Solution “Inleveropgave 2”

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Exercise 1.10 Let *Fun* be a collection of function symbols with $ar(f) \in \mathbb{N}$ the *arity* of f (required number of arguments). Assume that *Fun* contains at least one 0-ary function symbol (i.e., a constant), and define *Term* as the least set satisfying

$$Term = \{f(t_1, \ldots, t_n) \mid f \in Fun,\ \mathrm{ar}(f) = n,\ t_1, \ldots, t_n \in Term\} \ .$$

For $s, t \in Term$ with $s = f(t_1, \ldots, t_n)$, $t = g(t'_1, \ldots, t'_m)$, we define

$$d(s,t) = \begin{cases} 1 & \text{if } f \neq g; \\ \frac{1}{2}\max\{d(t_i, t'_i) \mid 1 \leq i \leq n = \mathrm{ar}(f)\} & \text{if } f = g \text{ and } ar(f) > 0; \text{ and} \\ 0 & \text{if } f = g \text{ and } ar(f) = 0. \end{cases}$$

Prove that $(Term, d)$ is an ultrametric space. Is *Term* complete?

Solution To prove that $(Term, d)$ is an ultrametric space we should establish that the postulates (M1), (M2) and (M4) (cf. Definition 1.15 in [1]) are satisfied by the function d defined above.

(M1) $d(x, y) = 0 \ \Leftrightarrow\ x = y$

We have to prove that $d(s, t) = 0 \ \Leftrightarrow\ s = t$ for all $s, t \in Term$. We prove the two directions of the bi-implication separately:

($\Rightarrow$) To prove that the implication from left to right holds for all $s, t \in Term$, let $s = f(t_1, \ldots, t_n)$ with $t_1, \ldots, t_n \in Term$ ($n = ar(f)$), and let $t = g(t'_1, \ldots, t'_m)$ with $t'_1, \ldots, t'_m \in Term$ ($m = ar(g)$). We proceed by induction on the structure of s:

(Basis) Suppose $ar(f) = 0$, and suppose $d(s, t) = 0$. Then $f = g$, for if $t = g(t'_1, \ldots, t'_m)$ for some $g \neq f$ and terms $t'_1, \ldots, t'_m \in Term$, then, according to the definition of d, $d(s, t) = 1 \neq 0$. It follows that $s = t$.

(Step) Suppose $ar(f) > 0$, and suppose $d(s, t) = 0$. Furthermore, suppose that for all $1 \leq i \leq n$ and for all terms $u \in Term$ the implication $d(t_i, u) = 0 \Rightarrow t_i = u$ holds (IH). By a similar argument as in the case (Basis), we have that $f = g$. Moreover,

$$d(s,t) = \tfrac{1}{2}\max\{d(t_i, t'_i) \mid 1 \leq i \leq n\} \ , \tag{1}$$

so from $d(s, t) = 0$ and Eqn. (1) it follows that $d(t_i, t'_i) = 0$ for all $1 \leq i \leq n$, and hence, by the induction hypothesis, $t_i = t'_i$. From $f = g$ and $t_i = t'_i$ ($1 \leq i \leq$) it follows that $s = f(t_1, \ldots, t_n) = g(t'_1, \ldots, t'_n) = t$.

($\Leftarrow$) Note that it suffices to prove that $d(s, s) = 0$ for all $s \in Term$; we proceed by induction on the structure of s:

(Basis) If $ar(f) = 0$, then $d(s, s) = 0$, directly according to the definition of d.

(Step) Suppose that $ar(f) > 0$, and suppose that, for all $1 \leq i \leq n$, $d(t_i, t_i) = 0$ (IH). Then

$$d(s, s) = \tfrac{1}{2} \max\{d(t_i, t_i) \mid 1 \leq i \leq n\} \ , \tag{2}$$

and the right-hand side of this equation is easily seen to be 0 using the induction hypothesis.

(M2) $d(x, y) = d(y, x)$

We have to prove that $d(s, t) = d(t, s)$ for all $s, t \in \mathit{Term}$. Let $s = f(t_1, \ldots, t_n)$ with $t_1, \ldots, t_n \in \mathit{Term}$ ($n = ar(f)$), and let $t = g(t'_1, \ldots, t'_m)$ with $t'_1, \ldots, t'_m \in \mathit{Term}$ ($m = ar(g)$). We proceed with induction on the structure of s:

(Basis) If $ar(f) = 0$, then we distinguish two cases according to whether $f = g$ or $f \neq g$:

(a) If $f = g$, then $s = t$, so clearly $d(s, t) = d(t, s)$.

(b) If $f \neq g$, then, by the definition of d, $d(s, t) = 1 = d(t, s)$.

(Step) Suppose that $ar(f) > 0$, and suppose that $d(t_i, u) = d(u, t_i)$ for all $1 \leq i \leq n = ar(f)$ and for all terms $u \in \mathit{Term}$ (IH). Again we distinguish two cases according to whether $f = g$ or $f \neq g$:

(a) If $f = g$, then

$$\begin{aligned} d(s, t) &= \tfrac{1}{2} \max\{d(t_i, t'_i) \mid 1 \leq i \leq n\} && \text{(def. } d\text{)} \\ &= \tfrac{1}{2} \max\{d(t'_i, t_i) \mid 1 \leq i \leq n\} && \text{(IH)} \\ &= d(t, s) && \text{(def. } d\text{).} \end{aligned}$$

(b) If $f \neq g$, then, by the definition of d, $d(s, t) = 1 = d(t, s)$.

(M4) $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

We have to prove that $d(s, t) \leq \max\{d(s, u), d(u, t)\}$ for all $s, t, u \in \mathit{Term}$. Let $s = f(t_1, \ldots, t_n)$ for some $t_i \in \mathit{Term}$ ($1 \leq i \leq n = ar(f)$), let $t = g(t'_1, \ldots, t'_m)$ for some $t'_i \in \mathit{Term}$ ($1 \leq i \leq m = ar(g)$), and let $u = h(t''_1, \ldots, t''_l)$ for some $t''_i \in \mathit{Term}$ ($1 \leq i \leq k = ar(h)$). We proceed with induction on the structure of s:

(Basis) If $ar(f) = 0$, then we distinguish two cases according to whether $f = g$ or $f \neq g$:

(a) If $f = g$, then $s = t$, so $d(s, t) = 0 \leq \max\{d(s, u), d(u, t)\}$.

(b) If $f \neq g$, then either $f \neq h$ or $h \neq g$ (or both), so either $d(s, u) = 1$ or $d(u, t) = 1$ (or both), and hence $d(s, t) \leq 1 = \max\{d(s, u), d(u, t)\}$.

(Step) Suppose that $ar(f) > 0$, and suppose that $d(t_i, u) \leq \max\{d(t_i, v), d(v, u)\}$ for all $1 \leq i \leq n$ and for all terms $u, v \in \mathit{Term}$ (IH). Again we distinguish two cases according to whether $f = g$ or $f \neq g$:

(a) If $f = g$, then

$$\begin{aligned} & d(s, t) \\ &= \tfrac{1}{2} \max\{d(t_i, t'_i) \mid 1 \leq i \leq n\} && \text{(def. } d\text{)} \\ &\leq \tfrac{1}{2} \max\{\max\{d(t_i, t''_i), d(t''_i, d'_i)\} \mid 1 \leq i \leq n\} && \text{(IH)} \\ &= \max\left\{ \begin{array}{c} \tfrac{1}{2} \max\{d(t_i, t''_i) \mid 1 \leq i \leq n\}, \\ \tfrac{1}{2} \max\{d(t''_i, t'_i) \mid 1 \leq i \leq n\} \end{array} \right\} && \text{(prop. max)} \\ &= \max\{d(s, u), d(u, t)\} && \text{(def. } d\text{).} \end{aligned}$$

(b) If $f \neq g$, then either $f \neq h$ or $h \neq g$ (or both), so either $d(s,u) = 1$, or $d(u,t) = 1$ (or both), and hence $d(s,t) \leq 1 = \max\{d(s,u), d(u,t)\}$.

So, we have now established that (Term, d) is an ultrametric space.

Whether it is complete, depends on the contents of *Fun*.

On the one hand, if *Fun* includes only constants, then for every Cauchy sequence $(x_n)_n$ there exists $i \in \mathbb{N}$ such that $x_j = x_k$ for all $j, k \geq i$, in which the sequence is actually convergent with limit x_j. Hence, in this case (Term, d) is complete.

On the other hand, if *Fun* includes at least one function symbol f with $\mathit{ar}(f) > 0$, and a is the presupposed constant in *Fun*, then we can inductively define the Cauchy sequence $(s_n)_n$ inductivelyby $s_0 = a$ and $s_{n+1} = f(s_n, a, \ldots, a)$ (i.e., the first argument of f is s_n and all the other arguments are a. This Cauchy sequence has no limit in (Term, d), and hence, in this case, (Term, d) is not complete.

References

[1] J. W. de Bakker and E. P. de Vink. *Control Flow Semantics*. The MIT Press, 1996.