Derivations

Handout for “Semantics and Computational Models” (2IF55)

Bas Luttik

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Recall the definition of *transition system specification* (cf. Definition 1.4 in [1]). With every transition system specification we associate a notion of derivation.

Definition 1. Let $\mathcal{T} = (\mathit{Conf}, \mathit{Obs}, \rightarrow, \mathit{Spec})$ be a transition system specification. We define the notion of *Spec-derivation with conclusion* $c \xrightarrow{a} c'$ inductively as follows:

(i) if $c \xrightarrow{a} c'$ is an axiom named (Ax) in *Spec*, then

$$\text{(Ax)}\,\frac{}{c \xrightarrow{a} c'}$$

is a *Spec*-derivation with conclusion $c \xrightarrow{a} c'$; and

(ii) if $d_1 = \left\{ \dfrac{\vdots}{c_1 \xrightarrow{a_1} c_1'} \right\}, \ldots, d_k = \left\{ \dfrac{\vdots}{c_k \xrightarrow{a_k} c_k'} \right\}$ are *Spec*-derivations with conclusions $c_1 \xrightarrow{a_1} c_1'$, …, $c_k \xrightarrow{a_k} c_k'$, respectively, and $\text{(R)}\,\dfrac{c_1 \xrightarrow{a_1} c_1', \cdots, c_k \xrightarrow{a_k} c_k'}{c \xrightarrow{a} c'}$ is a rule in *Spec*, then

$$\text{(R)}\,\frac{\dfrac{\vdots}{c_1 \xrightarrow{a_1} c_1'} \cdots \dfrac{\vdots}{c_k \xrightarrow{a_k} c_k'}}{c \xrightarrow{a} c'}$$

is a *Spec*-derivation with conclusion $c \xrightarrow{a} c'$.

Example 2. Consider the transition system specification $\mathcal{T} = (\mathbb{N}, \{<\}, \rightarrow, \mathit{Spec})$ defined in Example 1 of [1]. Here is an example of a *Spec*-derivation:

$$\text{(R2)}\,\frac{\text{(R1)}\,\dfrac{\text{(Ax)}\,\dfrac{}{0 \xrightarrow{<} s(0)}}{s(0) \xrightarrow{<} s(s(0))} \qquad \text{(R1)}\,\dfrac{\text{(R1)}\,\dfrac{\text{(Ax)}\,\dfrac{}{0 \xrightarrow{<} s(0)}}{s(0) \xrightarrow{<} s(s(0))}}{s(s(0)) \xrightarrow{<} s(s(s(0)))}}{s(0) \xrightarrow{<} s(s(s(0)))}$$

The following proposition establishes a correspondence between the transition relation $\rightarrow$ of a transition system specification $\mathcal{T} = (\mathit{Conf}, \mathit{Obs}, \rightarrow, \mathit{Spec})$ and the transitions that are conclusions of *Spec*-derivations.

Proposition 3. If $\mathcal{T} = (\mathit{Conf}, \mathit{Obs}, \rightarrow, \mathit{Spec})$ is a transition system specification, then for all $c, c' \in \mathit{Conf}$ and $c \in \mathit{Obs}$ we have that $c \xrightarrow{a} c'$ iff there exists a *Spec*-derivation with conclusion $c \xrightarrow{a} c'$.

The following notion is useful in proofs about derivations.

Definition 4. Let $\mathcal{T} = (\mathit{Conf}, \mathit{Obs}, \rightarrow, \mathit{Spec})$ be a transition system specification. We define the *height* $h(d)$ of a *Spec*-derivation inductively as follows:

(i) if $d = \text{(Ax)}\dfrac{}{c \xrightarrow{a} c'}$, then $h(d) = 1$; and

(ii) if $d = \left\{ \text{(R)}\dfrac{\dfrac{\vdots}{c_1 \xrightarrow{a_1} c_1'} \quad \cdots \quad \dfrac{\vdots}{c_k \xrightarrow{a_k} c_k'}}{c \xrightarrow{a} c'} \right\}$, and $d_1, \ldots, d_k$ denote the proper subderivations of d with conclusions $c_1 \xrightarrow{a_1} c_1', \ldots, c_k \xrightarrow{a_k} c_k'$, respectively, then $h(d) = \max\{h(d_1), \ldots, h(d_k)\} + 1$

Exercise 1. Consider the transition system specification $=(\mathbb{N}, \{<\}, \rightarrow, \mathit{Spec})$ of Example 1 in [1]. Prove that $n \xrightarrow{<} m$ iff $n < m$.
Hint: for the implication from left to right you should use induction on the height of *Spec*-derivation with conclusion $n \xrightarrow{<} m$ that $n < m$, and for the implication from right to left you should consider natural numbers n and m such that $n < m$ and explain how to construct a derivation for $n \xrightarrow{<} m$.

References

[1] J. W. de Bakker and E. P. de Vink. *Control flow semantics.* The MIT Press, 1996.