;=\; \begin{pmatrix} \frac{2}{\sqrt{\pi}} \\ 0 \\ -\frac{2}{3}\sqrt{\frac{2}{\pi}} \\ 0 \\ -\frac{2}{15}\sqrt{\frac{2}{\pi}} \\ \vdots \end{pmatrix}.$$

(5) **h)** Determine the matrix M of the linear transformation $f \mapsto z\,f$ in the orthonormal basis given above. Again, neglect all basis functions with $n \geq 5$.
Tip: Recall the result of problem 1(f).

Solution: Rewrite the recursion as $z\,T_{n-1}(z) = \frac{1}{2}\left(T_n(z) + T_{n-2}(z)\right)$ and shift it in n by one: $z\,T_n(z) = \frac{1}{2}\left(T_{n+1}(z) + T_{n-1}(z)\right)$ for $n \geq 1$. Furthermore, recall $z\,T_0(z) = T_1(z)$. Thus, one can write

$$M = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots \\ \sqrt{2} & 0 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(5) **i)** Prove or disprove, that the following definition is an inner product for function space $\mathcal{C}^\infty(H^1)$?

$$\langle f|g\rangle := \int_0^\pi \left(f^*(\theta)\,g(\theta) + 1\right) d\theta\ .$$

Solution: The proposed inner product is not linear! Example: $\langle f|2g\rangle \neq 2\,\langle f|g\rangle$.

2 Fourier Transformation

We adhere to the following definition of a Fourier transformation:

$$\hat{f}(\omega) \;=\; \mathcal{F}\left[f\right](\omega) \;:=\; \int\limits_{-\infty}^{\infty} f(x)\, e^{-i\,\omega\,x}\, dx \;.$$

Inverse Fourier transformation:

$$f(x) \;=\; \mathcal{F}^{-1}\left[\hat{f}\right](x) \;:=\; \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\,\omega\,x}\, d\omega \;.$$

(4) **a)** Show that

$$\int\limits_{-\infty}^{\infty} f(x)\, dx \;=\; \hat{f}(0) \;.$$

Solution: $\hat{f}(0) \;=\; \int_{-\infty}^{\infty} f(x)\, e^{-i\,0\,x}\, dx \;=\; \int_{-\infty}^{\infty} f(x)\, dx$, since $e^{-i0x} = 1$.

(5) **b)** Show that

$$\int\limits_{-\infty}^{\infty} x\, f(x)\, dx \;=\; i\,\hat{f}'(0) \;.$$

Note, that $\hat{f}'(0)$ denotes the derivative of the Fourier transform $\hat{f}$ at $\omega = 0$.
Solution: $\hat{f}'(\omega) \;=\; \int_{-\infty}^{\infty} \partial_\omega f(x)\, e^{-i\,\omega\,x}\, dx \;=\; \int_{-\infty}^{\infty} f(x)\,(-i\,x)\, e^{-i\,\omega\,x}\, dx$. Hence, for $\omega = 0$ we $i\,\hat{f}'(0) \;=\; \int_{-\infty}^{\infty} f(x)\, i\,(-i)\, x\, e^{-i\,0\,x}\, dx \;=\; \int_{-\infty}^{\infty} f(x)\, x\, dx$.

(6) **c)** Show that for $h(x) := f(a\,x)$ with $a \in \mathbb{R}$ and $a \neq 0$, the Fourier transform is given by

$$\hat{h}(\omega) = \frac{1}{|a|}\,\hat{f}\left(\frac{\omega}{a}\right) \;.$$

Solution: The solution is given by a simple substitution$z = a\,x$.
For $a > 0$ we have $\int_{-\infty}^{\infty} f(a\,x)\, e^{-i\,\omega\,x}\, dx = \int_{-\infty}^{\infty} f(z)\, e^{-i\,\frac{\omega\,z}{a}}\, \frac{dz}{a} = \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right)$.
For $a < 0$ we have $\int_{-\infty}^{\infty} f(a\,x)\, e^{-i\,\omega\,x}\, dx = \int_{\infty}^{-\infty} f(z)\, e^{-i\,\frac{\omega\,z}{a}}\, \frac{dz}{-|a|} = \int_{-\infty}^{\infty} f(z)\, e^{-i\,\frac{\omega\,z}{a}}\, \frac{dz}{|a|} = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right)$.
Hence, for both cases we can write: $\hat{h}(\omega) = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right)$

(6) **d)** Consider for $\lambda > 0$ the function

$$g : x \mapsto \begin{cases} 0 & \text{for } x < 0 \\ \lambda\, e^{-\lambda\,x} & \text{for } x \geq 0 \end{cases} \;.$$

Derive the Fourier transform $\hat{g}$.
Solution: $\hat{g}(\omega) = \int_{-\infty}^{\infty} g(x)\, e^{-i\,\omega\,x}\, dx = \int_0^{\infty} \lambda\, e^{-\lambda\,x}\, e^{-i\,\omega\,x}\, dx = \int_0^{\infty} \lambda\, e^{-(\lambda+i\,\omega)\,x}\, dx = -\frac{\lambda\, e^{-(\lambda+i\,\omega)\,x}}{\lambda+i\,\omega}\Big|_0^{\infty} = \frac{\lambda}{\lambda+i\,\omega}$.

Consider the two cardinal B-spline functions

$$B_0 : x \mapsto \begin{cases} 1 & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_1 : x \mapsto \begin{cases} 1+x & \text{for } -1 \leq x \leq 0 \\ 1-x & \text{for } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad .$$

(5) **e)** Show, that $B_1 = B_0 * B_0$ where $*$ denotes the convolution

$$\big(f * g\big)(x) := \int\limits_{-\infty}^{\infty} f(y)\, g(x-y)\, dy \ .$$

Solution: We need to distinguish three cases:
1. The two B_0 kernels do not have any overlap for $|x| > 1$. In this case, B_1 is 0.
2. For $-1 \leq x < 0$, $B_0(x-y)$ is more left then $B_0(y)$. The two kernels overlap from $y = -\frac{1}{2}$ to $y = x + \frac{1}{2}$ so that $B_1(x) = \int_{-\frac{1}{2}}^{x+\frac{1}{2}} 1\, dx = (x + \frac{1}{2} + \frac{1}{2}) = x + 1$.
3. For $0 \leq x \leq 1$, $B_0(x-y)$ is more right than $B_0(y)$. The two kernels overlap from $y = x - \frac{1}{2}$ to $y = \frac{1}{2}$ so that $B_1(x) = \int_{x-\frac{1}{2}}^{\frac{1}{2}} 1\, dx = (\frac{1}{2} - (x - \frac{1}{2})) = 1 - x$.

(4) **f)** Determine the Fourier transform $\hat{B}_0$.
Solution: $\hat{B}_0(\omega) = \int_{-\infty}^{\infty} B_0(x)\, e^{-i\,\omega\, x}\, dx \ = \ \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i\,\omega\, x}\, dx \ = \ \frac{\sin(\omega/2)}{\omega/2}$

(5) **g)** Determine the Fourier transform $\hat{B}_1$.
Recall the Fourier theorem $\mathcal{F}[f * g] = \mathcal{F}[f]\ \mathcal{F}[g]$.
Solution: $\hat{B}_1(\omega) = \hat{B}_0^2(\omega) \ = \ \left(\frac{\sin(\omega/2)}{\omega/2}\right)^2$.

3 Distribution Theory

The Dirac point distribution, or more loosely speaking the Dirac δ-function, provides a tempered distribution with the following property.

$$T_\delta\,[\phi(x)] \;:=\; \int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx \;=\; \phi(0)\;.$$

(6) **a)** Determine the result of the following distribution acting on an arbitrary Schwarz function $\phi \in \mathcal{S}(\mathbb{R})$.

$$\int_{-\infty}^{\infty} \delta(\sinh x)\;\phi(x)\,dx\;.$$

Recall, that $\sinh x = \frac{1}{2}\,(e^x - e^{-x})$, $\sinh' = \cosh$, and $\cosh^2 x - \sinh^2 x = 1$.

Solution: Perform a substitution $z = \sinh x$ with $dz = \cosh x\,dx$ and, thus, $dx = \frac{dz}{\sqrt{1+z^2}}$. So $\int_{-\infty}^{\infty} \delta(\sinh x)\;\phi(x)\,dx = \int_{-\infty}^{\infty} \delta(z)\;\phi(\operatorname{arcsinh}(z))\,\frac{dz}{\sqrt{1+z^2}} = \phi(\operatorname{arcsinh}(0))\,\sqrt{1+0^2} = \phi(0)$.

(4) **b)** We consider the function $g\colon \mathbb{R} \to \mathbb{R}$ in problem 2(d). Show that g satisfies the ordinary differential equation $g' + \lambda\,g = 0$ *almost everywhere*. Explain what the annotation "almost everywhere" means in this case.

Solution:

$$g'(x) = \begin{cases} 0 & x < 0 \\ \text{undefined} & x = 0 \\ -\lambda^2\,e^{-\lambda\,x} & x > 0 \end{cases}.$$

Hence, $g'(x) + \lambda g(x)$ add up to 0 except at position $x = 0$, where g' is not defined.

(6) **c)** Show that, in the distributional sense, T_g satisfies the ordinary differential equation

$$T_g' + \lambda\,T_g = \lambda T_\delta\;,$$

in which the right hand side denotes the Dirac point distribution defined above.

Solution: $T_g'\,[\phi(x)] = -T_g\,[\phi'(x)] = -\int_{-\infty}^{\infty} g(x)\,\phi'(x)\,dx = -\int_0^{\infty} \lambda\,e^{-\lambda\,x}\,\phi'(x)\,dx = -\,\lambda\,e^{-\lambda\,x}\,\phi(x)\big|_0^{\infty} + \int_0^{\infty}(-\lambda^2)\,e^{-\lambda\,x}\,\phi(x)\,dx = \lambda\,\phi(0)\;-\;\lambda\int_0^{\infty}\lambda\,e^{-\lambda\,x}\,\phi(x)\,dx = \lambda\,T_\delta\,[\phi(x)]\;-\;\lambda\,T_g\,[\phi(x)]$.

(4) **d)** Derive the Fourier transform of the ordinary differential equation

$$T_g' + \lambda\,T_g = \lambda T_\delta\;,$$

by applying the Fourier transformation $\mathcal{F}$ to both sides of the equation, and show that $\hat{g}$ is a solution.

Solution: We know that the Fourier transformation is linear $\mathcal{F}\,[\alpha\,f + \beta\,g] = \alpha\mathcal{F}\,[f] + \beta\,\mathcal{F}\,[g]$, that $\mathcal{F}\,[f']\,(\omega) = i\,\omega\,\hat{f}$, and that $\mathcal{F}\,[\delta] = 1$. Thus, we obtain $i\,\omega\,\hat{g} + \lambda\,\hat{g} = \lambda$. Recall $\hat{g} = \frac{\lambda}{\lambda + i\,\omega}$, which clearly satisfies the equation.