Solution Exercise 1.15

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Before we embark on solving Exercise 1.15, we first establish a lemma that will be useful in its solution:

Lemma. $\mathcal{O}(D \mid s_1 \,;\, s_2) = \mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2)$ for all $s_1, s_2 \in \mathit{Stat}$.

Proof. We shall prove the property using the $\varepsilon \leq \frac{1}{2}\varepsilon$ method, i.e., we define

$$\varepsilon = \sup\{d(\mathcal{O}(D \mid s_1 \,;\, s_2), \mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2)) \mid s_1, s_2 \in \mathit{Stat}\} \ ,$$

and prove for all s_1, s_2 that $d(\mathcal{O}(D \mid s_1 \,;\, s_2), \mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2)) \leq \frac{1}{2}\varepsilon$. Let $a \in \mathit{Act}$ and $r \in \mathit{Res}$ such that $s_1 \xrightarrow{a}_D r$; then, by the rule (Seq) of the operational semantics for $\mathcal{L}_{rec}$, $s_1 \,;\, s_2 \xrightarrow{a}_D r \,;\, s_2$. We now distinguish two cases, according to whether $r = \mathrm{E}$ or $r \in \mathit{Stat}$.

1. If $r = \mathrm{E}$, then $r \,;\, s_2 = s_2$, so

   $$\mathcal{O}(D \mid s_1 \,;\, s_2) = \Phi(\mathcal{O})(D \mid s_1 \,;\, s_2) = a \cdot \mathcal{O}(D \mid s_2) \ ,$$

   and

   $$\begin{aligned} \mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2) &= \Phi(\mathcal{O})(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2) \\ &= (a \cdot \mathcal{O}(D \mid \mathrm{E})) \,;\, \mathcal{O}(D \mid s_2) \\ &= (a \cdot \epsilon) \,;\, \mathcal{O}(D \mid s_2) \\ &= a \,;\, \mathcal{O}(D \mid s_2) \\ &= a \cdot \mathcal{O}(D \mid s_2) \ , \end{aligned}$$

   so $\mathcal{O}(D \mid s_1 \,;\, s_2) = \mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2)$, and therefore

   $$d(\mathcal{O}(D \mid s_1 \,;\, s_2), \mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2)) = 0 \leq \tfrac{1}{2}\varepsilon \ .$$

2. If $r \in \mathit{Stat}$, then

   $$\mathcal{O}(D \mid s_1 \,;\, s_2) = \Phi(\mathcal{O})(D \mid s_1 \,;\, s_2) = a \cdot \mathcal{O}(D \mid r \,;\, s_2) \ ,$$

   and

   $$\begin{aligned} \mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2) &= \Phi(\mathcal{O})(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2) \\ &= (a \cdot \mathcal{O}(D \mid r)) \,;\, \mathcal{O}(D \mid s_2) \\ &= a \cdot (\mathcal{O}(D \mid r) \,;\, \mathcal{O}(D \mid s_2)) \ , \end{aligned}$$

   so

   $$\begin{aligned} & d(\mathcal{O}(D \mid s_1 \,;\, s_2), \mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2)) \\ & \quad = d(a \cdot \mathcal{O}(D \mid r \,;\, s_2), a \cdot (\mathcal{O}(D \mid r) \,;\, \mathcal{O}(D \mid s_2))) \\ & \quad = \tfrac{1}{2} d(\mathcal{O}(D \mid r \,;\, s_2), \mathcal{O}(D \mid r) \,;\, \mathcal{O}(D \mid s_2)) \\ & \quad \leq \tfrac{1}{2}\varepsilon \ . \end{aligned}$$

We conclude that $d(\mathcal{O}(D \mid s_1 \,;\, s_2), \mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2)) \leq \frac{1}{2}\varepsilon$ for all $s_1, s_2 \in \mathit{Stat}$, from which it follows that $\varepsilon \leq \frac{1}{2}\varepsilon$, and hence $\varepsilon = 0$. It follows that, for all $s_1, s_2 \in \mathit{Stat}$,

$$d(\mathcal{O}(D \mid s_1 \,;\, s_2), \mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2)) = 0 \enspace ,$$

and hence $\mathcal{O}(D \mid s_1 \,;\, s_2) = \mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2)$. $\square$

Exercise 1.15

(a) To prove that $\mathcal{O}$ is a fixed point of Ψ, we prove that $\Psi(\mathcal{O})(D \mid s) = \mathcal{O}(D \mid s)$ for all $s \in \mathit{Stat}$ with induction on $\mathit{wgt}(s)$. So let $s \in \mathit{Stat}$, and suppose that for all $s' \in \mathit{Stat}$ such that $\mathit{wgt}(s') < \mathit{wgt}(s)$ it holds that $\Psi(\mathcal{O})(D \mid s') = \mathcal{O}(D \mid s')$ (the Induction Hypothesis). We distinguish cases according to the syntactic form of s:

1. If $s = a$ for some $a \in \mathit{Act}$, then

$$\Psi(\mathcal{O})(D \mid s) = a$$

and, since $s \xrightarrow{a}_D \mathrm{E}$, also

$$\mathcal{O}(D \mid s) = a \cdot \mathcal{O}(D \mid \mathrm{E}) = a \cdot \epsilon = a \enspace ,$$

so $\Psi(\mathcal{O})(D \mid s) = \mathcal{O}(D|s)$.

2. Suppose that $s = x$ for some $x \in \mathit{PVar}$, and suppose that $x \xrightarrow{a}_D r$. Then by Lemma 1.8(a) $D(x) \xrightarrow{a}_D r$, so

$$\mathcal{O}(D \mid x) = a \cdot \mathcal{O}(D \mid r) = \mathcal{O}(D \mid D(x)) \enspace .$$

Since $\mathit{wgt}(D(x)) < \mathit{wgt}(x)$, we get by the Induction Hypothesis that

$$\Psi(\mathcal{O})(D \mid D(x)) = \mathcal{O}(D \mid D(x)) \enspace .$$

Hence

$$\Psi(\mathcal{O})(D \mid x) = \Psi(\mathcal{O})(D \mid D(x)) = \mathcal{O}(D \mid D(x)) = \mathcal{O}(D \mid x) \enspace .$$

3. Suppose that $s = s_1;s_2$. Then, since $\mathit{wgt}(s_1) < \mathit{wgt}(s)$, we get with an application of the Induction Hypothesis that $\Psi(\mathcal{O})(D \mid s_1) = \mathcal{O}(D \mid s_1)$, and therefore, also using the lemma proved above:

$$\begin{aligned}\Psi(\mathcal{O})(D \mid s_1 \,;\, s_2) &= \Psi(\mathcal{O})(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2)\\ &= \mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(D \mid s_2)\\ &= \mathcal{O}(D \mid s_1 \,;\, s_2) \enspace .\end{aligned}$$

We have now proved that $\mathcal{O}$ is a fixed point of the $\frac{1}{2}$-contractive mapping Ψ. Since $\mathcal{D}$ is actually *defined* as the unique fixed point of Ψ, it follows that $\mathcal{O}(\pi) = \mathcal{D}(\pi)$ for all $\pi \in \mathcal{L}_{rec}$.

(b) We can also prove that $\mathcal{O}(\pi) = \mathcal{D}(\pi)$ for all $\pi \in \mathcal{L}_{rec}$ with the $\varepsilon \leq \frac{1}{2}\varepsilon$ method. Define

$$\varepsilon = \sup\{d(\mathcal{O}(\pi), \mathcal{D}(\pi)) \mid \pi \in \mathcal{L}_{rec}\} \enspace .$$

We prove that $d(\mathcal{O}(D \mid s), \mathcal{D}(D \mid s)) \leq \frac{1}{2}\varepsilon$ for all $s \in \mathit{Stat}$ with induction on $\mathit{wgt}(s)$. So let $s \in \mathit{Stat}$, and suppose that for all $s' \in \mathit{Stat}$ such that $\mathit{wgt}(s') < \mathit{wgt}(s)$ it holds that $d(\mathcal{O}(D \mid s'), \mathcal{D}(D \mid s')) \leq \frac{1}{2}\varepsilon$ (the Induction Hypothesis). We distinguish cases according to the syntactic form of s:

1. If $s = a$ for some $a \in Act$, then $s \xrightarrow{a}_D \mathrm{E}$, so $\mathcal{O}(D \mid s) = a \cdot \mathcal{O}(D \mid \mathrm{E}) = a \cdot \epsilon = a$, and, from the definition of $\mathcal{D}$, $\mathcal{D}(D \mid s) = a$. Hence,

$$d(\mathcal{O}(D \mid s), \mathcal{D}(D \mid s)) = d(a, a) = 0 \le \tfrac{1}{2}\varepsilon \ .$$

2. Suppose $s = x$ for some $x \in PVar$, and suppose that $x \xrightarrow{a}_D r$. Then, by Lemma 1.8(a) in [1], $D(x) \xrightarrow{a}_D r$, so

$$\mathcal{O}(D \mid x) = a \cdot \mathcal{O}(D \mid r) = \mathcal{O}(D \mid D(x)) \ .$$

Since, by the definition of $\mathcal{D}$, $\mathcal{D}(D \mid x) = \mathcal{D}(D \mid D(x))$, and $wgt(D(x)) < wgt(x)$, it follows by the Induction Hypothesis that

$$d(\mathcal{O}(D \mid s), \mathcal{D}(D \mid s)) = d(\mathcal{O}(D \mid D(x)), \mathcal{D}(D \mid D(x))) \le \tfrac{1}{2}\varepsilon \ .$$

3. Suppose that $s = s_1 \,;\, s_2$. Then, by the lemma established above,

$$\begin{aligned} & d(\mathcal{O}(D \mid s_1 \,;\, s_2), \mathcal{D}(D \mid s_1 \,;\, s_2)) \\ & \quad = d(\mathcal{O}(D \mid s_1) \,;\, \mathcal{O}(s_2), \mathcal{D}(D \mid s_1) \,;\, \mathcal{D}(s_2)) \end{aligned}$$

which, by Lemma 1.41 in [1],

$$\le \max\{d(\mathcal{O}(D \mid s_1), \mathcal{D}(D \mid s_1)), \tfrac{1}{2} d(\mathcal{O}(D \mid s_2), \mathcal{D}(D \mid s_2))\} \ .$$

Now note that, since $wgt(s_1) < wgt(s)$, we get by the Induction Hypothesis that $d(\mathcal{O}(D \mid s_1), \mathcal{D}(D \mid s_1)) \le \frac{1}{2}\varepsilon$, and $d(\mathcal{O}(D \mid s_2), \mathcal{D}(D \mid s_2)) \le \varepsilon$ follows from the definition of ε. So we have

$$\max\{d(\mathcal{O}(D \mid s_1), \mathcal{D}(D \mid s_1)), \tfrac{1}{2} d(\mathcal{O}(D \mid s_2), \mathcal{D}(D \mid s_2))\} \le \tfrac{1}{2}\varepsilon \ ,$$

and we conclude $d(\mathcal{O}(D \mid s_1 \,;\, s_2), \mathcal{D}(D \mid s_1 \,;\, s_2)) \le \frac{1}{2}\varepsilon$.

We have established that $d(\mathcal{O}(\pi), \mathcal{D}(\pi)) \le \frac{1}{2}\varepsilon$ for all $\pi \in \mathcal{L}_{rec}$, from which it follows that $\varepsilon \le \frac{1}{2}\varepsilon$ and hence $\varepsilon = 0$. Therefore, $d(\mathcal{O}(\pi), \mathcal{D}(\pi)) = 0$, and hence $\mathcal{O}(\pi) = \mathcal{D}(\pi)$ for all $\pi \in \mathcal{L}_{rec}$.

References

[1] J. W. de Bakker and E. P. de Vink. *Control Flow Semantics.* The MIT Press, 1996.