technische universiteit eindhoven

Image Structure

Luc Florack

October 2003

introduction: what is an image?

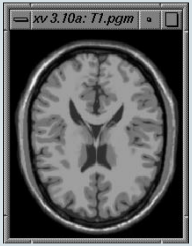
“Pictures are pictures relative to some input device (format), in themselves they are but records. A record may be a different picture to different input devices.”

—Jan Koenderink

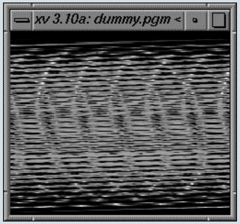
example 1: digital image

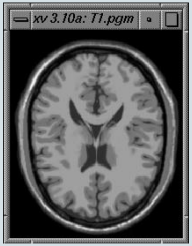
```
P2
# CREATOR: XV Version 3.10a Rev:  12/29/94
181 217
255

  0   0   0   2   6  12  24  36  42  45  49  53  59  64  67  72
 75  82  89  96  99  99 101 100  96 100 103 102 104 107 100  97
104  97  88  88  88  88  92  92  89  78  82  87  77  71  72  70
 70  71  69  61  55  53  51  51  51  49  47  46  44  41  36  30
```

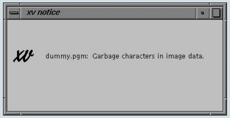
✂ ✂ ✂ ✂ ✂ ✂ ✂ ✂ ✂ ✂ ✂ ✂ ✂ *et cetera:* 181×217 *numbers* ✂ ✂ ✂ ✂ ✂ ✂ ✂ ✂ ✂ ✂ ✂ ✂



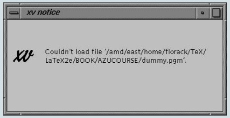
181 217 ➤ 217 181



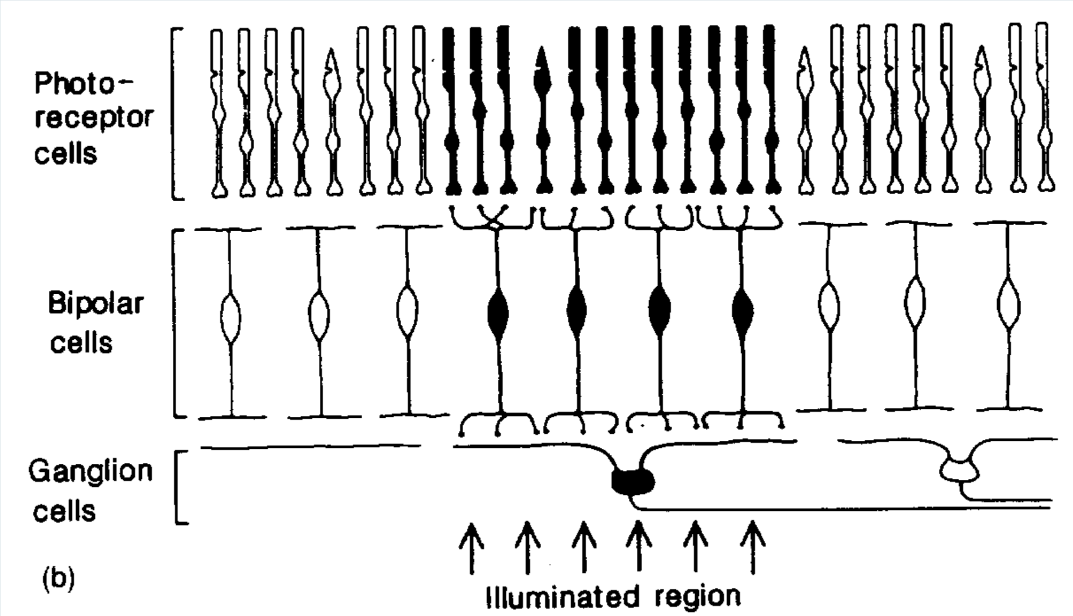




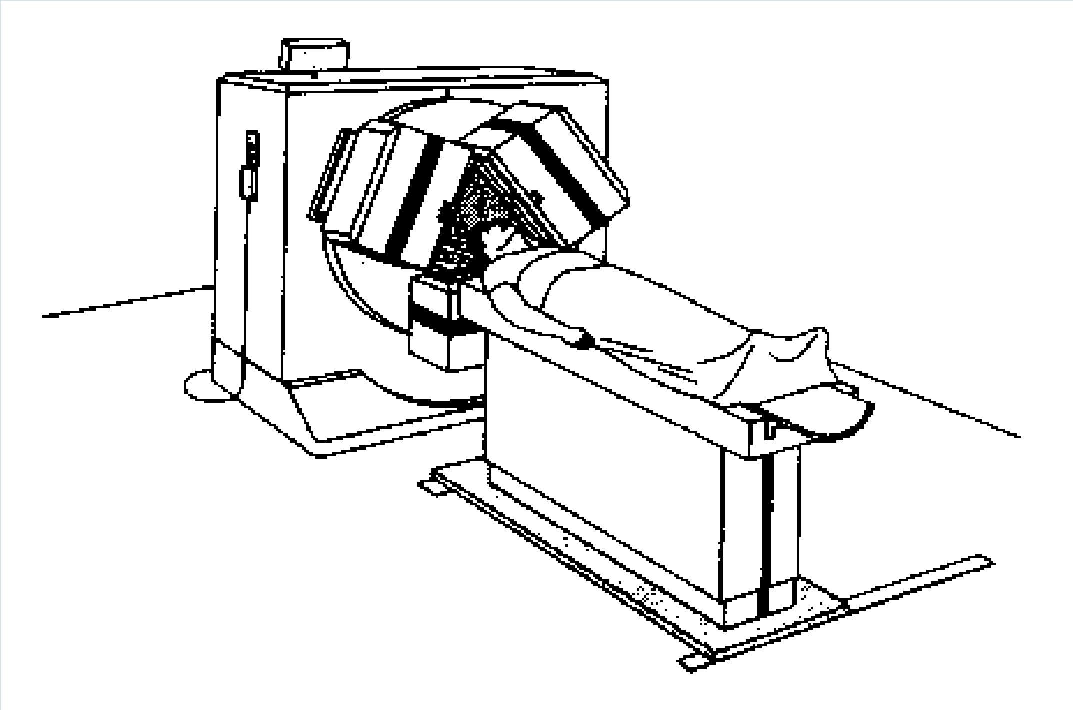
P2 ➤ P1



example 2: retinal image



example 3: patient scan



examples 1–3

record	input device
❶ image file	xv display program
❷ retinal irradiance	receptive fields
attenuated X-rays ❸ e^+ emission radiofrequency waves	CT scanner SPECT scanner MR scanner

generic situation: “duality”

record	input device (format)
source field = raw image	detector device = filter

example 1: classical image representation

☞ "hidden" duality: image independent of filter paradigm:

- image ≡ raw image
- f_{ij}, $(i, j) \in \mathbb{Z}^2$
- $f(x, y)$, $(x, y) \in \mathbb{R}^2$

☞ operations defined directly on raw image

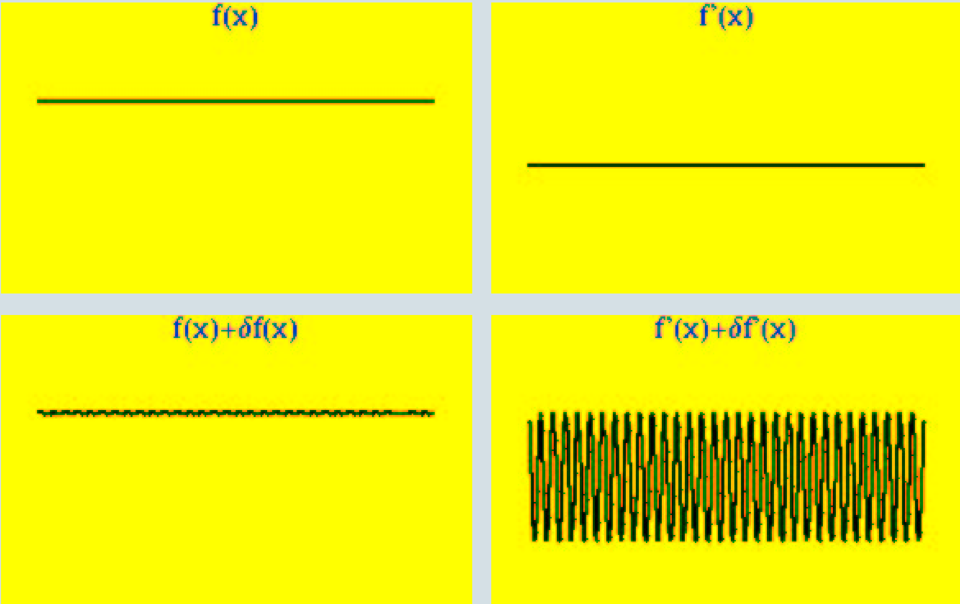
example 2: classical differentiation

☞ $f(x, y) \mapsto f'(x, y)$ ill-posed

☞ $f_{ij} \mapsto f'_{ij}$ ill-defined (no "infinitesimals" on discrete grid)

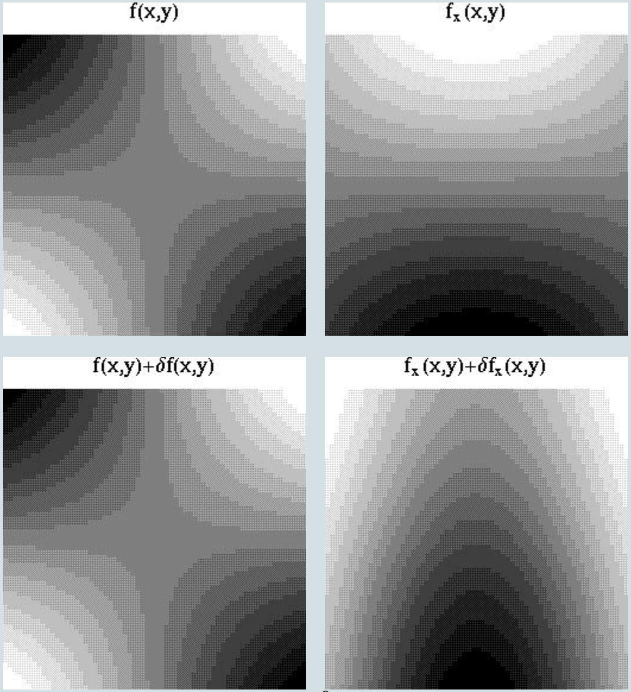
ill-posedness

$$f(x) + \delta f(x) \mapsto f'(x) + \delta f'(x)$$



ill-posedness

$$f(x,y) + \delta f(x,y) \mapsto f_x(x,y) + \delta f_x(x,y)$$



ill-posedness: possible causes

✘ lack of smoothness?

no:

- continuous space: $f, \delta f \in C^\infty$ in all examples!
- discrete grid: what *is* smoothness?

✘ wrong function topology?

yes:

- ∇ is well-posed if

$$f \approx g \quad \text{iff} \quad f(x) \approx g(x) \;\wedge\; \nabla f(x) \approx \nabla g(x)$$

or

$$\delta f \approx 0 \quad \text{iff} \quad \delta f(x) \approx 0 \;\wedge\; \nabla \delta f(x) \approx 0$$

but. . . :

- constraints on δf unrealistic: no control over noise!

ill-posedness: possible solutions

✘ smoothing (regularisation)?

odd:

- if lack of smoothness is not the problem then how could smoothing solve it?
- smoothness—therefore smoothing—ill-defined for discrete data
- one should not tamper with the evidence!

✔ duality!

conceptually correct:

- evidence = “read-only”
- evidence “generates” image
- generator = input device (format) = filter

ill-posedness: possible solutions

☞ ill-posedness ≈ discontinuity:

$$\frac{d}{dx}: \quad f(x) + \delta f(x) \mapsto f'(x) + \delta f'(x)$$

define

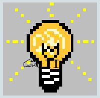
$$\delta f(x) = \varepsilon\, g\left(\frac{x}{\delta}\right) \quad \text{with} \quad 0 < \delta, \varepsilon \ll 1$$

and

$$g(x), g'(x), g''(x), g'''(x), \ldots \leq 1 \quad \forall\, x$$

then

$$\begin{aligned} \delta f'(x) &= \frac{\varepsilon}{\delta}\, g'\left(\frac{x}{\delta}\right) \\ \delta f''(x) &= \frac{\varepsilon}{\delta^2}\, g''\left(\frac{x}{\delta}\right) \\ \delta f'''(x) &= \frac{\varepsilon}{\delta^3}\, g'''\left(\frac{x}{\delta}\right) \qquad \text{etc.} \end{aligned}$$



Kant/Russell

"Ich werde ... jederzeit nur unter den Bedingungen der Sinnlichkeit vergleichen müssen, und so werden Raum und Zeit nicht Bestimmungen der Dinge an sich, sondern der Erscheinungen sein: was die Dinge an sich sein mögen, weiß ich nicht, und brauche es auch nicht zu wissen, weil mir doch niemals ein Ding anders, als in der Erscheinung vorkommen kann."

—IMMANUEL KANT

"It has appeared that, if we take any common object of the sort that is supposed to be known by the senses, what the senses immediately tell us is not the truth about the object as it is apart from us, but only the truth about certain sense-data which, so far as we can see, depend upon the relations between us and the object. Thus what we directly see and feel is merely 'appearance', which we believe to be a sign of some 'reality' behind."

—BERTRAND RUSSELL

duality: Kant/Russell formalised

> *"But if the reality is not what appears, have we any means of knowing whether there is any reality at all? And if so, have we any means of finding out what it is like?"*
>
> *"...every principle of simplicity urges us to adopt the natural view, that there really are objects other than ourselves and our sense-data which have an existence not dependent upon our perceiving them."*
>
> —BERTRAND RUSSELL

❶ 'Dinge an sich', 'reality' ➧ *raw images*

$$f \in \Sigma$$

❷ 'Sinnlichkeit', 'senses' ➧ *filters*

$$\phi \in \Delta$$

❸ 'Erscheinungen', 'sense-data', 'appearance' ➧ *samples*

$$\{f, \phi\} \longrightarrow \langle f | \phi \rangle \in \mathbb{R}$$

duality

raw image: $f \in \Sigma$

★

filter: $\phi \in \Delta$

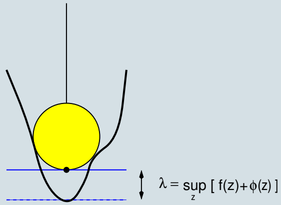
⇊

sample: $\{f, \phi\} \longrightarrow \langle f | \phi \rangle \in \mathbb{R}$

duality: instances

❶ Morphological Duality:

$$\langle f|\phi\rangle = \begin{cases} \sup_z & [\,f(z)+\phi(z)\,] \\ \inf_z & [\,f(z)-\phi(z)\,] \end{cases}$$



❷ Topological Duality:

$$\langle f|\phi\rangle = \int dz\, f(z)\,\phi(z)$$

duality: instances

❸ Pseudo-Topological Duality:

$$\langle f\,|\,\phi\rangle = \frac{1}{\mu}\ln\int dz\; e^{\mu f(z)}\,\phi(z)$$

☞ PTD is sort of "in-between" TD and MD:

- $\lim_{\mu\to 0}$ PTD = TD
- $\lim_{\mu\to\pm\infty}$ PTD = $\text{MD}_{\pm}$

samples versus scans

☞ a *scan* is a coherent set of samples obtained by scanning a region of interest;

☞ 2 options: one can

❶ translate the source, or

❷ translate the detector *in opposite sense*

samples versus scans

- T_x: "translation over x", acting on y yields $x + y$
- T_x acting on $f(y)$ yields $f(x + y)$
- T_x acting on $\phi(y)$ yields $\phi(y - x)$

MD: dilation & erosion

$$f \oplus \phi\,(x) = \sup_y \left[\, f(x+y) + \phi(y)\,\right] = \sup_y \left[\, f(y) + \phi(y-x)\,\right]$$

$$f \ominus \phi\,(x) = \inf_y \left[\, f(x+y) - \phi(y)\,\right] = \inf_y \left[\, f(y) - \phi(y-x)\,\right]$$

TD: correlation

$$f \star \phi\,(x) = \int dy\, f(x+y)\,\phi(y) = \int dy\, f(y)\,\phi(y-x)$$

✍ henceforth: TD

generalisation

- $\theta(y)$: any transformation acting on “empty space”
- $\theta^* f$—*pull back*—acts on Σ
- $\theta_* \phi$—*push forward*—acts on Δ
- by definition:

$$\langle \theta^* f | \phi \rangle \stackrel{\text{def}}{=} \langle f | \theta_* \phi \rangle$$

☞ admissible θ’s constitute a group Θ

☞ abstract infix notation:

$$\{f, \theta, \phi\} \longrightarrow \langle f | \theta | \phi \rangle$$

groups and semigroups

semigroup

- ❶ for all $\eta, \theta \in \Theta$: $\eta \circ \theta \in \Theta$
- ❷ for all $\delta, \eta, \theta \in \Theta$: $(\delta \circ \eta) \circ \theta = \delta \circ (\eta \circ \theta)$
- ❸ there exists a unique id $\in \Theta$ such that for all $\theta \in \Theta$: id $\circ\, \theta = \theta \circ$ id $= \theta$

group

- ❹ for all $\theta \in \Theta$ there exists a unique $\theta^{\text{inv}} \in \Theta$ such that $\theta \circ \theta^{\text{inv}} = \text{id}$

abelian (semi)group

- ❺ for all $\eta, \theta \in \Theta$: $\eta \circ \theta = \theta \circ \eta$

push forward & pull back

$$\theta^* f \stackrel{\text{def}}{=} f \circ \theta$$

&

$$\underbrace{\langle \theta^* f | \phi \rangle \stackrel{\text{def}}{=} \langle f | \theta_* \phi \rangle}_{\textbf{change of dummies}} \stackrel{\text{def}}{=} \langle f | \, \theta \, | \phi \rangle$$

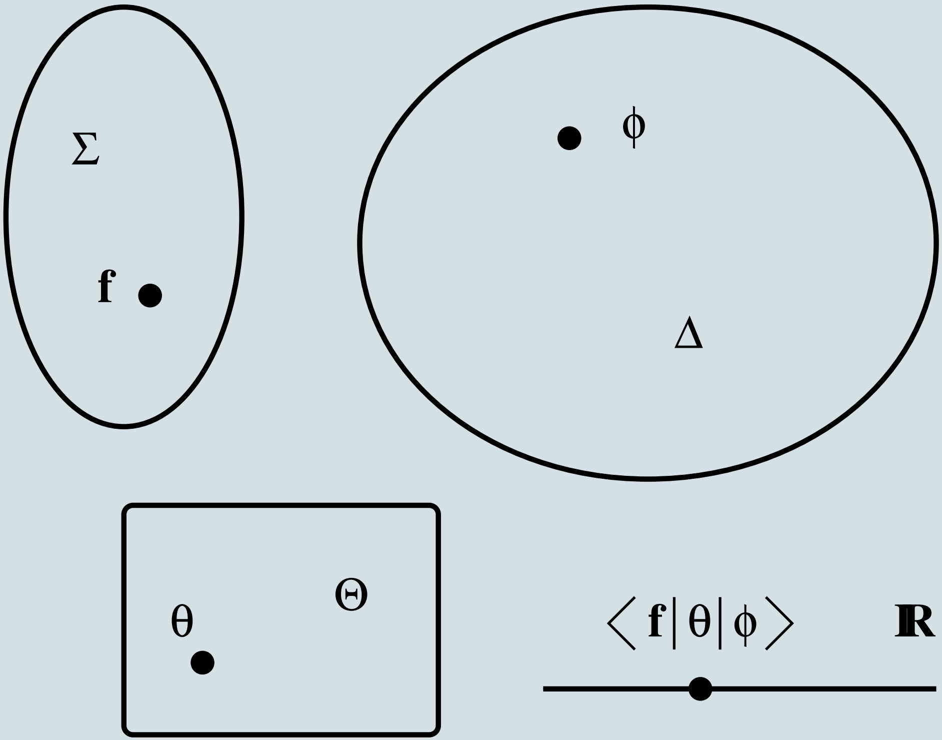
$\Downarrow$

$$\theta_* \phi = \underbrace{\left| \det \nabla \theta^{\text{inv}} \right|}_{\textbf{jacobian}} \phi \circ \theta^{\text{inv}}$$

✍ θ^* = pull back

✍ θ_* = push forward

overview: $f \in \Sigma$, $\phi \in \Delta$, $\theta \in \Theta$, $\langle f|\theta|\phi\rangle \in \mathbb{R}$



covariance: active and passive view

$$\langle \theta^* f | \theta_*^{\text{inv}} \phi \rangle = \langle f | \phi \rangle$$

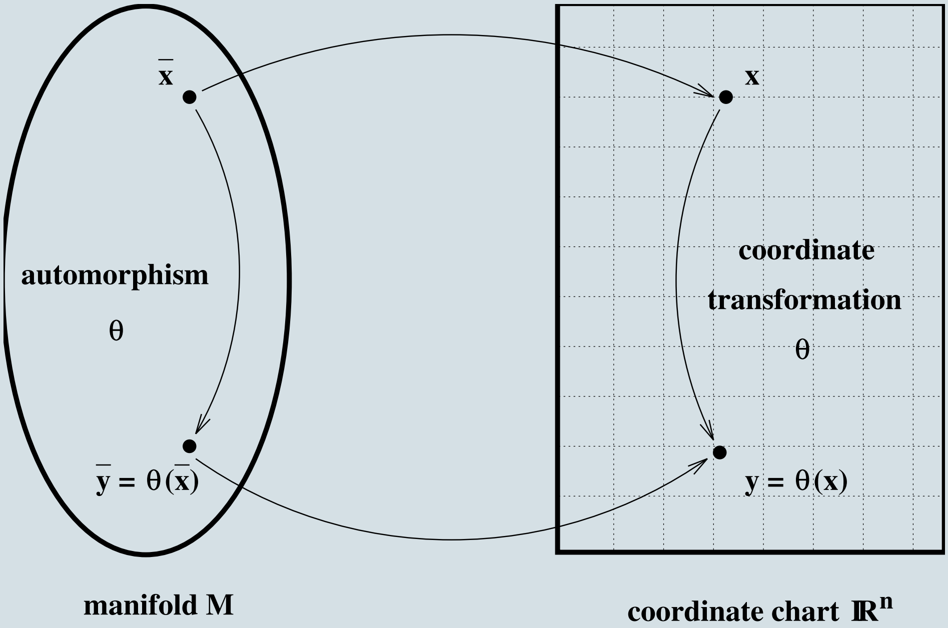
active view

- $\theta : \mathbf{M} \to \mathbf{M}$
- "warping of spacetime (and everything in it)"

passive view

- $\theta : \mathbb{R}^n \to \mathbb{R}^n$
- "coordinate transformation" = "relabelling of spacetime"

covariance: active and passive view



Active view: θ "warps"
the spacetime manifold

Passive view: θ "relabels"
the spacetime manifold

Einstein/Schlick

"Spacetime does not claim existence on its own, but only as a structural quality of the field."

—ALBERT EINSTEIN

"Space and time are not measurable in themselves: they only form a framework into which we arrange physical events."

—MORITZ SCHLICK

- $\theta(x)$ "transformation of a void" ???
- $\theta^* f$ "raw image warping"
- $\theta_* \phi$ corresponding filter adaptation, such that

$$\langle \theta^* f | \phi \rangle = \langle f | \theta_* \phi \rangle$$

group Θ

- Θ = symmetry group of "empty spacetime"
- "classical spacetime":
  - ❶ homogeneity = spacetime shift invariance
  - ❷ isotropy = spatial rotation invariance
  - ❸ spacetime scale invariance
- $\theta \in \Theta$ = measurement (control) parameter:
  - ❶ location and moment of interest
  - ❷ orientation
  - ❸ resolution

$$\theta \underbrace{\begin{bmatrix} t' \\ \vec{x}' \end{bmatrix}}_{\textbf{dummy}} \stackrel{\text{def}}{=} \exp\left[\begin{array}{c|c} \mu & \emptyset \\ \hline \emptyset & \lambda \boldsymbol{I} \end{array}\right] \left[\begin{array}{c|c} 1 & \emptyset \\ \hline \emptyset & \boldsymbol{R} \end{array}\right] \underbrace{\begin{bmatrix} t' \\ \vec{x}' \end{bmatrix}}_{\textbf{dummy}} + \begin{bmatrix} t \\ \vec{x} \end{bmatrix}$$

group Θ

☞ transformation of spacetime:

$$\theta(z) = \mathbf{A}z + x$$

☞ pull back $f \in \Sigma$:

$$\theta^* f(z) = f\left(\mathbf{A}z + x\right)$$

☞ push forward $\phi \in \Delta$:

$$\theta_* \phi(z) = \frac{1}{\det \mathbf{A}} \phi\left(\mathbf{A}^{-1}(z - x)\right)$$

device space Δ

❶ group Θ?

❷ device space Δ?

assumptions:

❶ genericity: Δ "sufficiently large"

❷ plausibility: Δ physically realizable

❸ consistency: Δ "closed"

Δ: consistency

❶ genericity: Δ "sufficiently large"

❷ plausibility: Δ physically realizable

❸ consistency: Δ "closed":

$$f \in \Sigma \quad \Rightarrow \quad f \star \phi \in \Sigma \quad \forall \phi \in \Delta$$

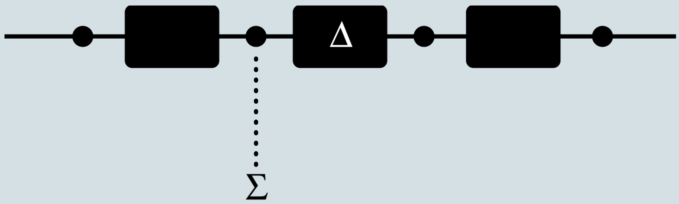
$$(f \star \phi) \star \psi = f \star (\phi * \psi)$$

algebraic closure:

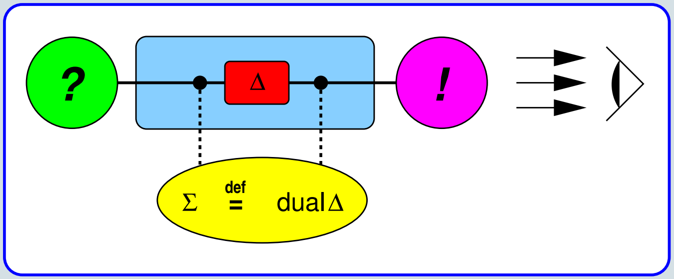
$$\phi, \psi \in \Delta \quad \Rightarrow \quad \phi * \psi \in \Delta$$

Δ: consistency

$$\textbf{if } f \in \Sigma \textbf{ then } f \star \phi \in \Sigma \textbf{ for all } \phi \in \Delta$$



$\approx$



☞ one piece Δ representative of entire chain

algebra

algebra

- group with 1 external operation and 2 internal operations

❶ "scalar multiplication"

❷ "addition"

❸ "multiplication" (or "concatenation")

❶ + ❷ vector space

❸ algebra

algebra

algebra

- ❶ $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in \mathcal{A}$
- ❷ $a \circ (b + c) = a \circ b + a \circ c$ for all $a, b, c \in \mathcal{A}$
- ❸ $(a + b) \circ c = a \circ c + b \circ c$ for all $a, b, c \in \mathcal{A}$
- ❹ $\lambda(a \circ b) = (\lambda a) \circ b = a \circ (\lambda b)$ for all $a, b \in \mathcal{A}, \lambda \in \mathbb{R}$

algebra with identity

- for all $a \in \mathcal{A}$ there esists a unique $e \in \mathcal{A}$ such that $e \circ a = a \circ e = a$

regular algebra

- algebra with identity
- for all $a \in \mathcal{A}$ there exists a unique a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = e$

commutative algebra

- $a \circ b = b \circ a$ for all $a, b \in \mathcal{A}$

device space Δ

❶ genericity: Δ “sufficiently large”

❷ plausibility: Δ physically realizable

❸ consistency: Δ “closed”

☞ possible choice: Schwartz functions:

$$\Delta \stackrel{\text{def}}{=} \mathcal{S}(\mathbb{R}^n)$$

☞ tempered distributions:

$$\Sigma \stackrel{\text{def}}{=} \mathcal{S}'(\mathbb{R}^n)$$

$$\Delta \stackrel{\text{def}}{=} \mathcal{S}(\mathbb{R}^n)$$

$$\phi \in \mathcal{S}(\mathbb{R}^n)$$

iff

$$\phi \in C^\infty(\mathbb{R}^n)$$

and

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \nabla_\beta \phi(x)| < \infty$$

$\Sigma \stackrel{\text{def}}{=} \mathcal{S}'(\mathbb{R}^n)$

$$T \in \mathcal{S}'(\mathbb{R}^n)$$

iff

$$|T[\phi]| \leq c \sup_{x \in \mathbb{R}^n} |x^\alpha \nabla_\beta \phi(x)|$$

special case: regular tempered distributions

Riesz representation formula

$$F[\phi] = \int dx\, f(x)\, \phi(x)$$

prototype non-regular tempered distribution:

$$\delta[\phi] = \phi(0)$$

☞ notation: F has Riesz representation $f(x)$

☞ common practice: Riesz representation in *all* cases

- $F = \delta$ has "Riesz representation" $f(x) = \delta(x)$
- δ important in "reverse engineering"

derivatives of tempered distributions

$$\nabla F[\phi] = -F[\nabla\phi] = \langle f|\nabla|\phi\rangle$$

- ☞ well-posed
- ☞ operationally well-defined
- ☞ correspondence principle:

$$\textbf{if } F \sim f \in C^1 \textbf{ then } \nabla F \sim \nabla f \in C^0$$

image definition ❶

$$\{f, \theta, \phi\} \longrightarrow \langle f|\theta|\phi\rangle \text{ with } \theta(z) = Az + x$$

❶ raw image $f \in \mathcal{S}'(\mathbb{R}^n)$

❷ spacetime parameters $\theta \in \Theta$

❸ filter $\phi \in \mathcal{S}(\mathbb{R}^n)$

image definition

$$\{f, \theta, \phi\} \longrightarrow \langle f|\theta|\phi\rangle \textbf{ with } \theta(z) = Az + x$$

problems

- ☞ $\Delta = \mathcal{S}(\mathbb{R}^n)$ too large for implementation
- ☞ no notion of “differential order”
- ☞ need for “0th order” = “point operator”
- ☞ id $\notin \mathcal{S}(\mathbb{R}^n)$

image definition: point operator

$$\{f,\theta,\phi\} \longrightarrow \langle f|\theta|\phi\rangle \quad \text{with} \quad \theta(z) = Az + x$$

ansatz: $\phi \in \mathcal{S}^+(\mathbb{R})$

- ☞ consistency: ϕ generates autoconvolution algebra
- ☞ unique solution:

$$\phi(z;x,\Lambda) = \frac{1}{\sqrt{2\pi}^n}\frac{1}{\sqrt{\det\Lambda}}\exp\left[-\frac{1}{2}(z-x)^\mu\,\Lambda^{-1}_{\mu\nu}\,(z-x)^\nu\right]$$

image definition: point operator

$$\phi(z;x,\Lambda) = \frac{1}{\sqrt{2\pi}^n}\frac{1}{\sqrt{\det\Lambda}}\exp\left[-\frac{1}{2}(z-x)^\mu\,\Lambda^{-1}_{\mu\nu}\,(z-x)^\nu\right]$$

☞ $\phi(z;x,\Lambda) = \theta_*\phi(z)$ with

$$\phi(z) = \frac{1}{\sqrt{2\pi}^n}\exp\left[-\frac{1}{2}z^2\right] \quad \text{and} \quad \theta(z) = \sqrt{\Lambda}\,X\,z + x$$

☞ $X\,X^{\mathrm{T}} = X^{\mathrm{T}}\,X = \mathrm{id}$: "hyperrotation"

☞ Λ: symmetric, positive definite matrix

☞ classical spacetime group: *strict* subset of affine group:

$$X = \left[\begin{array}{c|c} 1 & \emptyset \\ \hline \emptyset & \boldsymbol{R} \end{array}\right] \quad \text{and} \quad \sqrt{\Lambda} = \left[\begin{array}{c|c} \tau & \emptyset \\ \hline \emptyset & \sigma\boldsymbol{I} \end{array}\right]$$

image definition ❷

$$\{f,\theta,\phi\} \longrightarrow \langle f|\theta|\phi\rangle \quad \text{with} \quad \theta(z) = Az + x$$

❶ raw image $f \in \mathcal{G}'(\mathbb{R}^n)$

❷ spacetime parameters $\theta \in \Theta$

$$\underbrace{\theta\begin{bmatrix} t\,' \\ \vec{x}\,' \end{bmatrix}}_{\textbf{dummy}} \stackrel{\text{def}}{=} \exp\left[\begin{array}{c|c} \mu & \emptyset \\ \hline \emptyset & \lambda\,\boldsymbol{I} \end{array}\right]\left[\begin{array}{c|c} 1 & \emptyset \\ \hline \emptyset & \boldsymbol{R} \end{array}\right]\underbrace{\begin{bmatrix} t\,' \\ \vec{x}\,' \end{bmatrix}}_{\textbf{dummy}} + \begin{bmatrix} t \\ \vec{x} \end{bmatrix}$$

❸ filter $\phi \in \mathcal{G}(\mathbb{R}^n)$

$$\phi(\vec{x}\,',t\,';\vec{x},t,\sigma,\tau) = \frac{1}{\sqrt{2\pi\sigma^2}^{\,n-1}}\exp\left[-\frac{1}{2}\frac{\|\vec{x}\,'-\vec{x}\|^2}{\sigma^2}\right]\frac{1}{\sqrt{2\pi\tau^2}}\exp\left[-\frac{1}{2}\frac{(t\,'-t)^2}{\tau^2}\right]$$

image definition ❷

summary

☞ an image I is defined by a triple $\{f, \phi, \Theta\}$:

- $f \in \Sigma$: raw image
- $\phi \in \Delta$: filter/point operator
- Θ: spacetime symmetry group

together with a duality paradigm $\Sigma = \text{dual}\,\Delta$:

- $I \stackrel{\text{def}}{=} \langle f|\Theta|\phi\rangle \stackrel{\text{def}}{=} \{\langle f|\theta|\phi\rangle \mid \theta \in \Theta\}$

☞ duality options:

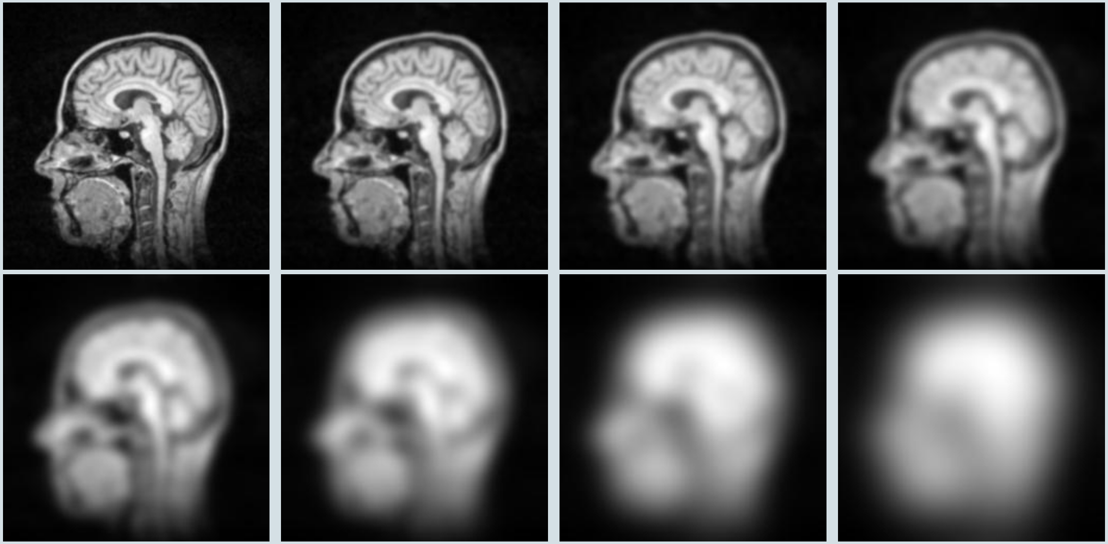
- topological:
  ❶ Schwartz' theory ("image processing")
  ❷ scale-space theory ("image representation")
- morphological: dilation & erosion scale-space based on QSF
- other options

image definition: Gaussian scale-space

summary

- ☞ an image is operationally defined by correlation of raw data and Gaussian point operator
- ☞ an image is a smooth function of spacetime $(t, \vec{x}) \in \mathbb{R}^n$
- ☞ an image depends on spatiotemporal scale parameters $(\tau, \sigma) \in \mathbb{R}^+ \times \mathbb{R}^+$
- ☞ image derivatives are operationally defined by correlation of raw data and transposed derivatives of the Gaussian point operator
- ☞ image differentiation is well-posed

scale-space image



data, models, and images: PDE approach

- PDE framework:

Σ	$\sim$	initial conditions
Δ + duality	$\sim$	propagators
Θ	$\sim$	symmetry group

- example:
  - topological duality $\longleftrightarrow$ linear scale-space:

$$\partial_s u = \Delta u$$

  - morphological duality $\longleftrightarrow$ dilation & erosion scale-space:

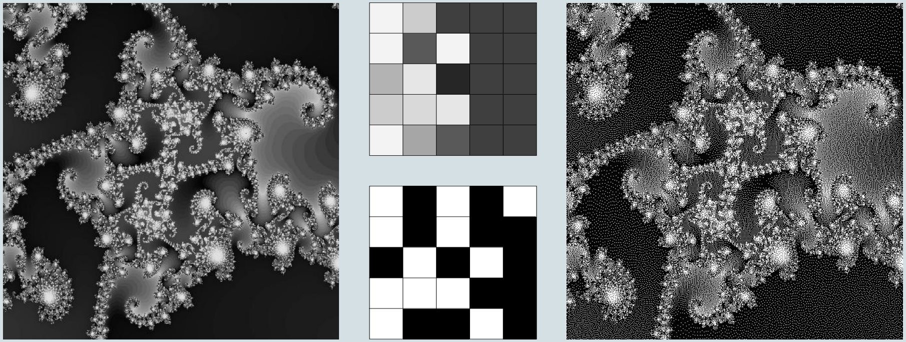
$$\partial_s u = \pm \|\nabla u\|^2$$

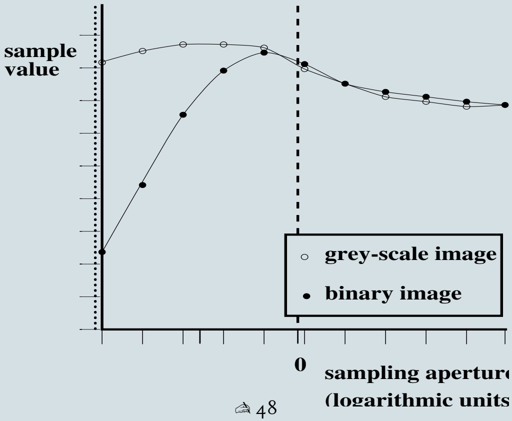
  - other types of duality: other PDE's, *e.g.*

$$\partial_s u = \frac{1}{1+|\mu|} \left(\Delta u + \mu \|\nabla u\|^2\right)$$

technische universiteit eindhoven

scale-space image: “deep structure”





analogies

image representation: analogies

❶ Brownian motion

- Einstein's argument (diffusion)
- functional integration

❷ regularisation

❸ entropy

$$S[u,f] = \int dx\, f(x)\, u(x) - \frac{1}{2} \sum_{i=0}^{\infty} \int dx\, \frac{t^i}{i!}\, u^{(i)}(x)\, u^{(i)}(x)$$

deep structure: study an image as a family

❶ spurious resolution

❷ iso-intensity linking

❸ catastrophe theory

deep structure

❶ spurious resolution

non-enhancement principle in nD:

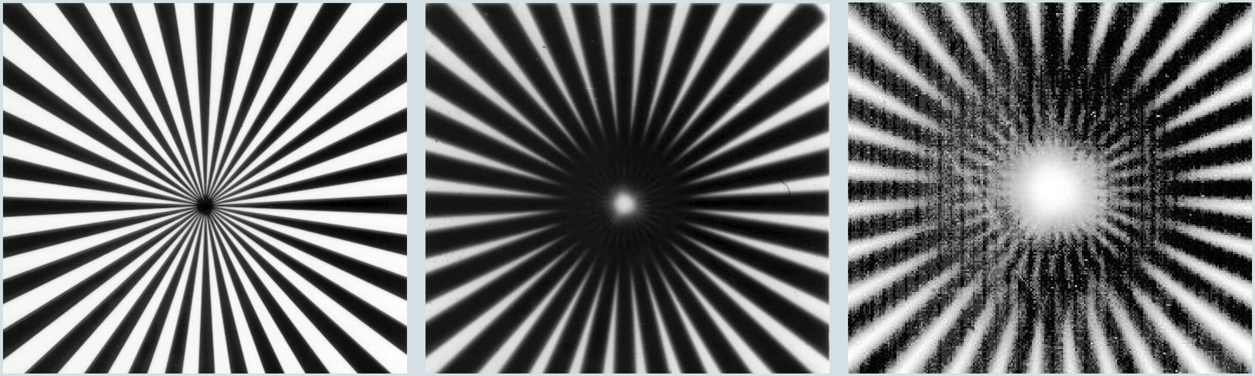
$$\frac{\partial f}{\partial s} = \sum_{i,j=1}^{n} \alpha^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_{i=1}^{n} \beta^i \frac{\partial f}{\partial x^i}$$

α symmetric positive definite matrix

☞ homogeneity, isotropy & parametrization: $\alpha(x; s) = \mathbf{I}$

deep structure

❶ spurious resolution



deep structure

❷ iso-intensity linking

- basic principle: $\mathcal{L}_v f(x;s) = 0$

$$\frac{\partial f}{\partial s} v^0 + \frac{\partial f}{\partial x^i} v^i = 0$$

- gauge condition: "proximity"

$$\vec{v} \parallel \nabla f \quad \text{and} \quad v^0 = \|\nabla f\|^2$$

- solution: linking field

$$(v^0; \vec{v}) = \left(\|\nabla f\|^2; -\Delta f \, \nabla f\right)$$

deep structure

❸ catastrophe theory

☞ control parameter: scale

☞ constraint: diffusion equation

morsification of generic events in 2D

❶ annihilation event

$$f(x, y; s) = x^3 + 6xs + \alpha(y^2 + 2s)$$

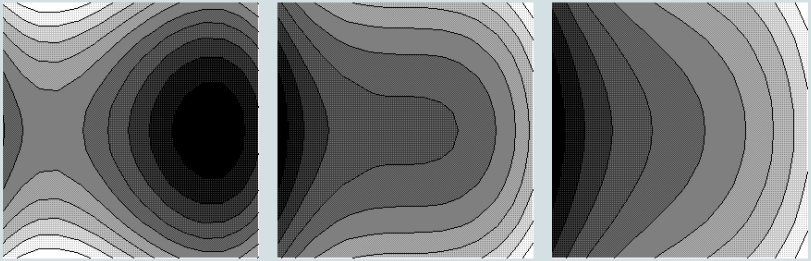
❷ creation event

$$f(x, y; s) = x^3 - 6xs - 6xy^2 + \alpha(y^2 + 2s)$$

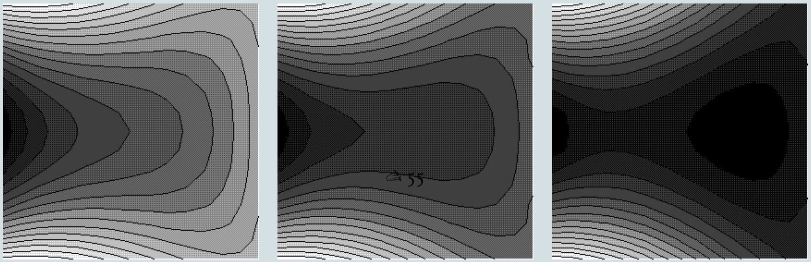
deep structure

❸ catastrophe theory

$$f(x, y; s) = x^3 + 6xs + \alpha(y^2 + 2s)$$

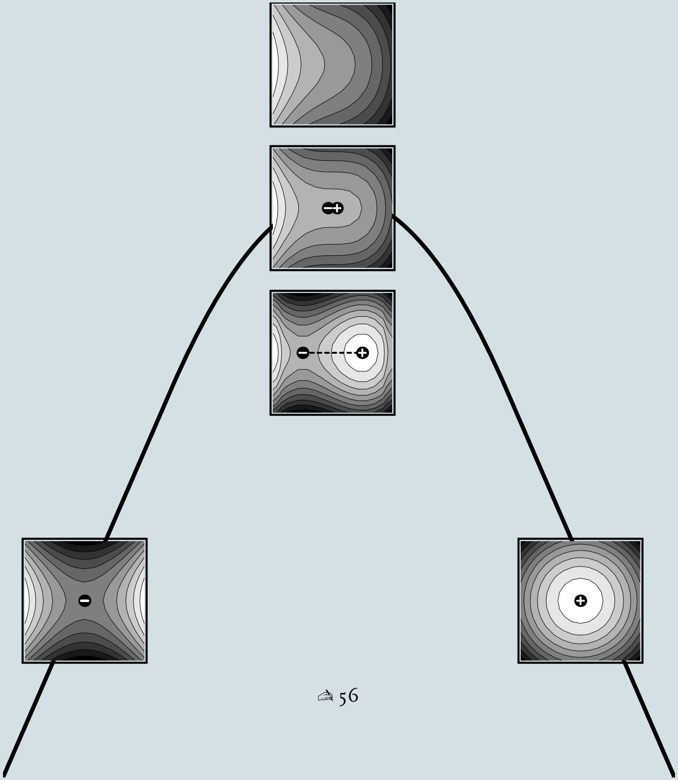


$$f(x, y; s) = x^3 - 6xs - 6xy^2 + \alpha(y^2 + 2s)$$



deep structure

❸ catastrophe theory



multiscale local jet

ansatz: $u(x;t) = e^{t\Delta + x\cdot\nabla} f(x=0)$

truncation:

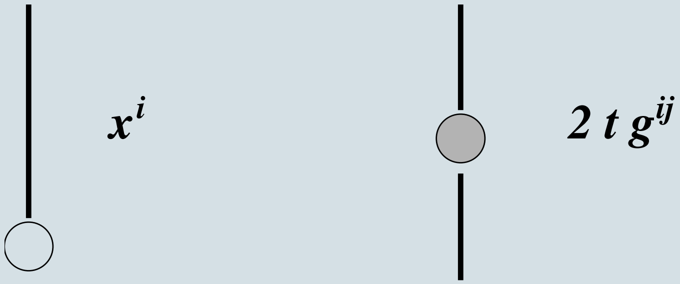
$$u_k(x;t) = \sum_{m=0}^{k} u_{i_1\ldots i_m} P^{i_1\ldots i_m}(x;t)$$

- homogeneity constraint: discard $x^\alpha t^j$ if $|\alpha| + 2j > k$
- $u_{i_1\ldots i_m} = (-1)^m F[\phi_{i_1\ldots i_m}]$
  - $m \leq k$: measurements
  - $m > k$: metamerism
- $P^{i_1\ldots i_m}(x;t)$: image model

multiscale local jet

multiscale local jet of order k:

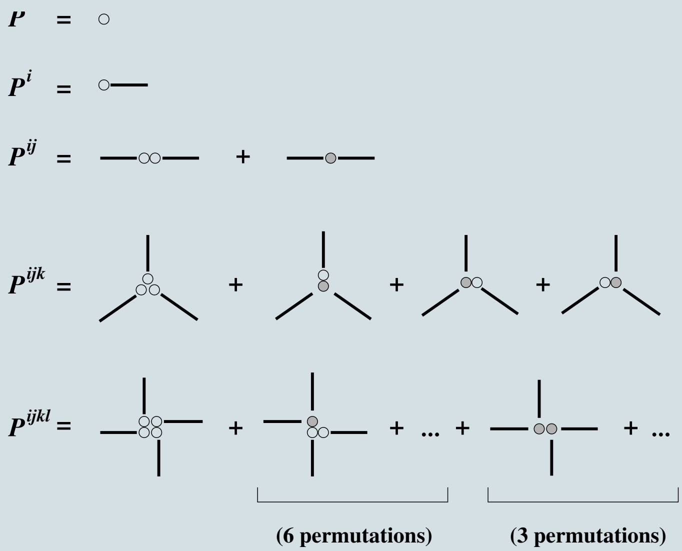
$$u_k(x;t) = \sum_{m=0}^{k} u_{i_1 \dots i_m} P^{i_1 \dots i_m}(x;t)$$



multiscale local jet

multiscale local jet of order k:

$$u_k(x;t) = \sum_{m=0}^{k} u_{i_1 \ldots i_m} \, P^{i_1 \ldots i_m}(x;t)$$

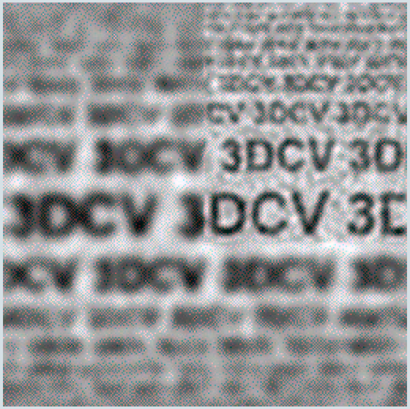


multiscale local jet

when to use $P^{i_1 \ldots i_m}(x;t)$

☞ theory & applications, such as

- local study of deep structure, e.g. catastrophe germs in scale-space
- interpolation, e.g. iso-intensity linking with sub-pixel precision
- extrapolation, *e.g.* "deblurring"
- other sub-pixel algorithms



temporal causality

❶ off-line analysis:

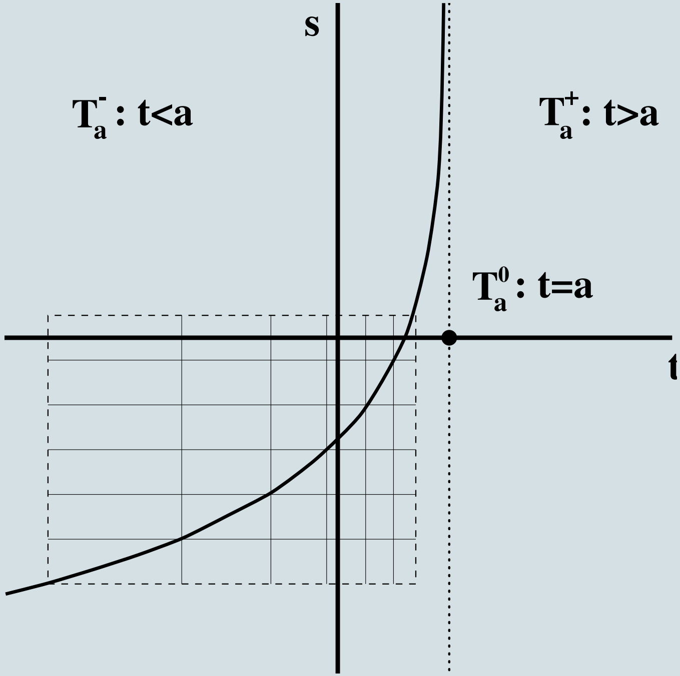
- no time horizon
- no causality problem

❷ real-time analysis:

- time horizon ("present")
- causality problem
- *e.g.* vision, stock market behaviour, etc.

temporal causality

manifest temporal causality: map past onto real axis isomorphically



temporal causality

manifest temporal causality: map past onto real axis isomorphically

☞ "classical time": affine invariance

❶ temporal shift invariance

❷ temporal scale invariance

$$\frac{ds_a(t)}{dt} \propto \frac{1}{a-t}$$

$$s_a(t) = -\ln\frac{a-t}{\tau} \; i.e. \; t_a(s) = a - \tau e^{-s}$$

☞ τ: intrinsic delay: $s_a(a-\tau) = 0$

temporal causality

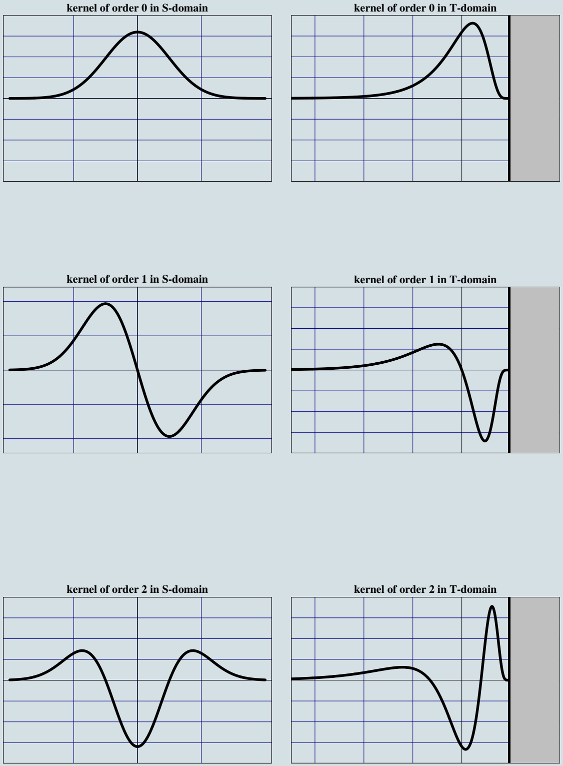
$$s_a(t) = -\ln \frac{a-t}{\tau} \quad i.e.\ t_a(s) = a - \tau e^{-s}$$

$$\begin{aligned} t_a^* f(s) &\stackrel{\text{def}}{=} (f \circ t_a)(s) \\ t_{*a}\phi(t) &\stackrel{\text{def}}{=} (\dot{s}_a\, \phi \circ s_a)(t) \end{aligned}$$

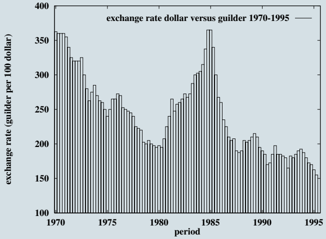
$$\begin{array}{ccc} \phi \in \Delta & \xrightarrow{t_{a*}} & \varphi_a \in t_{a*}\Delta \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_a} \\ s & \xrightarrow{t_a} & t \\ {\scriptstyle \pi_a}\uparrow & & \uparrow{\scriptstyle \pi} \\ f_a \in t_a^*\Sigma & \xleftarrow{t_a^*} & f \in \Sigma \end{array}$$

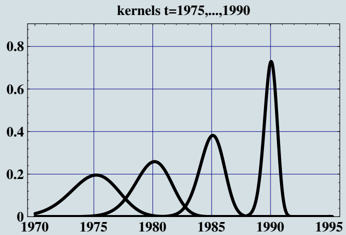
- ☞ $t_a^* f$: “causal source”
- ☞ $t_{a*}\phi$: “causal filter”
- ☞ $\langle f | t_a | \phi \rangle$: “causal sample” (unbiased view)

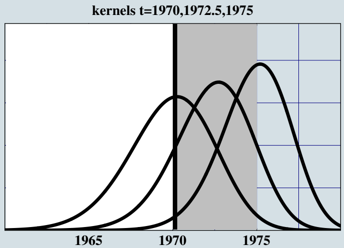
temporal causality

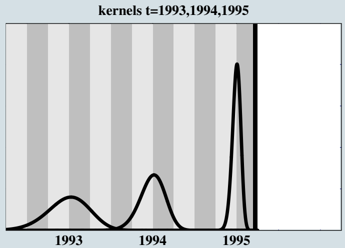


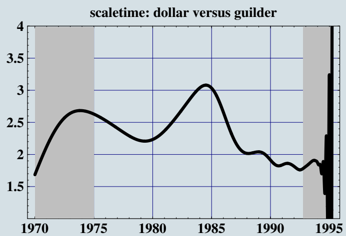
temporal causality

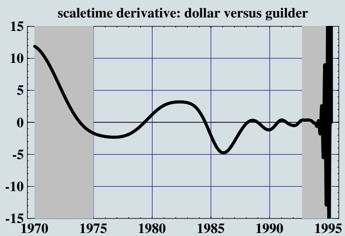












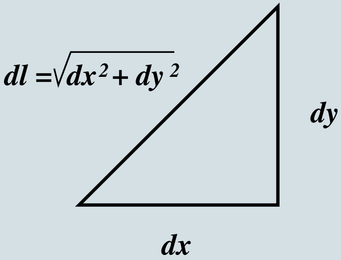
temporal causality

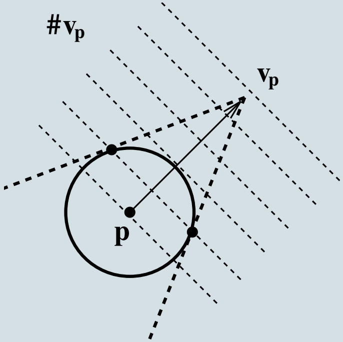
summary

- ☞ causality and "scale-time" demands are compatible
- ☞ causal signals depend on the present moment as well as on 2 free parameters:
  1. temporal resolution
  2. temporal delay
- ☞ a causal signal is C^∞ but not C^ω
- ☞ the present moment is a logarithmic singularity
- ☞ all derivatives vanish smoothly towards the present moment
- ☞ thus one cannot "Taylor" into the future

tensor calculus

Euclidean metric



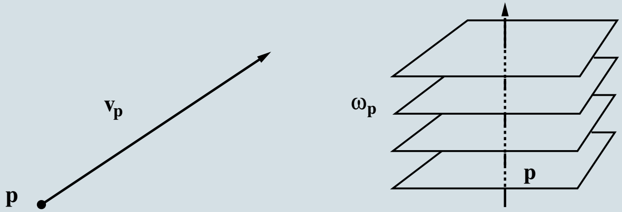


classical picture: $d\ell^2 \stackrel{\text{def}}{=} g_{ij}\, dx^i dx^j$

modern picture: $G \stackrel{\text{def}}{=} g_{ij}\, dx^i \otimes dx^j$

tensor calculus

vectors, covectors, tensors



dual basis: $\widetilde{e}^i[\vec{e}_j] \stackrel{\text{def}}{=} \delta^i_j$

holonomic basis: $\widetilde{e}^i = dx^i \quad \vec{e}_j = \dfrac{\partial}{\partial x^j}$

vectors: $\vec{v} \stackrel{\text{def}}{=} v^i\, \vec{e}_i$

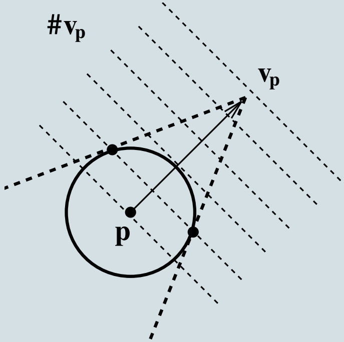
covectors: $\widetilde{\omega} \stackrel{\text{def}}{=} \omega_i\, \widetilde{e}^i$

tensors: $T \stackrel{\text{def}}{=} T^{i_1 \ldots i_l}_{j_1 \ldots j_k}\, \vec{e}_{i_1} \otimes \ldots \otimes \vec{e}_{i_l} \otimes \widetilde{e}^{j_1} \otimes \ldots \otimes \widetilde{e}^{j_k}$

technische universiteit eindhoven

tensor calculus

metric representations



metric: $G(\blacksquare\blacksquare) = g_{ij}\, dx^i \otimes dx^j$

dual metric: $H(\blacksquare\blacksquare) = g^{ij}\, \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$

Kronecker: $I(\blacksquare\blacksquare) = \delta^i_j\, dx^i \otimes \frac{\partial}{\partial x^j}$

tensor calculus

metric representations

abstraction: $\mathbf{G} \stackrel{\text{def}}{=} \{G, H, I\}$

i.e. $\mathbf{G}($(co)vector, (co)vector$)$ equals

❶ $G($vector, vector$)$ or

❷ $H($covector, covector$)$ or

❸ $I($vector, covector$)$

depending on argument type (vector/covector)

tensor calculus

sharp and flat: index raising and lowering

(co)vector	definition	coordinates
$\tilde{v} = \sharp\vec{v}$	$G(\vec{v}, \square)$	$v_i = g_{ij}\, v^j$
$\vec{v} = \flat\tilde{v}$	$H(\tilde{v}, \square)$	$v^i = g^{ij}\, v_j$

contraction/scalar product:

$$
\begin{aligned}
\tilde{v}[\vec{w}] &= v_i w^i \\
\tilde{v}[\flat\tilde{w}] &= v_i g^{ij} w_j \\
\sharp\vec{v}[\vec{w}] &= g_{ij} v^j w^i \quad \stackrel{\text{def}}{=} \quad \vec{v} \cdot \vec{w} \\
\sharp\vec{v}[\flat\tilde{w}] &= g_{ij} v^j g^{ik} w_k
\end{aligned}
$$

tensor calculus

tensor representations in metric space

abstraction: $\mathbf{T} \stackrel{\text{def}}{=} \{T, \sharp, \flat\}$

- example: if $T = T^i_{jk}\, dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$ then $\mathbf{T}$((co)vector, (co)vector, (co)vector) equals

☞ T(vector, vector, covector) or

☞ T($\flat$covector, vector, $\sharp$vector) or

☞ et cetera

depending on argument type (vector/covector)

tensor calculus

dual linear transformation

if $v'^i = A^i{}_j v^j$ then $\omega'_j = B_j{}^i \omega_i$ with

$$B^{\mathrm{T}} \stackrel{\text{def}}{=} A^{\text{inv}}$$

tensoriality criterion

$$T'^{i_1 \ldots i_l}{}_{j_1 \ldots j_k} = A^{i_1}{}_{p_1} \ldots A^{i_l}{}_{p_l} B_{j_1}{}^{q_1} \ldots B_{j_k}{}^{q_k} T^{p_1 \ldots p_l}{}_{q_1 \ldots q_k}$$

coordinate transformation: $x \mapsto x'$

☞ $A^i{}_j = \dfrac{\partial x'^i}{\partial x^j}$ and $B_j{}^i = \dfrac{\partial x^i}{\partial x'^j}$

tensor calculus

important tensors

- ☞ metric $g_{ij}, g^{ij}, \delta^i_j$
- ☞ filter derivatives $\phi_{i_1 \ldots i_m}$
- ☞ image derivatives $L_{i_1 \ldots i_m} = (-1)^m \, F[\phi_{i_1 \ldots i_m}]$
- ☞ scale-space polynomials $P^{i_1 \ldots i_m}$

caution

- ☞ $\phi_{i_1 \ldots i_m} = \partial_{i_1 \ldots i_m} \phi$ and $L_{i_1 \ldots i_m} = \partial_{i_1 \ldots i_m} L$ holds only in Cartesian frames
- ☞ $\partial_{i_1 \ldots i_m}$ does not "transform as a cotensor"
- ☞ to be defined: covariant derivative

tensor calculus

covariant derivative

connection coefficients: $\dfrac{\partial}{\partial x^k}\,\vec{e}_i \stackrel{\text{def}}{=} \Gamma^j_{ik}\,\vec{e}_j$

duality: $\dfrac{\partial}{\partial x^k}\,\widetilde{e}^j = -\Gamma^j_{ik}\,\widetilde{e}^i$

covariant derivative: $D_k T^{i_1\ldots i_p}_{j_1\ldots j_q} = \partial_k T^{i_1\ldots i_p}_{j_1\ldots j_q} + \text{“affinity terms”}$

example: $\partial_k\left\{T^i_j\,\vec{e}_i \otimes \widetilde{e}^j\right\} = \underbrace{\left\{\partial_k T^i{}_j + \Gamma^i{}_{\mu k} T^\mu{}_j - \Gamma^\mu{}_{jk} T^i{}_\mu\right\}}_{D_k T^i_j}\,\vec{e}_i \otimes \tilde{e}^j$

tensor calculus

covariant derivative

☞ metric is "covariantly constant": $\nabla G = 0$, i.e.

☞ $D_k g_{ij} = 0$

☞ $D_k \delta_i^j = 0$

☞ $D_k g^{ij} = 0$

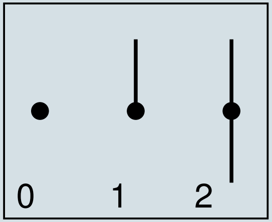
☞ in Cartesian frames: $\Gamma^i_{jk} = 0$, i.e. $D_k = \partial_k$

☞ Γ^i_{jk} is not a tensor

tensor calculus

diagrammar

image derivatives



☞ recall diagrammar scale-space polynomials

tensor calculus

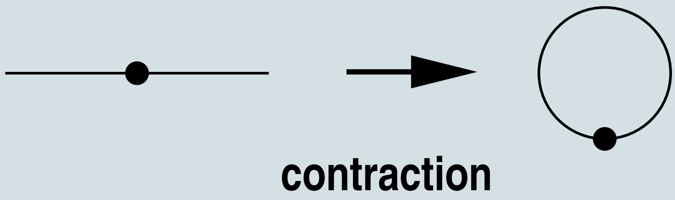
important observations

- ☞ covariance: a tensor equation holds in any coordinate system
- ☞ tensors form a linear space
- ☞ a product of tensors is itself a tensor

tensor calculus

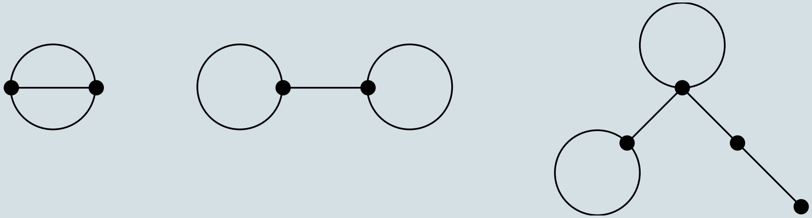
contraction: $T^i_j \mapsto T^i_i$

- ☞ contraction of a tensor of rank $k+2$ yields a tensor of rank k
- ☞ index manifestation: equate 1 upper and 1 lower index
- ☞ metric enables contraction of equal-type indices
- ☞ “diagrammar”: connect dangling branches



tensor calculus: differential invariants

full contraction yields a scalar



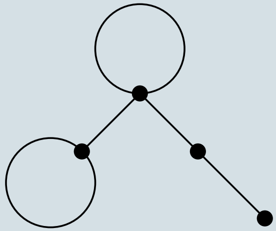
☞ $L_{ijk}L^{ijk}$, $L^{i}_{ij}L^{jk}_{k}$, $L^{i}_{ijk}L^{jl}_{l}L^{k}_{m}L^{m}$

☞ tensor index notation is convenient:

- condensed description
- manifest covariance

tensor calculus: differential invariants

example ❶



$$L^{i}_{ijk} L^{jl}_{l} L^{k}_{m} L^{m} =$$

$$(((((L_{xxxx} + L_{xxyy})(L_{xxx} + L_{xyy})) + ((L_{xxxy} + L_{xyyy})(L_{xxy} + L_{yyy})))L_{xx})$$

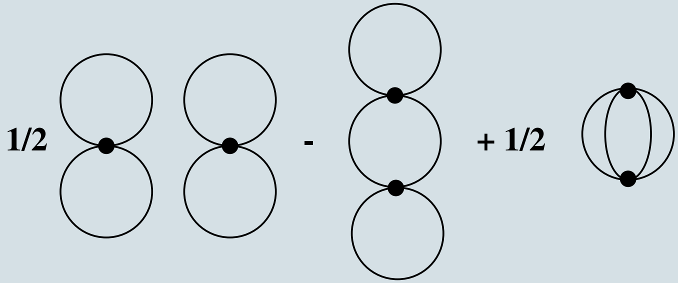
$$+((((L_{xxxy} + L_{xyyy})(L_{xxx} + L_{xyy})) + ((L_{xxyy} + L_{yyyy})(L_{xxy} + L_{yyy})))L_{xy}))L_{x}$$

$$+(((((L_{xxxx} + L_{xxyy})(L_{xxx} + L_{xyy})) + ((L_{xxxy} + L_{xyyy})(L_{xxy} + L_{yyy})))L_{xy})$$

$$+((((L_{xxxy} + L_{xyyy})(L_{xxx} + L_{xyy})) + ((L_{xxyy} + L_{yyyy})(L_{xxy} + L_{yyy})))L_{yy}))L_{y}$$

tensor calculus: differential invariants

example ❷



$$\frac{1}{2}L_{ij}^{ij}L_{kl}^{kl} - L_{ijk}^{i}L_{l}^{jkl} + \frac{1}{2}L_{ijkl}L^{ijkl} =$$
$$L_{xxxx}L_{yyyy} - 4L_{xxxy}L_{xyyy} + 3L_{xxyy}L_{xxyy}$$

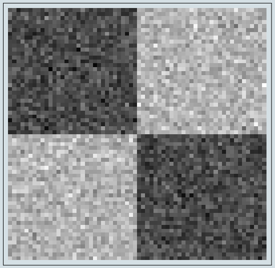
tensor calculus: differential invariants

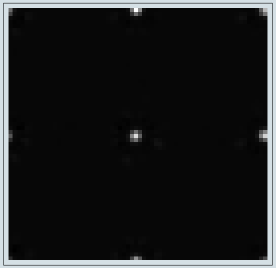
example ❸

$$I = \frac{1}{2} L_{ij}^{ij} L_{kl}^{kl} - L_{ijk}^{i} L_{l}^{jkl} + \frac{1}{2} L_{ijkl} L^{ijkl}$$

$$J = \frac{1}{8} L_{ij}^{ij} L_{kl}^{kl} L_{mn}^{mn} - \frac{1}{8} L_{ij}^{ij} L_{klmn} L^{klmn} - \frac{1}{4} L_{ij}^{ij} L_{klm}^{k} L_{n}^{lmn} + \frac{1}{4} L_{i}^{ijk} L_{lm}^{jk} L_{n}^{lmn}$$

$$D \stackrel{\text{def}}{=} -I^3 + 27 J^2 = -(L_{xxxx} L_{yyyy} - 4 L_{xxxy} L_{xyyy} + 3 L_{xxyy}^2)^3 =$$
$$+27(L_{xxxx}(L_{xxyy} L_{yyyy} - L_{xyyy}^2) - L_{xxxy}(L_{xxxy} L_{yyyy} - L_{xxyy} L_{xyyy}) + L_{xxyy}(L_{xxxy} L_{xyyy} - L_{xxyy}^2))^2$$





tensor calculus: Levi-Civita tensor

$$\varepsilon_{i_1 \ldots i_d} \stackrel{\text{def}}{=} \sqrt{g}\,[\,i_1 \ldots i_d\,]$$

$$[i_1 \ldots i_d] \stackrel{\text{def}}{=} \begin{cases} +1 & (i_1 \ldots i_d) \text{ even} \\ -1 & (i_1 \ldots i_d) \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$g \stackrel{\text{def}}{=} \det g_{ij}$$

$$\varepsilon^{i_1 \ldots i_d} = \frac{1}{\sqrt{g}}\,[\,i_1 \ldots i_d\,]$$

☞ g is a relative scalar

☞ ε is a pseudo-tensor

tensor calculus

tensoriality criterion generalised

$$P'^{i_1\dots i_l}{}_{j_1\dots j_k} = f(\det B)$$
$$A^{i_1}{}_{p_1} \dots A^{i_l}{}_{p_l} B_{j_1}{}^{q_1} \dots B_{j_k}{}^{q_k} P^{p_1\dots p_l}{}_{q_1\dots q_k}$$

odd relative tensors: $f(\det B) = |\det B|^w$

even relative tensors: $f(\det B) = (\det B)^w$

absolute tensors: $f(\det B) = 1$

pseudo-tensors: $f(\det B) = \mathrm{sgn}\,(\det B)$

- ☞ g = even relative scalar of weight 2
- ☞ $\sqrt{g}$ = odd relative scalar of weight 1
- ☞ $\varepsilon_{i_1\dots i_d}$ and $\varepsilon^{i_1\dots i_d}$ = absolute pseudo-tensors
- ☞ index lowering/raising does not affect weight

tensor calculus: pseudo-scalars

full contraction with one ε-tensor yields a pseudo-scalar

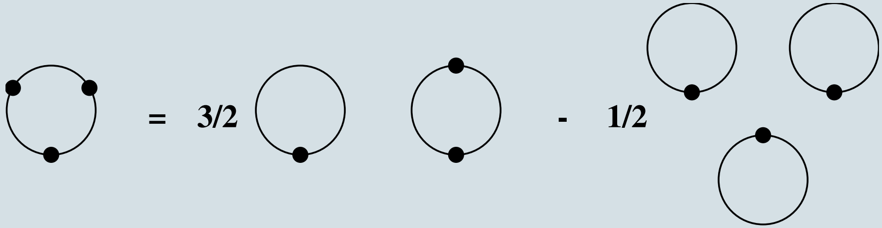
☞ in 2D: $\varepsilon^{ij} L_j L_k L_i^k$

products of multiple ε-tensors can be simplified

☞ in 2D: $\varepsilon^{ij} \varepsilon^{kl} L_i L_{jk} L_l = L_i L_j^i L^j - L_i L^i L_j^j$

completeness & irreducibility

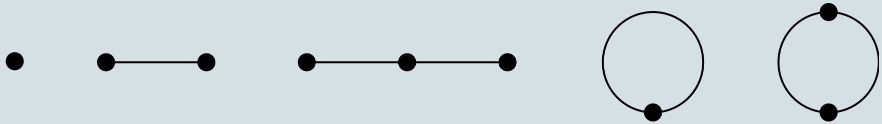
diagrammatic representation is not unique



☞ in 2D: $L_{ij}L^j_kL^{ki} = \frac{3}{2}L^i_iL_{jk}L^{jk} - \frac{1}{2}L^i_iL^j_jL^k_k$

completeness & irreducibility

Hilbert: "complete irreducible invariants"



☞ set of polynomial invariants that

❶ cannot be reduced

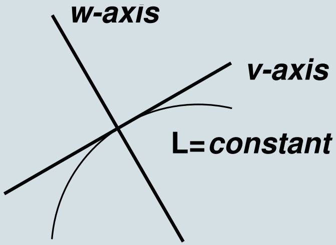
❷ forms a complete system

☞ possibly "syzygies"

gauge coordinates

"gradient gauge":

set $d-1$ 1st order and $\frac{1}{2}(d-1)(d-2)$ mixed 2nd order derivatives to 0



"Hessian gauge":

set all $\frac{1}{2}d(d-1)$ mixed 2nd order derivatives to 0

☞ all partial derivatives in gauge coordinates constitute a complete system

☞ $\dim \mathrm{SO}(d) = \frac{1}{2}d(d-1)$

gauge coordinates

example ❶ $L_{vw}L_w^2$ in 2D

- ☞ $-\varepsilon^{ij}L_iL_{jk}L^k$ or
- ☞ $L_xL_y(L_{xx}-L_{yy})-(L_x^2-L_y^2)L_{xy}$

example ❷ $L_{vv}L_w^2$ in 2D

- ☞ $-\varepsilon^{ij}\varepsilon^{kl}L_iL_{jk}L_l$ or
- ☞ $L_iL^iL_j^j - L_iL_j^iL^j$ or
- ☞ $L_x^2L_{yy}+L_y^2L_{xx}-2L_xL_yL_{xy}$

gauge coordinates

example ❸ $L_{pp}L_{qq}$ in 2D

- ☞ $\frac{1}{2}\varepsilon^{ij}\varepsilon^{kl}L_{ik}L_{jl}$ or
- ☞ $\frac{1}{2}\left(L_i^i L_j L^j - L_{ij}L^{ij}\right)$ or
- ☞ $L_{xx}L_{yy} - L_{xy}^2$

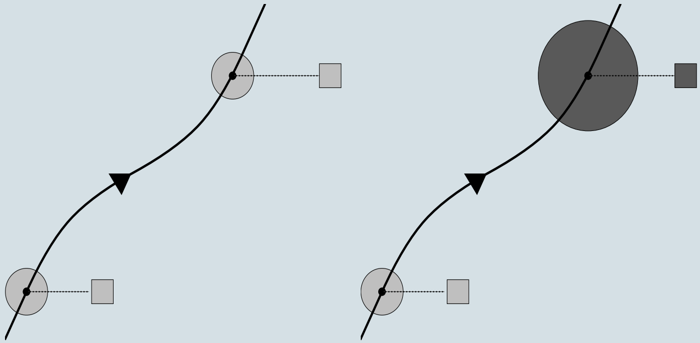
kinematic image structure

"classical model": zero Lie derivative $\mathcal{L}_v f = 0$

☞ $v = (1; \mathbf{v})$ with $\mathbf{v} \in \mathbb{R}^d$

❶ scalar: $\mathcal{L}_v f = \dfrac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f$

❷ density: $\mathcal{L}_v f = \dfrac{\partial f}{\partial t} + \operatorname{div}(\mathbf{v} f)$



kinematic image structure

"classical model": problems

- ☞ consistency problem: $v = (\mathbf{1}; \mathbf{v})$ is not always possible:

$$\frac{\partial f}{\partial t} \neq 0 \text{ and } \nabla f = 0$$

- ☞ ambiguity: 1 equation in d components per base point
- ☞ $\mathcal{L}_v f$ ill-posed & ill-defined

kinematic image structure

definition: $\mathcal{L}_v F[\phi] \stackrel{\text{def}}{=} F[\mathcal{L}_v^{\mathrm{T}}\phi]$

- ☞ for a scalar image the filters transform as densities: $\mathcal{L}_v^{\mathrm{T}}\phi = -\operatorname{div}(v\phi)$
- ☞ for a density image the filters transform as scalars: $\mathcal{L}_v^{\mathrm{T}}\phi = -v \cdot \nabla\phi$

kinematic image structure

ansatz: $\mathcal{L}_v F[\phi] = 0$

- ☞ $v = (v^0; \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^d$
- ☞ gauge invariance: 1 equation in $d+1$ components per base point
- ☞ temporal gauge: $v^0 = 1$ requires "conservation of topological detail"
- ☞ spatial gauge: *e.g.* $\mathcal{L}_{*v} F[\phi] = 0$ produces normal flow if $*v \cdot v \stackrel{\text{def}}{=} 0$
- ☞ "temporal gauge" inspired by conservation hypothesis
- ☞ "spatial gauge" must be determined from external insight (semantics)
- ☞ computational problem: how to compute v

kinematic image structure

computation

❶ use M-th order polynomial v_M instead of v: $v_M(x) = v(x) + \mathcal{O}(\|x\|^{M+1})$

❷ consider all spatiotemporal derivatives of $\mathcal{L}_{v_M} F[\phi] = 0$ of orders $0, \ldots, M$ (*no higher*)

❸ get components of v_M out of integral

❹ rewrite $x^\alpha D_\beta \phi(x)$ in terms of derivatives only (completeness)

❺ result: linear system for v_M in which coefficients are image derivatives of orders $1, \ldots, 2M + 1$

❻ fix the gauge and solve

❼ v_{M+1} refines *all* components of v_M