EXAMINATION:
MATHEMATICAL TECHNIQUES FOR IMAGE ANALYSIS

Course code: 8D020.
Date: Thursday April 8th, 2010.
Time: 14h00 – 17h00.
Place: AUD 13

Read this first!

- Write your name and student identification number on each paper.
- The exam consists of 3 problems on 5 pages. The maximum credit for each item is indicated in parenthesis.
- Motivate your answers. The use of course notes and calculator is allowed. The use of the problem companion, opgaven- en tentamenbundel, is not allowed.
- You may provide your answers in Dutch or (preferably) in English.

Good Luck!

1 Linear Algebra

Consider the set $\mathcal{C}^\infty(H^1)$ of $\mathbb{C}$-valued, infinitely differentiable functions on a unit half-cirlce H^1.
We parametrize functions $f \in \mathcal{C}^\infty(H^1)$ either by an angular coordinate $\theta \in [0, \pi]$ or by the corresponding projection onto the z-axis, being $z = \cos\theta$ (see Figure 1).
We equip the function space $\mathcal{C}^\infty(H^1)$ with the inner product

$$\langle f|g\rangle := \int_0^\pi f^*(\theta)\, g(\theta)\, d\theta \ , \ \text{for } f, g \in \mathcal{C}^\infty(H^1) \ , \qquad (1)$$

with f^* denoting the complex-conjugate of f.
The corresponding measure is given by $\|f\| := \sqrt{\langle f|f\rangle}$.
For our calculations we utilize the orthogonal basis functions

$$b_n : \ \theta \mapsto \cos(n\,\theta), \ \text{for } n \in \{0, 1, 2, \dots\} \, .$$

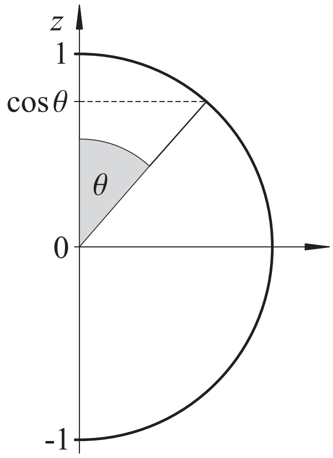

Figure 1: Half-cirlce H^1 parameterized by angle $\theta \in [0, \pi]$ or projection $z = \cos\theta\ \in [-1, 1]$.

(4) **a)** With $e^{i\,\theta} = \cos\theta + i\,\sin\theta$, proof the trigonometric identity

$$\cos^2\theta + \sin^2\theta \;=\; 1 \qquad \text{for } \theta \in \mathbb{R}.$$

Note, that $\cos^2\theta$ stands for $(\cos\theta)^2$ and $\sin^2\theta$ stands for $(\sin\theta)^2$.

(4) **b)** Proof Moivre's formula

$$(\cos\theta + i\sin\theta)^n \;=\; \cos(n\,\theta) + i\sin(n\,\theta)$$

for $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$.

(6) **c)** Show, that the inner product

$$\langle f|g\rangle := \int_{-1}^{1} f^*(\arccos z)\, g(\arccos z)\, \frac{dz}{\sqrt{1-z^2}}\ ,\ \text{for } f, g \in \mathcal{C}^\infty(H^1)\ . \tag{2}$$

is equivalent to the inner product in equation (1). Note, that arccos is the inverse function of cos. Hence, $z = \cos\theta$, $\theta = \arccos z$, and $\theta = \arccos(\cos\theta)$ for all $\theta \in [0, \pi]$.

(5) **d)** Express the first three basis functions b_n with $n = 0, 1, 2$ as polynomials $T_n(z)$ of z. Remark: the polynomials $T_n(z)$ are the so-called Chebyshev polynomials of the first kind.

(5) **e)** Verify the orthogonality relation for all $n = \{0, 1, 2, 3, \ldots\}$.

$$\int_{-1}^{1} T_n^*(z)\, T_m(z)\, \frac{dz}{\sqrt{1-z^2}} \;=\; \begin{cases} \pi & , n = m = 0 \\ \frac{\pi}{2} & , n = m \neq 0\ . \\ 0 & , n \neq m \end{cases} \tag{3}$$

Tip: Remember the relation between $T_n(z)$ and $b_n(\theta)$.

(6) **f)** Derive the recursion relation

$$T_n(z) = 2\,z\,T_{n-1}(z) - T_{n-2}(z)$$

for all $n \in \{2, 3, 4, \dots\}$.
Tip: First proof the trigonometric relation $\cos(n\;\theta) = 2\,\cos(\theta)\,\cos((n-1)\;\theta) - \cos((n-2)\;\theta)$.

(5) **g)** Determine the expansion of function $v : z \mapsto \sqrt{1-z^2}$ in the orthonormal basis

$$\begin{pmatrix}1\\0\\0\\0\\0\\\vdots\end{pmatrix} = \frac{1}{\sqrt{\pi}}\,T_0\,, \quad \begin{pmatrix}0\\1\\0\\0\\0\\\vdots\end{pmatrix} = \sqrt{\frac{2}{\pi}}\,T_1\,, \quad \begin{pmatrix}0\\0\\1\\0\\0\\\vdots\end{pmatrix} = \sqrt{\frac{2}{\pi}}\,T_2\,, \quad \begin{pmatrix}0\\0\\0\\1\\0\\\vdots\end{pmatrix} = \sqrt{\frac{2}{\pi}}\,T_3\,, \quad \begin{pmatrix}0\\0\\0\\0\\1\\\vdots\end{pmatrix} = \sqrt{\frac{2}{\pi}}\,T_4\,, \dots$$

Neglect all basis functions with $n \geq 5$.

(5) **h)** Determine the matrix M of the linear transformation $f \mapsto z\,f$ in the orthonormal basis given above. Again, neglect all basis functions with $n \geq 5$.
Tip: Recall the result of problem 1(f).

(5) **i)** Prove or disprove, that the following definition is an inner product for function space $\mathcal{C}^\infty(H^1)$?

$$\langle f|g\rangle := \int_0^\pi \big(f^*(\theta)\,g(\theta) + 1\big)\,d\theta\ .$$

2 Fourier Transformation

We adhere to the following definition of a Fourier transformation:

$$\hat{f}(\omega) \;=\; \mathcal{F}\,[f](\omega) \;:=\; \int_{-\infty}^{\infty} f(x)\,e^{-i\,\omega\,x}\,dx\ .$$

Inverse Fourier transformation:

$$f(x) \;=\; \mathcal{F}^{-1}\Big[\hat{f}\Big](x) \;:=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,e^{i\,\omega\,x}\,d\omega\ .$$

(4) **a)** Show that

$$\int_{-\infty}^{\infty} f(x)\,dx \;=\; \hat{f}(0)\ .$$

(5) **b)** Show that

$$\int_{-\infty}^{\infty} x\, f(x)\, dx \;=\; i\, \hat{f}'(0)\ .$$

Note, that $\hat{f}'(0)$ denotes the derivative of the Fourier transform $\hat{f}$ at $\omega = 0$.

(6) **c)** Show that for $h(x) := f(a\,x)$ with $a \in \mathbb{R}$ and $a \neq 0$, the Fourier transform is given by

$$\hat{h}(\omega) = \frac{1}{|a|}\, \hat{f}\left(\frac{\omega}{a}\right)\ .$$

(6) **d)** Consider for $\lambda > 0$ the function

$$g : x \mapsto \begin{cases} 0 & \text{for } x < 0 \\ \lambda\, e^{-\lambda\, x} & \text{for } x \geq 0 \end{cases}\ .$$

Derive the Fourier transform $\hat{g}$.

Consider the two cardinal B-spline functions

$$B_0 : x \mapsto \begin{cases} 1 & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_1 : x \mapsto \begin{cases} 1 + x & \text{for } -1 \leq x \leq 0 \\ 1 - x & \text{for } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}\ .$$

(5) **e)** Show, that $B_1 = B_0 * B_0$ where $*$ denotes the convolution

$$\big(f * g\big)(x) := \int_{-\infty}^{\infty} f(y)\, g(x - y)\, dy\ .$$

(4) **f)** Determine the Fourier transform $\hat{B}_0$.

(5) **g)** Determine the Fourier transform $\hat{B}_1$.
Recall the Fourier theorem $\mathcal{F}[f * g] = \mathcal{F}[f]\ \mathcal{F}[g]$.

3 Distribution Theory

The Dirac point distribution, or more loosely speaking the Dirac δ-function, provides a tempered distribution with the following property.

$$T_\delta\left[\phi(x)\right] \;:=\; \int\limits_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx \;=\; \phi(0)\ .$$

(6) **a)** Determine the result of the following distribution acting on an arbitrary Schwarz function $\phi \in \mathcal{S}(\mathbb{R})$.

$$\int\limits_{-\infty}^{\infty} \delta(\sinh x)\ \phi(x)\,dx\ .$$

Recall, that $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$, $\sinh' = \cosh$, and $\cosh^2 x - \sinh^2 x = 1$.

(4) **b)** We consider the function $g\colon \mathbb{R} \to \mathbb{R}$ in problem 2(d). Show that g satisfies the ordinary differential equation $g' + \lambda\, g = 0$ *almost everywhere*. Explain what the annotation "almost everywhere" means in this case.

(6) **c)** Show that, in the distributional sense, T_g satisfies the ordinary differential equation

$$T_g' + \lambda\, T_g = \lambda T_\delta\ ,$$

in which the right hand side denotes the Dirac point distribution defined above.

(4) **d)** Derive the Fourier transform of the ordinary differential equation

$$T_g' + \lambda\, T_g = \lambda T_\delta\ ,$$

by applying the Fourier transformation $\mathcal{F}$ to both sides of the equation, and show that $\hat{g}$ is a solution.