Expressiveness modulo Bisimilarity of Regular Expressions with Parallel Composition

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According to a standard result from automata theory, every non-deterministic finite automaton (NFA) is, modulo language equivalence, denoted by a regular expression. It is well-known that this result fails modulo bisimilarity. In this paper, we first prove that adding an operation for pure interleaving to the theory of regular expressions modulo bisimilarity strictly increases its expressiveness. Then, we prove that replacing the operation for pure interleaving by ACP-style parallel composition gives a further increase in expressiveness. Finally, we prove that the theory of regular expressions with ACP-style parallel composition and encapsulation is expressive enough to express all NFAs modulo bisimilarity. Our results extend the expressiveness results obtained by Bergstra, Bethke and Ponse for process algebras with (the binary variant of) Kleene's star operation.

1 Introduction

A well-known result in automata and formal language theory is that every NFA can be denoted by a regular expression modulo language equivalence (see, e.g., [8]). Milner, in [10], considered regular expressions as a means to describe so-called *regular behaviours*. He studied the correspondence between regular behaviours and regular expressions and observed that the aforementioned result, establishing the correspondence between NFAs and regular expressions modulo language equivalence, does not hold modulo bisimilarity. He left it as an open problem to find a structural property on NFAs that characterises those that are denoted with a regular expression modulo bisimilarity. Such a structural property was found only recently by Baeten, Corradini and Grabmayer [1].

In this paper we consider the expressiveness of regular expressions from another angle. Instead of trying to characterise the subclass of NFAs denoted, up to bisimilarity, by a regular expression, we study to what extent the expressiveness of regular expressions increases when notions of parallel composition are added.

Our first contribution is to show that adding an operation for pure interleaving to regular expressions strictly increases their expressiveness modulo bisimilarity. A crucial step in our proof consists of characterising the strongly connected components in NFAs denoted by regular expressions. The characterisation allows us to prove a property pertaining to the exit transitions from such strongly connected components. If interleaving is added, then it is possible to denote NFAs violating this property.

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Our second contribution is to show that replacing the operation for pure interleaving by ACP-style parallel composition [5], which implements a form of synchronisation by communication between components, leads to a further increase in expressiveness. To this end, we first characterise the strongly connected components in NFAs denoted by regular expressions with interleaving, from which we deduce a property on the exit transitions from such strongly connected components, and then we present an expression in the theory of regular expressions with ACP-style parallel composition that denotes an NFA violating this property.

Our third contribution is to establish that adding ACP-style parallel composition and encapsulation to the theory of regular expressions actually yields a theory in which every NFA can be expressed up to isomorphism, and hence, since bisimilarity is coarser than isomorphism, also up to bisimilarity. Every expression in the resulting theory, in turn, denotes an NFA, so this result can be thought of as a process-theoretic counterpart of the correspondence between NFAs and regular expressions from automata theory.

The results in this paper are inspired by the results of Bergstra, Bethke and Ponse pertaining to the relative expressiveness of process algebras with a binary variant of Kleene's star operation. In [3] they establish an expressiveness hierarchy on the extensions of the process theories $\mathrm{BPA}(\mathcal{A})$, $\mathrm{BPA}_{\delta}(\mathcal{A})$, $\mathrm{PA}(\mathcal{A})$, $\mathrm{PA}_{\delta}(\mathcal{A})$, $\mathrm{ACP}(\mathcal{A},\gamma)$, and $\mathrm{ACP}_{\tau}(\mathcal{A},\gamma)$ with binary Kleene star. The reason that their results are based on extensions with the binary version of the Kleene star is that they want to avoid the process-theoretic complications arising from the notion of intermediate termination (we say that a state in an NFA is intermediately terminating if it is terminating but also admits a transition). Most of the expressiveness results in [3] are included in [4], with more elaborate proofs.

Casting our contributions mentioned above in process-theoretic terminology, we establish a strict expressiveness hierarchy on the process theories $\mathrm{BPA}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ (regular expressions) modulo bisimilarity, $\mathrm{PA}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ (regular expressions with interleaving) modulo bisimilarity) and $\mathrm{ACP}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ (regular expressions with ACP-style parallel composition and encapsulation) modulo bisimilarity. The differences between the process theories $\mathrm{BPA}_{\delta}(\mathcal{A})$, $\mathrm{PA}_{\delta}(\mathcal{A})$ and $\mathrm{ACP}(\mathcal{A},\gamma)$ considered [3, 4] and the process theories $\mathrm{BPA}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, $\mathrm{PA}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $\mathrm{ACP}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ considered in this paper are as follows: we write **0** for the constant deadlock which is denoted by δ in [3, 4], we include the unary Kleene star instead of its binary variant, and we include a constant **1** denoting the successfully terminated process. The first difference is, of course, cosmetic, and with the addition of the constant **1** the unary and binary variants of Kleene's star are interdefinable. So, our results pertaining to the relative expressiveness of $\mathrm{BPA}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, $\mathrm{PA}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $\mathrm{ACP}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ extend the expressiveness hierarchy of [3, 4] with the constant **1**.

In [4] the expressiveness proofs are based on identifying cycles and exit transitions from these cycles. There are two reasons why the proofs in [3] and [4] cannot easily be adapted to a setting with a unary Kleene star and **1**. First, since we have a unary Kleene star there are cycles without any exit transitions. Second, the inclusion of the empty process **1** gives intermediate termination, which, combined with the previously described different behaviour of cycles, forces us to consider the more general structure of strongly connected component.

The paper is organised as follows. In Section 2 we present the process theories $\mathrm{BPA}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, $\mathrm{PA}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $\mathrm{ACP}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ and recall the notion of strongly connected component. In Section 3 we prove that $\mathrm{PA}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is more expressive than $\mathrm{BPA}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. In Section 4 we prove that $\mathrm{ACP}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ is more expressive than $\mathrm{PA}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. In Section 5 we prove that every NFA is denoted, up to isomorphism, by an $\mathrm{ACP}^{*}_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expression. We end the paper in Section 6 with some conclusions.

2 Preliminaries

In this section, we present the relevant definitions for the process theory $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ and its subtheories $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. We give their syntax and operational semantics, and the notion of (strong) bisimilarity. We also introduce some auxiliary technical notions that we need in the remainder of the paper, most notably that of strongly connected component. The expressions of the process theory $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ are precisely the well-known regular expressions from the theory of automata and formal languages, but we shall consider the transition systems (automata) associated with them modulo bisimilarity instead of modulo language equivalence.

The process theory $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ is parametrised by a non-empty set $\mathcal{A}$ of *actions*, and a *communication function* γ on $\mathcal{A}$, i.e., an associative and commutative binary partial operation $\gamma : \mathcal{A}\times\mathcal{A} \rightharpoonup \mathcal{A}$. $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ incorporates a form of synchronisation between the components of a parallel composition by allowing certain actions to engage in a *communication* resulting in another action. The communication function γ then defines which actions may communicate and what is the result. The details of this feature will become clear when we present the operational semantics of parallel composition.

The set of $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expressions $\mathcal{P}_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)}$ is generated by the following grammar:

$$p ::= \mathbf{0} \mid \mathbf{1} \mid a \mid p\cdot p \mid p+p \mid p^* \mid p \parallel p \mid p \mathbin{\|\!\!\lfloor} p \mid p \mid p \mid \partial_H(p) \ ,$$

with a ranging over $\mathcal{A}$ and H ranging over subsets of $\mathcal{A}$.

The process theory $\mathrm{ACP}(\mathcal{A},\gamma)$ (excluding the constants $\mathbf{0}$ and $\mathbf{1}$, but including a constant δ with exactly the same behaviour as $\mathbf{0}$, and without the operation *) originates with [5]. The extension of $\mathrm{ACP}(\mathcal{A},\gamma)$ with a constant $\mathbf{1}$ was investigated by [9, 2, 14] (in these articles, the constant was denoted ε). The extension of $\mathrm{ACP}(\mathcal{A},\gamma)$ with the binary version of the Kleene star was first proposed in [3].

The constants $\mathbf{0}$ and $\mathbf{1}$ respectively stand for the deadlocked process and the successfully terminated process, and the constants $a\in\mathcal{A}$ denote processes of which the only behaviour is to execute the action a. An expression of the form $p\cdot q$ is called a *sequential composition*, an expression of the form $p+q$ is called an *alternative composition*, and an expression of the form p^* is called a *star expression*. An expression of the form $p \parallel q$ is called a *parallel composition*, and an expression of the form $\partial_H(p)$ is called an *encapsulation*.

From the names for the constructions in the syntax of $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$, the reader probably has already an intuitive understanding of the behaviour of the corresponding processes. We proceed to formalise the operational behaviour by means of a collection of operational rules in the style of Plotkin's Structural Operational Semantics [13]. Note how the communication function in rule 14 is employed to model a form of communication between parallel components: if one of the components of a parallel composition can execute a transition labelled with a, the other can execute a transition labelled with b, and the communication function γ is defined on a and b, then the parallel composition can execute a transition labelled with $\gamma(a,b)$. (It may help to think of the action a as standing for the event of sending some datum d, the action b as standing for the event of receiving datum d, and the action $\gamma(a,b)$ as standing for the event that two components communicate datum d.) The $\mathcal{A}$-labelled transition relation $\rightarrow_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)}$ and the termination relation $\downarrow_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)}$ on $\mathcal{P}_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)}$ are the least relations $\rightarrow\ \subseteq \mathcal{P}_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)}\times\mathcal{A}\times\mathcal{P}_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)}$ and $\downarrow\ \subseteq \mathcal{P}_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)}$ satisfying the rules in Table 1.

The triple $\mathcal{T}_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)} = (\mathcal{P}_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)}, \rightarrow_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)}, \downarrow_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)})$, consisting of the $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expressions together with the $\mathcal{A}$-labelled transition relation and the termination predicate associated with them, is an example of an $\mathcal{A}$-labelled transition system space. In general, an *$\mathcal{A}$-labelled transition system space* $(S,\rightarrow,\downarrow)$ consists of a (non-empty) set S, the elements of which are called *states*, together with

$$1\,\frac{}{\mathbf{1}\downarrow} \qquad 2\,\frac{}{a \xrightarrow{a} \mathbf{1}} \qquad 3\,\frac{p \xrightarrow{a} p'}{p+q \xrightarrow{a} p'} \qquad 4\,\frac{q \xrightarrow{a} q'}{p+q \xrightarrow{a} q'} \qquad 5\,\frac{p\downarrow}{p+q\downarrow} \qquad 6\,\frac{q\downarrow}{p+q\downarrow}$$

$$7\,\frac{p \xrightarrow{a} p'}{p\cdot q \xrightarrow{a} p'\cdot q} \qquad 8\,\frac{p\downarrow \quad q \xrightarrow{a} q'}{p\cdot q \xrightarrow{a} q'} \qquad 9\,\frac{p\downarrow \quad q\downarrow}{p\cdot q\downarrow} \qquad 10\,\frac{p \xrightarrow{a} p'}{p^* \xrightarrow{a} p'\cdot p^*} \qquad 11\,\frac{}{p^*\downarrow}$$

$$12\,\frac{p \xrightarrow{a} p'}{p \parallel q \xrightarrow{a} p' \parallel q, \quad p \mathbin{\|\!\!\_} q \xrightarrow{a} p' \parallel q} \qquad 13\,\frac{q \xrightarrow{a} q'}{p \parallel q \xrightarrow{a} p \parallel q'} \qquad 14\,\frac{p\downarrow \quad q\downarrow}{p \parallel q\downarrow, \quad p \mid q\downarrow}$$

$$15\,\frac{p \xrightarrow{a} p' \quad q \xrightarrow{b} q' \quad \gamma(a,b) \text{ is defined}}{p \parallel q \xrightarrow{\gamma(a,b)} p' \parallel q', \quad p \mid q \xrightarrow{\gamma(a,b)} p' \parallel q'} \qquad 16\,\frac{p \xrightarrow{a} p' \quad a \notin H}{\partial_H(p) \xrightarrow{a} \partial_H(p')} \qquad 17\,\frac{p\downarrow}{\partial_H(p)\downarrow}$$

Table 1: Operational rules for $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$, with $a \in \mathcal{A}$ and $H \subseteq \mathcal{A}$.

an $\mathcal{A}$-labelled transition relation $\rightarrow \subseteq S \times \mathcal{A} \times S$ and a subset $\downarrow \subseteq S$. We shall in this paper consider two more examples of transition system spaces, obtained by restricting the syntax of $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ and making special assumptions about the communication function.

Next, we define the $\mathcal{A}$-labelled transition system space $\mathcal{T}_{\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})} = (\mathcal{P}_{\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}, \rightarrow_{\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}, \downarrow_{\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})})$ corresponding with the process theory $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. The set of $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions $\mathcal{P}_{\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$ consists of the $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ process expressions without occurrences of the constructs $\_ \mid \_$ and $\partial_H(\_)$. The $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ transition relation $\rightarrow_{\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$ on $\mathcal{P}_{\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$ and the termination predicate $\downarrow_{\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$ on $\mathcal{P}_{\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$ are the transition relation and termination predicate induced on $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions by the operational rules in Table 1 minus the rules 15–17. Alternatively (and equivalently) the transition relation $\rightarrow_{\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$ can be defined as the restriction of the transition relation $\rightarrow_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\emptyset)}$, with $\emptyset$ denoting the communication function that is everywhere undefined, to $\mathcal{P}_{\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$.

To define the $\mathcal{A}$-labelled transition system space $\mathcal{T}_{\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})} = (\mathcal{P}_{\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}, \rightarrow_{\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})})$ associated with the process theory $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, let $\mathcal{P}_{\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$ consist of all $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions without occurrences of the constructs $\_ \parallel \_$ and $\_ \mathbin{\|\!\!\_} \_$. The $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ transition relation $\rightarrow_{\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$ and the $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ termination predicate $\downarrow_{\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$ are the transition relation and the termination predicate induced on $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions by the operational rules in Table 1 minus the rules 12–17. That is, $\rightarrow_{\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$ and $\downarrow_{\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$ are simply obtained by restricting $\rightarrow_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)}$ and $\downarrow_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)}$ to $\mathcal{P}_{\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$.

Henceforth, we shall omit the subscripts $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$, $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ from transition relations and termination predicates whenever it is clear from the context which transition relation or termination predicate is meant. Furthermore, we shall often use $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$, $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, respectively, to denote the associated transition system spaces $\mathcal{T}_{\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)}$, $\mathcal{T}_{\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$ and $\mathcal{T}_{\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})}$.

Let $\mathcal{T} = (S, \rightarrow, \downarrow)$ be an $\mathcal{A}$-labelled transition system space. If $s, s' \in S$, then we write $s \longrightarrow s'$ if there exists $a \in \mathcal{A}$ such that $s \xrightarrow{a} s'$, and $s \not\longrightarrow s'$ if there exists no such $a \in \mathcal{A}$. We denote by $\rightarrow^+$ the transitive closure of $\rightarrow$, and by $\rightarrow^*$ the reflexive-transitive closure of $\rightarrow$. If $s \longrightarrow^* s'$ then we say that s' is *reachable* from s; the set of all states reachable from s is denoted by $[s]_{\rightarrow}$. We say that a state s is *normed* if there exists s' such that $s \longrightarrow^* s'$ and $s'\downarrow$. $\mathcal{T}$ is called *regular* if $[s]_{\rightarrow}$ is finite for all $s \in S$.

Lemma 2.1. The transition system spaces $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$, $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, and $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ are all regular.

Proof. By induction on the maximal depth of *-nestings it can be established that for every p in the respective transition system spaces the set $[p]_{\rightarrow}$ is finite (see also [3, Lemma 3.1]). $\square$

With every state s in $\mathcal{T}$ we can associate an *$\mathcal{A}$-labelled transition system* (or: a *non-deterministic automaton*) $([s]_{\rightarrow}, s, \rightarrow \cap ([s]_{\rightarrow} \times \mathcal{A} \times [s]_{\rightarrow}), \downarrow \cap [s]_{\rightarrow})$. Its states are the states reachable from s, its transition relation and termination predicate are obtained by restricting $\rightarrow$ and $\downarrow$ accordingly, and the state s is declared as the *initial state* of the transition system. If a transition system space is regular, then the transition system associated with a state in it is finite, i.e., it is an NFA in the terminology of automata theory. Thus, we get by Lemma 2.1 that the operational semantics of $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$, and, a fortiori, that of $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, associates an NFA with every process expression.

In automata theory, automata are usually considered as language acceptors and two automata are deemed indistinguishable if they accept the same languages. Language equivalence is, however, arguably too coarse in process theory, where the prevalent notion is bisimilarity [11, 12].

Definition 2.2. Let $\mathcal{T}_1 = (S_1, \rightarrow_1, \downarrow_1)$ and $\mathcal{T}_2 = (S_2, \rightarrow_2, \downarrow_2)$ be transition system spaces. A binary relation $\mathcal{R} \subseteq S_1 \times S_2$ is a *bisimulation* between $\mathcal{T}_1$ and $\mathcal{T}_2$ if it satisfies, for all $a \in \mathcal{A}$ and for all $s_1 \in S_1$ and $s_2 \in S_2$ such that $s_1 \mathrel{\mathcal{R}} s_2$ the following conditions:

(i) if there exists $s_1' \in S_1$ s.t. $s_1 \xrightarrow{a}_1 s_1'$, then there exists $s_2' \in S_2$ s.t. $s_2 \xrightarrow{a}_2 s_2'$ and $s_1' \mathrel{\mathcal{R}} s_2'$; and

(ii) if there exists $s_2' \in S_2$ s.t. $s_2 \xrightarrow{a}_2 s_2'$, then there exists $s_1' \in S_1$ s.t. $s_1 \xrightarrow{a}_1 s_1'$ and $s_1' \mathrel{\mathcal{R}} s_2'$; and

(iii) $s_1\downarrow_1$ if, and only if, $s_2\downarrow_2$.

States $s_1 \in S_1$ and $s_2 \in S_2$ are *bisimilar* (notation: $s_1 \leftrightarrows s_2$) if there exists a bisimulation $\mathcal{R}$ between $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $s_1 \mathrel{\mathcal{R}} s_2$.

To achieve a sufficient level of generality, we have defined bisimilarity as a relation between transition system spaces; to obtain a suitable notion of bisimulation between automata one should add the requirement that the initial states of the automata be related.

Based on the associated transition system spaces, we can now define what we mean when some transition system space is, modulo bisimilarity, less expressive than some other process theory.

Definition 2.3. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be transition system spaces. We say that $\mathcal{T}_1$ is *less expressive* than $\mathcal{T}_2$ (notation: $\mathcal{T}_1 \prec \mathcal{T}_2$) if every state in $\mathcal{T}_1$ is bisimilar to a state in $\mathcal{T}_2$, and, moreover, there is a state in $\mathcal{T}_2$ that is *not* bisimilar to some state in $\mathcal{T}_1$.

When we investigate the expressiveness of $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$, we want to be able to choose γ. So, we are actually interested in the expressiveness of the (disjoint) union of all transition spaces $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ with γ ranging over all communication functions. We denote this transition system space by $\bigcup_{\gamma}\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$. In this paper we shall then establish that $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A}) \prec \mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A}) \prec \bigcup_{\gamma}\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$.

In the remainder of this section we recall the notion of strongly connected component (see, e.g., [6]) and some further auxiliary notions that will play an important rôle in our analysis of the relative expressiveness of the specific transition system spaces introduced above.

Definition 2.4. A *strongly connected component* in a transition system space $\mathcal{T} = (S, \rightarrow, \downarrow)$ is a maximal subset C of S such that $s \longrightarrow^* s'$ for all $s, s' \in C$. A strongly connected component C is *trivial* if it consists of only one state, say $C = \{s\}$, and $s \not\longrightarrow s$; otherwise, it is *non-trivial*.

Note that every element of a transition system space is an element of precisely one strongly connected component of that space. Furthermore, if s is an element of a non-trivial strongly connected component, then $s \longrightarrow^+ s$. Since in a strongly connected component from every element every other element can be reached, we get as a corollary to Lemma 2.1 that strongly connected components in $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$, $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ are finite.

Let $\mathcal{T} = (S, \rightarrow, \downarrow)$ be a transition system space, let $s \in S$, and let $C \subseteq S$ be a strongly connected component in S. We say that C is *reachable* from s if $s \longrightarrow^* s'$ for all $s' \in C$.

Lemma 2.5. Let $\mathcal{T}_1 = (S_1, \rightarrow_1, \downarrow_1)$ and $\mathcal{T}_2 = (S_2, \rightarrow_2, \downarrow_2)$ be regular transition system spaces, and let $s_1 \in S_1$ and $s_2 \in S_2$ be such that $s_1 \leftrightarrows s_2$. If s_1 is an element of a strongly connected component C_1 in $\mathcal{T}_1$, then there exists a strongly connected component C_2 reachable from s_2 satisfying that for all $s_1' \in C_1$ there exists $s_2' \in C_2$ such that $s_1' \leftrightarrows s_2'$.

Proof. According to Lemma 2.1, the set $[s_2]_\rightarrow$ is finite; we use induction on $|[s_2]_\rightarrow|$.

If the unique strongly connected component containing s_2 satisfies the condition of the lemma, which, in particular, is clearly the case if $|[s_2]_\rightarrow| = 1$, then we are done. Otherwise, there exists a state in C_1, say s_1', for which there is no bisimilar state in the strongly connected component containing s_2, and since $s_1 \longrightarrow^+ s_1' \longrightarrow^+ s_1$, it follows that there exists s_2' distinct from s_2 such that $s_2 \longrightarrow^+ s_2'$ and $s_1 \leftrightarrows s_2'$. Clearly, $|[s_2']_\rightarrow| < |[s_2]_\rightarrow|$, so by the induction hypothesis there exists a strongly connected component C_2 reachable from s_2', and hence also from s_2, satisfying the condition of the lemma. □

3 Relative Expressiveness of $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$

In [3] it is proved that $\mathrm{BPA}^*_{\mathbf{0}}(\mathcal{A})$ (with the binary variant of the Kleene star) is less expressive than $\mathrm{PA}^*_{\mathbf{0}}(\mathcal{A})$ (also with the binary Kleene star). The proof in [3] is by arguing that the $\mathrm{PA}^*_{\mathbf{0}}(\mathcal{A})$ expression $(a \cdot b)^* c \parallel d$ is not bisimilar with a $\mathrm{BPA}^*_{\mathbf{0}}(\mathcal{A})$ expression. (Actually, the $\mathrm{PA}^*_{\mathbf{0}}(\mathcal{A})$ expression employed in [4] uses only a single action a, i.e., considers the $\mathrm{PA}^*_{\mathbf{0}}(\mathcal{A})$ expression $(a \cdot a)^* a \parallel a$; we use the actions b, c and d for clarity.) An alternative, and more general, proof that the $\mathrm{PA}^*_{\mathbf{0}}(\mathcal{A})$ expression above is not expressible in $\mathrm{BPA}^*_{\mathbf{0}}(\mathcal{A})$ is presented in [4]. There it is proved that the $\mathrm{PA}^*_{\mathbf{0}}(\mathcal{A})$ expression above fails the following general property, which is satisfied by all $\mathrm{BPA}^*_{\mathbf{0}}(\mathcal{A})$-expressible labelled transition systems:

> If C is a cycle in a transition system associated with a $\mathrm{BPA}^*_{\mathbf{0}}(\mathcal{A})$ expression, then there is at most one state $p \in C$ that has an exit transition.

(A cycle is a sequence $(p_1, \ldots, p_n)$ such that $p_i \longrightarrow p_{i+1}$ $(1 \leq i < n)$ and $p_n \longrightarrow p_1$; an exit transition from p_i is a transition $p_i \longrightarrow p_i'$ such that no element of the cycle is reachable from p_i'.)

Note that labelled transition systems associated with $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions do not satisfy the property above.

Example 3.1. Consider the $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression $(a \cdot (a + \mathbf{1}))^* \cdot b$; its associated transition system has the following cycle:

$$\{\mathbf{1} \cdot (a \cdot (a + \mathbf{1}))^* \cdot b, (a + \mathbf{1}) \cdot (a \cdot (a + \mathbf{1}))^* \cdot b\} \ .$$

Both states on the cycle have a b-transition off the cycle.

In this section we shall establish that $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is less expressive than $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. As in [4] we prove that $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressible labelled transition systems satisfy a general property that some labelled transition system expressible in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ fails to satisfy. We find it technically convenient, however, to base our relative expressiveness proofs on the notion of strongly connected component, instead of cycle. Note, e.g., that every process expression is an element of precisely one strongly connected component, while it may reside in more than one cycle. Furthermore, if $p \longrightarrow q$ and p and q are in distinct strongly connected components, then we can be sure that $p \longrightarrow q$ is an exit transition, while if p and q are on distinct cycles, then it may happen that p is reachable from q.

We proceed as follows. First, we shall carefully investigate the (syntax of process expressions in the) non-trivial strongly connected components. From our investigation we shall be able to conclude that non-trivial strongly connected components in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ satisfy a certain property pertaining to its exit transitions. Finally, we shall present a $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression that fails the property and conclude that indeed $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is less expressive than $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$.

3.1 Strongly Connected Components in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$

We shall now establish a syntactic characterisation of the non-trivial strongly connected components in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, proving that a non-trivial strongly connected component in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is either of the form $\{p_1 \cdot q^*, \ldots, p_n \cdot q^*\}$ with p_i $(0 \leq i \leq n)$ reachable from q and $\{p_1, \ldots, p_n\}$ not a strongly connected component, or of the form $\{p_1 \cdot q, \ldots, p_n \cdot q\}$ where $\{p_1, \ldots, p_n\}$ is a strongly connected component. To this end, let us first establish, by reasoning on the basis of the operational semantics, that process expressions in a non-trivial strongly connected component are necessarily sequential compositions. At the heart of the argument is a measure on process expressions that will enable us to prove that the process expressions in a non-trivial strongly connected are sequential compositions.

Definition 3.2. Let p a $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression; then $\#(p)$ is defined with recursion on the structure of p by the following clauses:

(i) $\#(\mathbf{0}) = \#(\mathbf{1}) = 0$, and $\#(a) = 1$;

(ii) $\#(p \cdot q) = \begin{cases} 0 & \text{if } q \text{ is a star expression} \\ \#(q)+1 & \text{otherwise;} \end{cases}$

(iii) $\#(p+q) = \max(\#(p), \#(q)) + 1$; and

(iv) $\#(p^*) = 1$.

We establish that $\#(\_)$ is non-increasing over transitions, and, in fact, in most cases decreases.

Lemma 3.3. If p and p' are $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions such that $p \longrightarrow^+ p'$, then $\#(p) \geq \#(p')$. Moreover, if $\#(p) = \#(p')$, then $p = p_1 \cdot q$ and $p' = p_1' \cdot q$ for some p_1, p_1' and q.

Proof. Let p and p' are $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions. Note that if the lemma holds in the special case that $p \longrightarrow p'$, then the general case follows with a straightforward induction on the length of a transition sequence from p to p'. In the remainder of the proof we concentrate on the special case, proving that $p \longrightarrow p'$ implies $\#(p) \geq \#(p')$, and $\#(p) = \#(p')$ implies $p = p_1 \cdot q$ and $p' = p_1' \cdot q$ for some p_1, p_1' and q. Let $a \in \mathcal{A}$ such that $p \xrightarrow{a} p'$; we reason by induction on a derivation according to the operational rules for $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ (rules 1–11) of the transition $p \xrightarrow{a} p'$.

2. Suppose that the last rule applied is rule 2; then $p = a$ and $q = \mathbf{1}$, so $\#(p) = 1 > 0 = \#(q)$.

3. Suppose that the last rule applied is rule 7. Then there exist p_1, p_1' and q such that $p = p_1 \cdot q$, with $p_1 \xrightarrow{a} p_1'$ and $p' = p_1' \cdot q$. There are two cases: if q is a star expression then $\#(p) = 0 = \#(p')$, and otherwise $\#(p) = \#(q) + 1 = \#(p')$.

4. Suppose that the last rule applied is rule 8. Then there exist p_1 and q such that $p = p_1 \cdot q$, with $p_1{\downarrow}$ and $q \xrightarrow{a} p'$. Note that, since there is a derivation of $q \xrightarrow{a} p'$ that is a proper subderivation of the considered derivation of $p \xrightarrow{a} p'$, we get by the induction hypothesis that $\#(q) \geq \#(p')$. There are again two cases. On the one hand, if q is a star expression, then the last rule applied in the derivation of $q \xrightarrow{a} p'$ is rule 10, so p' is a sequential composition, say $p' = p_1' \cdot q$, and q is a star expression; it follows that $\#(p) = 0 = \#(p')$. On the other hand, if q is not a star expression, then $\#(p) = \#(q) + 1 \geq \#(p') + 1 > \#(p')$.

6. Suppose that the last rule applied is rule 3. Then there exist p_1 and p_2 such that $p = p_1 + p_2$ and $p_1 \xrightarrow{a} p'$. Since there is a derivation of $p_1 \xrightarrow{a} p'$ that is a proper subderivation of the considered derivation of $p \xrightarrow{a} p'$, it follows by the induction hypothesis that $\#(p_1) \geq \#(p')$, and hence $\#(p) = \max(\#(p_1), \#(p_2)) + 1 \geq \#(p_1) + 1 \geq \#(p') + 1 > \#(p')$.
7. If the last rule applied is rule 4, then the it the proof that $\#(p) > \#(p')$ is analogous to the proof in the previous case.
10. If the last rule applied is rule 10, then there exist q and q' such that $p = q^*$, with $q \xrightarrow{a} q'$ and $p' = q' \cdot q^*$. It then follows immediately from the definition of $\#(\_)$ that $\#(p) = 1 > 0 = \#(q' \cdot q^*)$.

(Of course, since the rules 1, 9, 5, 6 and 11 do not have a transition as a conclusion, we need not consider them.) □

Let P be a set of process expressions, and let q be a process expression; by $P \cdot q$ we denote the set of process expressions

$$P \cdot q = \{p \cdot q \mid p \in P\} \ .$$

Lemma 3.4. If C is a non-trivial strongly connected component in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, then there exist a set of process expressions C' and a process expression q such that $C = C' \cdot q$.

Proof. Let $p \in C$. We first establish that there exist p_1 and q such that $p = p_1 \cdot q$. Note that $p \longrightarrow^+ p$, since C is a non-trivial strongly connected component. Hence, since obviously $\#(p) = \#(p)$, there exist p_1 and q such that $p = p_1 \cdot q$ by Lemma 3.3. It now remains to prove that for all $p' \in C$ there exists p'_1 such that $p' = p'_1 \cdot q$. Note that, since $p \longrightarrow^+ p' \longrightarrow^+ p$, $\#(p) \geq \#(p') \geq \#(p)$ by Lemma 3.3, and hence $\#(p) = \#(p')$. It then follows, also by Lemma 3.3, that there exists p'_1 such that $p' = p'_1 \cdot q$. □

We now give an inductive description of non-trivial strongly connected components in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. The basis for the inductive description is the following notion of basic strongly connected component.

Definition 3.5. A non-trivial strongly connected component $C = \{p_1, \ldots, p_n\}$ in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is *basic* if there exist $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions $p'_1, \ldots, p'_n$ and a $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression q such that $p_i = p'_i \cdot q^*$ $(1 \leq i \leq n)$ and $\{p'_1, \ldots, p'_n\}$ is not a strongly connected component in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$.

Lemma 3.6. Let $\{p_1 \cdot q^*, \ldots, p_n \cdot q^*\}$ be a basic strongly connected component. Then $q \longrightarrow^+ p_i$ for all $1 \leq i \leq n$.

Proof. By Definition 3.5, $\{p_1, \ldots, p_n\}$ is not a strongly connected component. By Definition 2.4, for all p_i, there are p_j such that $p_j \not\longrightarrow^+ p_i$. This implies that we cannot only apply operational rule 7 on the derivation $p_j \cdot q^* \longrightarrow^+ p_i \cdot q^*$. The only alternative rule is 8. There must be p'_j and p'_i such that $p'_j \cdot q^* \longrightarrow p'_i \cdot q^*$ is the last application of rule 8 in $p_j \cdot q^* \longrightarrow^+ p_i \cdot q^*$. This implies that $q^* \longrightarrow p'_i \cdot q^*$, and $p'_i \longrightarrow^* p_i \cdot q^*$, where the latter only has applications of operational rule 7. The former must be a transition using rule 10, thus $q \longrightarrow p'_i \longrightarrow^* p_i$. □

Proposition 3.7. Let C be a non-trivial strongly connected component in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. Then either C is basic, or there exist a non-trivial strongly connected component C' and a $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression q such that $C = C' \cdot q$.

Proof. By Lemma 3.4 there exists a set of states C' and a $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression q such that $C = C' \cdot q$. If C' is a non-trivial strongly connected component, then the proposition follows, so it remains to prove that if C' is not a non-trivial strongly connected component, then C is basic. Note that if C' is not a strongly connected component, then there are $p, p' \in C'$ such that $p \not\longrightarrow^+ p'$. Since C is a non-trivial strongly connected component and $C = C' \cdot q$, it holds that $p \cdot q \longrightarrow^+ p' \cdot q$. Using that $p \not\longrightarrow^+ p'$, it can be established with induction on the length of the transition sequence from $p \cdot q$ to $p' \cdot q$ that $q \longrightarrow^+ p' \cdot q$. It follows by Lemma 3.3 that $\#(q) \geq \#(p' \cdot q)$, and therefore, according to the definition of $\#(\_)$, q must be a star expression. We conclude that C is basic. □

3.2 $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A}) \prec \mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$

The crucial tool that will allow us to establish that $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is less expressive than $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ will be a special property of states with a transition out of their strongly connected component in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. Roughly, if C is a strongly connected component in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, then all states with a transition out of C, have the same transitions out of C.

Definition 3.8. Let C be a strongly connected component in the transition system space $\mathcal{T} = (S, \rightarrow, \downarrow)$ and let $s \in C$. An exit transition from s is a pair (a, s') such that $s \xrightarrow{a} s'$ and $s' \notin C$. We denote by $ET(s)$ the set of all *exit transitions* from s, i.e.,

$$ET(s) = \{(a, s') \mid s \xrightarrow{a} s' \wedge s' \notin C\} \ .$$

An element $s \in C$ is called an *exit state* if $s\downarrow$ or there exists an exit transition from s.

Example 3.9. Consider the strongly connected components in the transitions systems associated with the following $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions $\mathbf{1} \cdot (a \cdot b \cdot (c + \mathbf{1}))^* \cdot d$, depicted in Figure 1. It has a strongly connecting component with two exit states, both with an exit transition $(d, \mathbf{1})$.

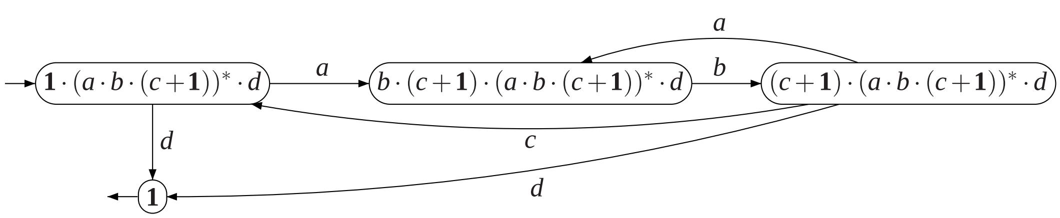

Figure 1: Example of a strongly connected component in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$.

Non-trivial strongly connected components in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ arise from executing the argument of a Kleene star. An exit state of a strongly connected component in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is then a state in which the execution has the option to terminate. Due to the presence of $\mathbf{0}$ in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ this is, however, not the only type of exit state in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ strongly connected components.

Example 3.10. Consider the strongly connected components in the transitions system associated with the following $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression $\mathbf{1} \cdot (a \cdot ((b \cdot \mathbf{0}) + \mathbf{1}))^* \cdot c$, depicted in Figure 2.
The strongly connected component contains two exit states, with two (distinct) exit transitions. One of these exit transitions leads to deadlock.

The preceding example illustrates that the special property of strongly connected components in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ that we are after should exclude exit transitions arising from a $\mathbf{0}$ from consideration. This is achieved in the following definitions.

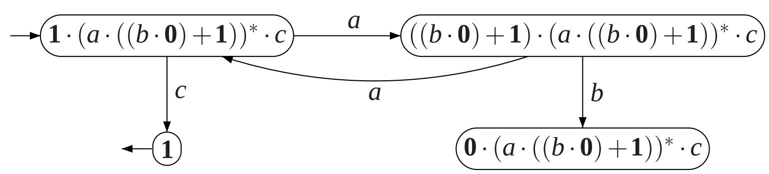

Figure 2: Example of a strongly connected component with normed exit transitions.

Definition 3.11. Let C be a strongly connected component and let $s\in C$. An exit transition (a,s') from s is *normed* if s' is normed. We denote by $ET_n(s)$ the set of normed exit transitions from s.

An exit state $s\in C$ is *alive* if $s\downarrow$ or there exists a normed exit transition from s.

Lemma 3.12. If $p\cdot q^* \longrightarrow^* r$, then either there exists p' such that $p\longrightarrow^* p'$ and $r=p'\cdot q^*$ or there exist p' and q' such that $p\longrightarrow^* p'$, $p'\downarrow$, $q\longrightarrow^* q'$, and $r=q'\cdot q^*$.

Proof. Induction on the length of the transition sequence, using every transition in the sequence is the conclusion of a derivation of which the last rule applied is either rule 7 or rules 8 and 10. □

Lemma 3.13. If C is a basic strongly connected component, then $ET_n(p)=\emptyset$ for all $p\in C$.

Proof. Since C is a basic strongly connected component, there exist $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions $p_1,\dots,p_n$ and q such that $C=\{p_1\cdot q^*,\dots,p_n\cdot q^*\}$ and $C'=\{p_1,\dots,p_n\}$ is not a strongly connected component. Consider $p_i\cdot q^*\in C$; to prove that $ET_n(p_i\cdot q^*)=\emptyset$, we suppose that $p_i\cdot q^*$ has a normed exit transition and derive a contradiction. So, let r and r' be $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions such that $r\notin C$, $p_i\cdot q^*\xrightarrow{a} r\longrightarrow^* r'\downarrow$, and $r \not\longrightarrow^* p_i\cdot q^*$. We proceed with a case distinction according to whether the last rule applied in the derivation of the transition $p_i\cdot q^*\xrightarrow{a} r$ is either rule 7 or rule 8.

From $p_i\cdot q^*\longrightarrow^* r'$ it follows by Lemma 3.12 that there exists a $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression s such that $r'=s\cdot q^*$, and since $r'\downarrow$, according to rule 9, also $s\downarrow$. Hence, since by Lemma 3.6 $q\longrightarrow^+ p_i\cdot q^*$, it immediately follows that $r'\longrightarrow^+ p_i\cdot q^*$. But then $r'\in C$ and, a fortiori, $r\in C$, contradicting our assumption that $r\notin C$. We conclude that $p_i\cdot q^*$ does not admit normed exit transitions, and hence $ET_n(p_i\cdot q^*)=\emptyset$. □

Lemma 3.14. Let C be a non-trivial strongly connected components in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, let $p\in C$, and let q be a $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ process expression such that $C\cdot q$ is a strongly connected component. Then $p\cdot q$ is an alive exit state in $C\cdot q$ iff p is an alive exit state in C and q is normed.

Proof. We prove the implications from left to right and from right to left separately.

(⇒) If $p\cdot q$ is an alive exit state in $C\cdot q$, then either $p\cdot q\downarrow$, or there exist $a\in\mathcal{A}$ and $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions $r\notin C\cdot q$ and r' such that $p\cdot q\xrightarrow{a} r\longrightarrow^* r'\downarrow$.

In the first case it is immediately clear from rule 9 that $p\downarrow$, so p is an alive exit state in C.

We proceed to prove that p is an alive exit state also in the second case. Note that from $p\cdot q\xrightarrow{a} r$ it follows that either $p\downarrow$ and $q\xrightarrow{a} r$ or there exists p' such that $p\xrightarrow{a} p'$ and $r=p'\cdot q$. If $p\downarrow$, then p is an alive exit state of C directly according to the definition. It remains to prove that $p'\notin C$ and that there exists a $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression p'' such that $p'\longrightarrow^* p''\downarrow$. Since $p'\in C$ would imply $r=p'\cdot q\in C\cdot q$, quod non, it follows that $p'\notin C$. The existence of a p'' such that $p'\longrightarrow^* p''\downarrow$ follows from the existence of an r' such that $p\cdot q\xrightarrow{a} r\longrightarrow^* r'\downarrow$.

(⇐) If p is an alive exit state in C, then there are two cases: either $p\downarrow$, or there exist an $a \in \mathcal{A}$ and $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions $r \notin C$ and r' such that $p \xrightarrow{a} r \longrightarrow^* r'\downarrow$. We consider these two cases separately.

Suppose that $p\downarrow$. If also $q\downarrow$, then $p \cdot q\downarrow$, and hence $p \cdot q$ is an alive exit state in $C \cdot q$. If $q\not\downarrow$, then there exist $a \in \mathcal{A}$ and a $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression q'' such that $q \xrightarrow{a} q'' \longrightarrow^* q'\downarrow$, due to the assumption that $q \longrightarrow^* q'\downarrow$, and $q\not\downarrow$. From $p \cdot q \xrightarrow{a} q''$ it follows by Lemma 3.3 that either $\#(p \cdot q) > \#(q'')$, or $\#(p \cdot q) = \#(q'')$ and there is a p' such that $q'' = p' \cdot q$. If $\#(p \cdot q) > \#(q'')$, then (a, q'') is an alive exit transition from $p \cdot q$, so $p \cdot q$ is an alive exit state in $C \cdot q$. The other case, that $\#(p \cdot q) = \#(q'')$ and there is a p' such that $q'' = p' \cdot q$, cannot occur, for $q \xrightarrow{a} p' \cdot q$ implies by Lemma 3.3 and the definition of $\#(\_)$ that q is a star expression, which is in contradiction with our assumption that $q\not\downarrow$.

Suppose there exist $a \in \mathcal{A}$ and $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions $r \notin C$ and r' such that $p \xrightarrow{a} r \longrightarrow^* r'\downarrow$. Then $p \cdot q \xrightarrow{a} r \cdot q \longrightarrow^* r' \cdot q \longrightarrow^* q'$, and since $r \notin C$ it follows that $r \cdot q \notin C \cdot q$. We conclude that $(a, r \cdot q)$ is a normed exit transition of $p \cdot q$. □

Lemma 3.15. Let C be a non-trivial strongly connected component in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, let $p \in C$, and let q be a normed $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ process expression such that $C \cdot q$ is a strongly connected component. Then

$$ET_n(p \cdot q) = \begin{cases} ET_n(p) \cdot q \cup \{(a, r) \mid r \notin C \cdot q \wedge r \text{ is normed} \wedge q \xrightarrow{a} r\} & \text{if } p\downarrow\text{; and} \\ ET_n(p) \cdot q & \text{if } p\not\downarrow. \end{cases}$$

Proof. We distinguish cases according to whether $p\downarrow$ or $p\not\downarrow$:

1. Suppose that $p\not\downarrow$.

   To see that $ET_n(p \cdot q) \subseteq ET_n(p) \cdot q$, consider an exit transition $(a, r) \in ET_n(p \cdot q)$. Then $r \notin C \cdot q$ and there exists r' such that $p \cdot q \xrightarrow{a} r \longrightarrow^* r'\downarrow$. Since $p\not\downarrow$, it follows that there exists p' such that $p \xrightarrow{a} p'$ and $r = p' \cdot q$. From $r \notin C \cdot q$, it follows that $p' \notin C$. Moreover, it is clear from $r \longrightarrow^* r'\downarrow$ that there exists a process expression p'' such that $p' \longrightarrow^* p''\downarrow$. Thus, we have established that $p' \notin C$ and $p \xrightarrow{a} p' \longrightarrow^* p''\downarrow$, so $(a, p') \in ET_n(p)$, and hence $(a, r) \in ET_n(p) \cdot q$.

   To see that $ET_n(p) \cdot q \subseteq ET_n(p \cdot q)$, consider and exit transition $(a, p') \in ET_n(p)$. Then $p' \notin C$ and there exists a process expression p'' such that $p \xrightarrow{a} p' \longrightarrow^* p''\downarrow$. Hence, using the assumption that $q \longrightarrow^* q'\downarrow$, we get $p \cdot q \xrightarrow{a} p' \cdot q \longrightarrow^* p'' \cdot q \longrightarrow^* p'' \cdot q'\downarrow$. From $p' \notin C$ it follows that $p' \cdot q \notin C \cdot q$, and therefore $(a, p' \cdot q) \in ET_n(p \cdot q)$.

2. Suppose that $p\downarrow$.

   To see that $ET_n(p \cdot q) \subseteq ET_n(p) \cdot q \cup \{(a, r) \mid r \notin C \cdot q \wedge \exists r'.\ q \xrightarrow{a} r \longrightarrow^* r'\downarrow\}$, consider an exit transition $(a, r) \in ET_n(p \cdot q)$. Then $r \notin C \cdot q$ and there exists r' such that $p \cdot q \xrightarrow{a} r \longrightarrow^* r'\downarrow$. From $p \cdot q \xrightarrow{a} r$ and $p\downarrow$ it follows that either $q \xrightarrow{a} r$ or there exists p' such that $p \xrightarrow{a} p'$ and $r = p' \cdot q$. In the first case it follows that $(a, r) \in \{(a, r) \mid r \notin C \cdot q \wedge \exists r'.\ q \xrightarrow{a} r \longrightarrow^* r'\downarrow\}$. In the second case, note that $p' \notin C$, for otherwise $r = p' \cdot q \in C \cdot q$ contradicting $(a, r) \in ET_n(p \cdot q)$. It is also clear from $p' \cdot q = r \longrightarrow^* r'\downarrow$ that there exists p'' such that $p' \longrightarrow^* p''\downarrow$. So $(a, p') \in ET_n(p)$.

   To see that $ET_n(p) \cdot q \subseteq ET_n(p \cdot q)$, consider an exit transition $(a, p') \in ET_n(p)$. Then $p' \notin C$ and there exists p'' such that $p \xrightarrow{a} p' \longrightarrow^* p''\downarrow$. Clearly, if $p' \notin C$, then $p' \cdot q \notin C \cdot q$ and, since $q \longrightarrow^* q'\downarrow$ according to the assumption of the lemma, $p \cdot q \xrightarrow{a} p' \cdot q \longrightarrow^* p'' \cdot q'\downarrow$. It follows that $(a, p' \cdot q) \in ET_n(p \cdot q)$.

   That $\{(a, r) \mid \exists r \notin C \cdot q \wedge r'.\ q \xrightarrow{a} r \longrightarrow^* r'\downarrow\} \subseteq ET_n(p \cdot q)$ is immediate. □

Proposition 3.16. Let C be a non-trivial strongly connected component in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. If p_1 and p_2 are alive exit states in C, then $ET_n(p_1) = ET_n(p_2)$.

Proof. Suppose that p_1 and p_2 are alive exit states; we prove by induction on the structure of non-trivial strongly connected components in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ as given by Proposition 3.7 that $ET_n(p_1) = ET_n(p_2)$ and $p_1\downarrow$ iff $p_2\downarrow$.

If C is basic, then by Lemma 3.13 $ET_n(p_1) = \emptyset = ET_n(p_2)$, and, since p_1 and p_2 are alive exit states, it also follows from this that both $p_1\downarrow$ and $p_2\downarrow$.

Suppose that $C = C' \cdot q$, with C' a non-trivial strongly connected component, and let $p_1', p_2' \in C'$ be such that $p_1 = p_1' \cdot q$ and $p_2 = p_2' \cdot q$. Since p_1 and p_2 are alive exit states, by Lemma 3.14 so are p_1' and p_2'. Hence, by the induction hypothesis, $ET_n(p_1') = ET_n(p_2')$ and $p_1'\downarrow$ iff $p_2'\downarrow$. From the latter it immediately follows that $p_1\downarrow$ iff $p_2\downarrow$. We now apply Lemma 3.15: if, on the one hand, $p_1\downarrow$ and $p_2\downarrow$, then

$$\begin{aligned} ET_n(p_1) &= ET_n(p_1') \cdot q \cup \{(a,r) \mid r \notin C \land \exists r'.\, q \xrightarrow{a} r \longrightarrow^* r'\downarrow\} \\ &= ET_n(p_2') \cdot q \cup \{(a,r) \mid r \notin C \land \exists r'.\, q \xrightarrow{a} r \longrightarrow^* r'\downarrow\} \\ &= ET_n(p_2)\ , \end{aligned}$$

and if, on the other hand, $p_1\not\downarrow$ and $p_2\not\downarrow$, then

$$ET_n(p_1) = ET_n(p_1') \cdot q = ET_n(p_2') \cdot q = ET_n(p_2)\ . \qquad \square$$

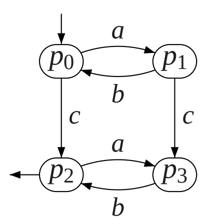

Figure 3: A $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ transition system that is not expressible in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$.

Theorem 3.17. $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is less expressive than $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$.

Proof. According to Definition 2.3 we should prove that every state in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is bisimilar to a state in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and that there exists a state in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ for which there is no bisimilar state in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$.

That every state in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is bisimilar to a state in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is immediately clear since $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is an operationally conservative extension of $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$.

To prove that there exists a state in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ for which there is no bisimilar state in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, consider the $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression $\mathbf{1} \cdot (a \cdot b)^* \parallel c$. Let us use the following abbreviations: $p_0 = \mathbf{1} \cdot (a \cdot b)^* \parallel c$, $p_1 = b \cdot (a \cdot b)^* \parallel c$, $p_2 = \mathbf{1} \cdot (a \cdot b)^* \parallel \mathbf{1}$, and $p_3 = b \cdot (a \cdot b)^* \parallel \mathbf{1}$. The transition system associated with this $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression is shown in Figure 3; for clarity we have labelled the states with abbreviations of the $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions. To establish that there is no $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression bisimilar to $\mathbf{1} \cdot (a \cdot b)^* \parallel c$, we assume that p is a $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression bisimilar to $\mathbf{1} \cdot (a \cdot b)^* \parallel c$ and derive a contradiction. Note that the set $C = \{p_0, p_1\}$ is a non-trivial strongly connected component in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $p_0 \mathrel{\underline{\leftrightarrow}} p$. Hence, by Lemma 2.5, there is a strongly connected component C' in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ reachable from p satisfying the condition that there exist $p_0', p_1' \in C'$ such that $p_0 \mathrel{\underline{\leftrightarrow}} p_0'$ and $p_1 \mathrel{\underline{\leftrightarrow}} p_1'$. From $p_0 \mathrel{\underline{\leftrightarrow}} p_0'$ and $p_0 \xrightarrow{c} p_2$ it follows that there exists a $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression p_2' such that $p_0' \xrightarrow{c} p_2'$ and $p_2 \mathrel{\underline{\leftrightarrow}} p_2'$. Similarly, from

$p_1 \leftrightarrows p'_1$ and $p_1 \xrightarrow{c} p_3$ it follows that there exists a $\text{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression p'_3 such that $p'_1 \xrightarrow{c} p'_3$ and $p_3 \leftrightarrows p'_3$. It is easy to see that $p'_2 \notin C'$, for $p'_2 \in C'$ would imply the existence of a transition sequence $p'_2 \longrightarrow^* p'_0 \xrightarrow{c} p'_2$ that clearly cannot be simulated by p_2. For similar reasons, $p'_3 \notin C'$. Further, note that from $p_2 \leftrightarrows p'_2$ and $p_2\downarrow$ it follows that p'_2 and p'_3 are alive, and from $p_3 \leftrightarrows p'_3$ and $p_3 \xrightarrow{b} p_2$ it follows that there exists p''_2 such that $p'_3 \xrightarrow{b} p''_2$ and $p_2 \leftrightarrows p''_2$, and therefore $p''_2\downarrow$. We conclude that both p'_0 and p'_1 are alive exit states. By Proposition 3.16 p'_0 and p'_1 have the same normed exit transitions, so, in particular, $p'_1 \xrightarrow{c} p'_3$. However, since $p'_2 \not\leftrightarrows p'_3$ we have a contradiction. □

4 Relative Expressiveness of $\text{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $\text{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$

The proof in [4] that $\text{PA}^*_\delta(\mathcal{A})$ is less expressive than $\text{ACP}^*(\mathcal{A},\gamma)$ uses the same expression as the one showing that $\text{BPA}^*_\delta(\mathcal{A})$ is less expressive than $\text{PA}^*_\delta(\mathcal{A})$, but it presupposes that $\gamma(c,d) = e$. It is claimed that the associated transition system fails the following general property of cycles in $\text{PA}^*_\delta(\mathcal{A})$:

> If C is a cycle reachable from a $\text{PA}^*_{\mathbf{0}}(\mathcal{A})$ process term and there is a state in C with a transition to $\mathbf{1}$, then all other states in C have only successors in C.

The claim, however, is incorrect, as illustrated by the following example. (To avoid having to introduce the syntax and operational semantics of $\text{PA}^*_\delta(\mathcal{A})$ formally, we present the example in the syntax of $\text{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. To translate it into the syntax of $\text{PA}^*_\delta(\mathcal{A})$ the occurrence of $\mathbf{1}\cdot$ can simply be removed, and ${}^*\cdot$ should simply be replaced by (the binary version of) *; we refer to [4] for the operational semantics of $\text{PA}^*_\delta(\mathcal{A})$.)

Example 4.1. Consider the $\text{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression $(a\cdot b\cdot(c+c\cdot c))^*\cdot d$, from which the cycle

$$C = \{\mathbf{1}\cdot(a\cdot b\cdot(c+c\cdot c))^*\cdot d,\ b\cdot(c+c\cdot c)\cdot(a\cdot b\cdot(c+c\cdot c))^*\cdot d,\ (c+c\cdot c)\cdot(a\cdot b\cdot(c+cc))^*\cdot d\}$$

is reachable. Clearly, the first expression in C can perform a d-transition to $\mathbf{1}$. Then, according to the property above, every other expression only has transitions to expressions in C. However,

$$(c+c\cdot c)\cdot(a\cdot b\cdot(c+c\cdot c))^*\cdot d \xrightarrow{c} c\cdot(a\cdot b\cdot(c+c\cdot c))^*\cdot d \notin C\ .$$

If we replace, in the property above, the notion of cycle by the notion of strongly connected component, then the resulting property does hold for $\text{PA}^*_{\mathbf{0}}(\mathcal{A})$, but it still fails for $\text{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$.

Example 4.2. Consider the $\text{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression $(a\cdot b)^* \parallel c$; it gives rise to the following non-trivial strongly connected component:

$$\{\mathbf{1}\cdot(a\cdot b)^* \parallel c,\ b\cdot(a\cdot b)^* \parallel c\}\ .$$

The expression $\mathbf{1}\cdot(a\cdot b)^* \parallel c$ can do a c-transition to $\mathbf{1}\cdot(a\cdot b)^* \parallel \mathbf{1}$, for which the termination predicate holds, but at the same time $b\cdot(a\cdot b)^* \parallel c$ has an exit transition $(c, b\cdot(a\cdot b)^* \parallel \mathbf{1})$.

In this section we shall establish that $\text{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is less expressive than $\text{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$. To this end, we apply the same method as in Section 3. The remainder of this section is organised as follows. First, we investigate the non-trivial strongly connected components associated with $\text{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions. Then, we conclude that a weakened version of the aforementioned property for strongly connected components holds in $\text{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, and present an $\text{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expression that does not satisfy it.

4.1 Strongly Connected Components in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$

To give a syntactic characterisation of the non-trivial strongly connected components in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, we reason again about the operational semantics. First, we extend the measure $\#(\_)$ from Section 3 to $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions.

Definition 4.3. Let p be a $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression; $\#(p)$ is defined with recursion on the structure of p by the clauses (i)–(iv) in Definition 3.2 with the following clauses added:

(v) $\#(p \mathbin{\|\!\!\lfloor} q) = 1$; and

(vi) $\#(p \parallel q) = 0$.

With the extension, the non-increasing measure $\#(\_)$ still in most cases decreases over transitions.

Lemma 4.4. If p and p' are $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions such that $p \longrightarrow^+ p'$, then $\#(p) \geq \#(p')$. Moreover, if $\#(p) = \#(p')$, then either $p = p_1 \cdot q$ and $p' = p'_1 \cdot q$, or $p = p_1 \parallel p_2$ and $p' = p'_1 \parallel p_2$, or $p = p_1 \parallel p_2$ and $p' = p_1 \parallel p'_2$ for some process expressions p_1, p_2, p'_1, p'_2, and q.

Proof. As in the proof of Lemma 3.3 we note that if the lemma holds in the special case that $p \longrightarrow p'$, then the general case follows with a straightforward induction on the length of a transition sequence from p to p'. In the remainder of the proof, we only consider the special case. Let $a \in \mathcal{A}$ such that $p \xrightarrow{a} p'$; as in the proof of Lemma 3.3 we reason by induction on a derivation according to the operational rules for $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ (rules 1–14) of the transition $p \xrightarrow{a} p'$. The arguments for rules 1–11 are the same as in the proof of Lemma 3.3, so we only present arguments for the remaining rules.

12. Suppose that the last rule applied is rule 12. Then there exist p_1, p'_1 and p_2 such that $p_1 \xrightarrow{a} p'_1$, $p' = p'_1 \parallel p_2$, and either $p = p_1 \parallel p_2$ or $p = p_1 \mathbin{\|\!\!\lfloor} p_2$. If $p = p_1 \parallel p_2$, then $\#(p) = 0 = \#(p')$, and if $p = p_1 \mathbin{\|\!\!\lfloor} p_2$, then $\#(p) = 1 > 0 = \#(p')$.
13. Suppose that the last rule applied is rule 13. Then there exist p_1, p_2 and p'_2 such that $p = p_1 \parallel p_2$, $p_2 \xrightarrow{a} p'_2$, and $p' = p_1 \parallel p'_2$, and hence $\#(p) = 0 = \#(p')$.

(Of course, since rule 14 does not have a transition as conclusion, we need not consider it.) □

Lemma 4.5. Let p, q and r be $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ process expressions such that $p \parallel q \longrightarrow^* r$. Then there exist $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ process expressions p' and q' such that $r = p' \parallel q'$, $p \longrightarrow^* p'$ and $q \longrightarrow^* q'$.

Proof. From $p \parallel q \longrightarrow r$ it is easily deduced, by distinguishing cases according to the last rule applied in a derivation of this transition, that either there exists a $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ process expression p' such that $r = p' \parallel q$ and $p \longrightarrow p'$, or there exists a $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ process expression q' such that $r = p \parallel q'$ and $q \longrightarrow q'$. The lemma follows from this by induction on the length of a transition sequence from $p \parallel q$ to r. □

Let P and Q be sets of process expressions; by $P \parallel Q$ we denote the set of process expressions

$$P \parallel Q = \{p \parallel q \mid p \in P \wedge q \in Q\} \ .$$

We also write $P \parallel q$ and $p \parallel Q$ for $P \parallel \{q\}$ and $\{p\} \parallel Q$, respectively.

The proof of the following lemma, characterising the syntactic form of non-trivial strongly connected components in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, is a straightforward adaptation and extension of the proof of Lemma 3.4, using Lemma 4.4 and Lemma 4.5 instead of Lemma 3.3.

Lemma 4.6. If C is a non-trivial strongly connected component in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, then either there exist a set of process expressions C' and a process expression q such that $C = C' \cdot q$, or there exist strongly connected components C_1 and C_2 in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, at least one of them non-trivial, such that $C = C_1 \parallel C_2$.

The notion of *basic* strongly connected component in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is obtained from Definition 3.5 by replacing $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ by $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ everywhere in the definition. In Proposition 3.7 we gave an inductive characterisation of non-trivial strongly connected components in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. There is a similar inductive characterisation of non-trivial strongly connected components in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, obtained by simply adding a case for parallel composition.

Proposition 4.7. Let C be a non-trivial strongly connected component in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. Then one of the following holds:

(i) C is a basic strongly connected component; or

(ii) there exist a non-trivial strongly connected component C' and a $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression q such that $C = C' \cdot q$; or

(iii) there exist strongly connected components C_1 and C_2, at least one of them non-trivial, such that $C = C_1 \parallel C_2$.

Proof. Let C be a non-trivial strongly connected component in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. According to Lemma 4.6 there are two cases: either there exist a set of process expressions C' and a process expression q such that $C = C' \cdot q$, or there exist strongly connected components C_1 and C_2, at least one of them non-trivial, such that $C = C_1 \parallel C_2$. In the second case, there is nothing left to prove. In the first case, applying the same reasoning as in the proof of Proposition 3.7, using Lemma 4.4 instead of Lemma 3.3, it can be argued that either C is basic, or C' is a non-trivial strongly connected component. □

Note that, in the above proposition, one of the strongly connected components C_1 and C_2 may be trivial in which case it corresponds to a single $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression.

4.2 $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A}) \prec \mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$

In Section 3 we deduced, from our syntactic characterisation of strongly connected components in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, the property that all alive exit states of a strongly connected component have the same sets of normed exit transitions. This property may fail for strongly connected components in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$: the transition system in Figure 3 is $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$-expressible, but the alive exit states p_0 and p_1 of the strongly connected component $\{p_0, p_1\}$ have different normed exit transitions. Note, however, that these normed exit transitions both end up in another strongly connected component $\{p_2, p_3\}$. It turns out that we can relax the requirement on normed exit transitions from strongly connected components in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ to get a requirement that holds for strongly connected components in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. The idea is to identify exit transitions if they have the same action and end up in the same strongly connected component.

Definition 4.8. Let $\mathcal{T} = (S, \rightarrow, \downarrow)$ be an $\mathcal{A}$-labelled transition system space. We define a binary relation $\sim$ on $\mathcal{A} \times S$ by $(a,s) \sim (a',s')$ iff $a = a'$ and s and s' are in the same strongly connected component in $\mathcal{T}$.

Since the relation of being in the same strongly connected component is an equivalence on states in a transition system space, it is clear that $\sim$ is an equivalence relation on exit transitions. The following lemma will give some further properties of the relation $\sim$ associated with $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$.

Lemma 4.9. Let p and q be $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions, and let a and b be actions.

(i) If $(a,p) \sim (b,q)$, then $(a, p \cdot r) \sim (b, q \cdot r)$.

(ii) If $(a,p) \sim (b,q)$, then $(a, p \parallel r) \sim (b, q \parallel r)$.

(iii) If $(a,p) \sim (b,q)$, then $(a, r \parallel p) \sim (b, r \parallel q)$.

Proof. Suppose that $(a,p) \sim (b,q)$. Then $a = b$, and p and q are in the same strongly connected component of $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. Then $p \longrightarrow^* q \longrightarrow^* p$, and hence $p \cdot r \longrightarrow^* q \cdot r \longrightarrow^* p \cdot r$, $p \parallel r \longrightarrow^* q \parallel r \longrightarrow^* p \parallel r$, and $r \parallel p \longrightarrow^* r \parallel q \longrightarrow^* r \parallel p$. From this it follows that $p \cdot r$ and $q \cdot r$, $p \parallel r$ and $q \parallel r$, and $r \parallel p$ and $r \parallel q$, are in the same strongly connected component, and hence $(a, p \cdot r) \sim (b, q \cdot r)$, $(a, p \parallel r) \sim (b, q \parallel r)$, and $(a, r \parallel p) \sim (b, r \parallel q)$. □

To formulate a straightforward corollary of this lemma we use the following notation: if E is a set of exit transitions E and p is a $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression, then $E \cdot p$, $E \parallel p$ and $p \parallel E$ are defined by

$$E \cdot p = \{(a, q \cdot p) \mid (a,q) \in E\} \ ,$$
$$E \parallel p = \{(a, q \parallel p) \mid (a,q) \in E\} \ , \text{ and}$$
$$p \parallel E = \{(a, p \parallel q) \mid (a,q) \in E\} \ .$$

We are now in a position to establish a property of strongly connected components in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ that will allow us to prove that $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is less expressive than $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$: a strongly connected component C in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ always has a special exit state from which, up to $\sim$, all exit transitions are enabled.

Lemma 4.10. Let C_1 and C_2 be sets of $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expressions. Then $C_1 \parallel C_2$ is a strongly connected component iff both C_1 and C_2 are strongly connected components. Moreover, $C_1 \parallel C_2$ is non-trivial iff at least one of C_1 and C_2 is non-trivial.

Proof. The lemma is a straightforward consequence of Lemma 4.5. □

Lemma 4.11. Let C_1 and C_2 be a strongly connected components in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, both with alive exit states. Then $C_1 \parallel C_2$ is a strongly connected component with alive exit states too, and, for all $p \in C_1$ and $q \in C_2$,

$$ET_n(p \parallel q) = (ET_n(p) \parallel q) \cup (p \parallel ET_n(q)) \ .$$

Proof. Suppose that C_1 and C_2 are strongly connected components with alive exit states. Then by Lemma 4.10 $C_1 \parallel C_2$ is a strongly connected component too. Further note that if p is an alive exit state of C_1 and q is an alive exit state of C_2, then $p \parallel q$ is an alive exit state of $C_1 \parallel C_2$. So it remains to prove that for all $p \in C_1$ and $q \in C_2$

$$ET_n(p \parallel q) = (ET_n(p) \parallel q) \cup (p \parallel ET_n(q)) \ .$$

To prove that $ET_n(p \parallel q) \subseteq (ET_n(p) \parallel q) \cup (p \parallel ET_n(q))$, consider an arbitrary $(a,r) \in ET_n(p \parallel q)$. Then $r \notin C_1 \parallel C_2$ and there exists a $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression r' such that $p \parallel q \xrightarrow{a} r \longrightarrow^* r'{\downarrow}$. From the operational rules for $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ with $\parallel$ in the conclusion it follows that we can distinguish two cases: either there exists p' such that $r = p' \parallel q$ and $p \xrightarrow{a} p'$, or there exists q' such that $r = p \parallel q'$ and $q \xrightarrow{a} q'$. The proofs for these cases are completely analogous; we only present details for the first case. Since $q \in C_2$, $p' \parallel q \notin C_1 \parallel C_2$ implies that $p' \notin C_1$, and $p' \parallel q = r \longrightarrow^* r'{\downarrow}$ implies the existence of a p'' such that $p' \longrightarrow^* p''{\downarrow}$. So $(a,p') \in ET_n(p)$, and hence $(a, p' \parallel q) \in ET_n(p) \parallel q$.

To prove that $(ET_n(p) \parallel q) \subseteq ET_n(p \parallel q)$ consider an arbitrary $(a,r) \in ET_n(p) \parallel q$. Then there exists a $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression p' such that $(a,p') \in ET_n(p)$ and $r = p' \parallel q$. From $(a,p') \in ET_n(p)$ it follows that $p' \notin C_1$ and there exists p'' such that $p' \longrightarrow^* p''{\downarrow}$. Since $p' \notin C_1$, we have that $p' \not\longrightarrow^* p$ and hence, by Lemma 4.5, $p' \parallel q \not\longrightarrow^* p \parallel q$. Further note that, by our assumption that C_2 contains an alive exit state, there exists q' such that $q \longrightarrow^* q'{\downarrow}$, so $p' \parallel q \longrightarrow^* p' \parallel q'{\downarrow}$. We conclude that $(a, p' \parallel q) \in ET_n(p \parallel q)$.

The proof that $(p \parallel ET_n(q)) \subseteq ET_n(p \parallel q)$ is completely analogous to the proof that $(ET_n(p) \parallel q) \subseteq ET_n(p \parallel q)$, so the proof of the lemma is now complete. □

To formulate the special property of strongly connected components in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ that will allow us to prove that some $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expressions do not have a counterpart in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, we need the notion of maximal alive exit state.

Definition 4.12. Let $\mathcal{T} = (S, \rightarrow, \downarrow)$ be an $\mathcal{A}$-labelled transition system space, let $\sim \subseteq \mathcal{A} \times S$ be the equivalence relation associated with $\mathcal{T}$ according to Definition 4.8, let C be a strongly connected component in $\mathcal{T}$, and let $s \in C$ be an alive exit state. We say that s is *maximal* (modulo $\sim$) if for all alive exit states $s' \in C$ and for all $e' \in ET_n(s')$ there exists an exit transition $e \in ET_n(s)$ such that $e \sim e'$.

The following proposition establishes the property with which we shall prove that $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is less expressive than $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$.

Proposition 4.13. If C is a strongly connected component in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and C has an alive exit state, then C has a maximal alive exit state.

Proof. Suppose that C is a strongly connected component in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ with an alive exit state. If C is trivial, then C is a singleton, say $C = \{p_m\}$, and, clearly, if p_m is alive, then it is maximal. So, for the remainder of the proof we may assume that C is non-trivial. We proceed to prove with induction on the structure of C, as given by Proposition 4.7, that C has maximal alive exit state p_m such that, in addition, $p_m\downarrow$ if $p\downarrow$ for some $p \in C$. We distinguish three cases:

1. If C is basic, then, by Lemma 3.13, $ET_n(p) = \emptyset$ for all $p \in C$. We have assumed that C contains an alive exit state, say p_m, which is then vacuously maximal and satisfies $p_m\downarrow$.
2. Suppose that $C = C' \cdot q$ with C' a non-trivial strongly connected component and q a $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression. Let $p' \cdot q$, for some $p' \in C$, be some alive exit state in C; then, by Lemma 3.14, p' is an alive exit state in C' and $q \longrightarrow^* q'\downarrow$ for some $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression q. Hence, by the induction hypothesis, C' has a maximal alive exit state $p_m \in C'$ such that $p_m\downarrow$ if $p\downarrow$ for some $p \in C'$. By Lemma 3.14 $p_m \cdot q$ is an alive exit state in C. If $p \cdot q\downarrow$ for some $p \in C$, then $p\downarrow$ and $q\downarrow$, so $p_m\downarrow$, and hence $p_m \cdot q\downarrow$.

   For this case it therefore remains to prove that $p_m \cdot q$ is maximal. To this end, consider an alive exit state $p \cdot q \in C$ and let (a, r) be a normed exit transition from $p \cdot q$. Then by Lemma 3.15 either there exist p' such that $r = p' \cdot q$ and (a, p') is a normed exit transition from p, or $p\downarrow$ and there exists a $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression r' such that $q \xrightarrow{a} r \longrightarrow^* r'$. In the first case, since p_m is maximal, there exists an exists a $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression p'_m such that (a, p'_m) is a normed exit transition from p_m and $(a, p'_m) \sim (a, p_m)$; it follows by Lemma 3.15 that $(a, p'_m \cdot q)$ is a normed exit transition from $(a, p_m \cdot q)$, and by Lemma 4.9 $(a, p_m \cdot q) \sim (a, r)$. In the second case, since $p'\downarrow$ implies $p_m\downarrow$, (a, r) is a normed exit transition from $p_m \cdot q$.
3. Suppose that $C = C_1 \parallel C_2$ with C_1 and C_2 strongly connected components. Since C has an alive exit state, both C_1 and C_2 have alive exit states.

   There are now three cases: only C_1 is non-trivial, only C_2 is non-trivial, or C_1 and C_2 are both non-trivial. We only present details for the third case; for the other cases the reasoning is similar.

   Let p_m and q_m be maximal alive exit states in C_1 and C_2, respectively. We argue that $p_m \parallel q_m$ is a maximal alive exit state in C, and that if $p \parallel q\downarrow$ for some $p \parallel q \in C$, then $p_m \parallel q_m\downarrow$.

   That $p_m \parallel q_m\downarrow$ if $p \parallel q\downarrow \in C$ is immediate by operational rule 14 and the induction hypothesis.

   To see that $p_m \parallel q_m$ is maximal, suppose that $e \in ET_n(p \parallel q)$ for some $p \parallel q \in C$. Then by Lemma 4.11 there are two cases: either there exists $e' \in ET_n(p)$ such that $e = e' \parallel q$, or there exists $e' \in ET_n(q)$ such that $e = p \parallel e'$. In the first case, if $e' \in ET_n(p)$, then by the induction hypothesis there exists $e'' \in ET_n(p_m)$ such that $e'' \sim e'$. By Lemma 4.11, $e'' \parallel q_m \in ET_n(p_m \parallel q_m)$ and since q_m and q are in the same strongly connected component C_2, it is clear that $e'' \parallel q_m \sim e$. In the second case, if $e' \in ET_n(p_m)$ we find by a analogous reasoning e'' such that $p_m \parallel e'' \in ET_n(p_m \parallel q_m)$ and $p_m \parallel e'' \sim e$. □

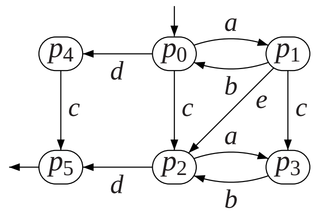

Figure 4: An $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ transition system that is not expressible in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$.

Theorem 4.14. $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is less expressive than $\bigcup_\gamma \mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$.

Proof. According to Definition 2.3 we should prove that every state in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is bisimilar to a state in $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ and that there exists a state in $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ for which there is no bisimilar state in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$.

That every state in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is bisimilar to a state in $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ is immediate since $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is included in $\bigcup_\gamma \mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ as $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\emptyset)$.

To prove that there exists a state in $\bigcup_\gamma \mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ for which there is no bisimilar state in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$, consider the $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expression $\mathbf{1}\cdot(a\cdot b)^*\cdot d \parallel c$ with γ satisfying $\gamma(b,c)=\gamma(c,b)=e$ and γ undefined everywhere else. Let us use the following abbreviations:

$$\begin{aligned}
p_0 &= \mathbf{1}\cdot(a\cdot b)^*\cdot d \parallel c \ \ ,\\
p_1 &= \mathbf{1}\cdot b\cdot(a\cdot b)^*\cdot d \parallel c \ \ ,\\
p_2 &= \mathbf{1}\cdot(a\cdot b)^*\cdot d \parallel \mathbf{1} \ \ ,\\
p_3 &= \mathbf{1}\cdot b\cdot(a\cdot b)^*\cdot d \parallel \mathbf{1} \ \ ,\\
p_4 &= \mathbf{1} \parallel c \ \ , \text{ and}\\
p_5 &= \mathbf{1} \parallel \mathbf{1} \ \ .
\end{aligned}$$

The transition system associated with p_0 is shown in Figure 4, and for clarity we have labelled the states with the corresponding abbreviations.

To establish that there is no $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression bisimilar to p_0, we assume that p is such a $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression and derive a contradiction. Note that the set $C=\{p_0,p_1\}$ is a strongly connected component in $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$. Hence, since $p_0 \leftrightarrows p$, by Lemma 2.5 there is a strongly connected component C' in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ reachable from p satisfying the condition that there exist $p'_0, p'_1 \in C'$ such that $p_0 \leftrightarrows p'_0$ and $p_1 \leftrightarrows p'_1$. By Proposition 4.13, C' has a maximal alive exit state. From $p_0 \leftrightarrows p'_0$ it follows that there exists $p'_4 \notin C'$ such that $p_0 \xrightarrow{d} p'_4$ and $p_4 \leftrightarrows p'_4$. From $p_1 \leftrightarrows p'_1$ it follows that there exists $p'_2 \notin C'$ such that $p'_1 \xrightarrow{e} p'_2$ and $p_2 \leftrightarrows p'_2$. So, on the one hand, both p'_0 and p'_1 are alive exit states. On the other hand, p'_0 does not have a normed exit transition labelled with an e and p_1 does not have a normed exit transition labelled with a d. We conclude that C' does not have a maximal alive exit state, and thus arrive at a contradiction. We conclude that p_0 is not $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$-expressible. □

5 Every Finite Transition System is $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$-expressible

Recall the well-known result from automata theory that for every NFA there exists a regular expression describing the language of that NFA. The result can be rephrased using process-theoretic terminology by saying that, modulo language equivalence, every finite transition system is equivalent to the transition system associated with a $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression. It was observed by Milner in [10] that the result is not true modulo bisimilarity: there exist finite transition systems that are not bisimilar to the transition system associated with a $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression.

Note that our proof of Theorem 3.17 has Milner's observation as an immediate consequence: the transition system associated with the $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ expression used in the proof is finite, but it is not $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$-expressible. Similarly, by Theorem 4.14, there are finite transition systems that are not expressible in $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. The question remains whether it is possible to express every finite transition system in $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$. In this section we shall address this question. We shall prove that every finite transition system is expressible in $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$, for suitable choices of $\mathcal{A}$ and γ, even up to isomorphism. Note that it can be proved that this result can only be obtained if $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ includes encapsulation; again by characterising the exit transitions of strongly connected components. As we recall, the counterexample used to prove Theorem 4.14, did not use encapsulation, hence the expressiveness of $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ excluding encapsulation is somewhere in between that of $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$.

Before we formally prove the result, let us first explain the idea informally, and illustrate it with an example. Suppose that $\mathcal{T}$ is a finite transition system that we want to describe in a suitable instance of $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$. The $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expression $p_{\mathcal{T}}$ that we shall associate with $\mathcal{T}$ will have one parallel component for every state of the transition system; this parallel component represents the behaviour in the state (i.e., which outgoing transitions it has to which other states and whether it is terminating). At any time, one of those parallel components, the one corresponding with the "current state," has control. An a-transition from that current state to a next state corresponds with a communication between two components. We make essential use of $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$'s facility to let the action a be the result of communication.

Example 5.1. Let $\mathcal{T}$ be the finite transition system in Figure 5.

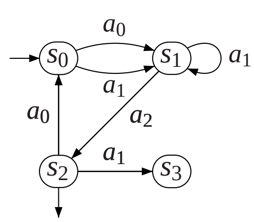

Figure 5: A finite transition system.

We associate with every state s_i of $\mathcal{T}$ an $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expression p_i as follows:

$$p_0 = \Big(enter_0 \cdot (leave_{0,1} + leave_{1,1})\Big)^* \ ,$$
$$p_1 = \Big(enter_1 \cdot {a_1}^* \cdot (leave_{2,2})\Big)^* \ ,$$
$$p_2 = \Big(enter_2 \cdot (leave_{1,3} + \mathbf{1})\Big)^* \ ,$$
$$p_3 = \Big(enter_3 \cdot \mathbf{0}\Big)^* \ .$$

Every p_i has an $enter_i$ transition to gain control, and by executing a $leave_{k,j}$ it may then release control to p_j with action a_k as effect. We define the communication function such that an $enter_i$ action communicates with a $leave_{k,i}$ action, resulting in the action a_k. Loops in the transition system (such as the loop on state s_1) require special treatment as they should not release control.

Let p'_0 be the result of executing the $enter_0$-transition from p_0. We define $p_{\mathcal{T}}$, the $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expression that simulates $\mathcal{T}$, as the parallel composition of p'_0, p_1, p_2 and p_3, encapsulating the control actions $enter_i$ and $leave_{k,i}$, i.e.,

$$p_{\mathcal{T}} = \partial_{\{enter_i, leave_{k,i} \mid 0 \le i \le 3,\ 0 \le k \le 2\}}(p'_0 \parallel p_1 \parallel p_2 \parallel p_3) \ .$$

We proceed to define the association of an $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expression $p_{\mathcal{T}}$ with a finite transition system $\mathcal{T}$ in full generality.

Let $\mathcal{T}$ be a finite transition system. Then $\mathcal{T}$ has a finite set of states $S = \{s_1, \ldots, s_n\}$ and a finite transition relation $\rightarrow$. Furthermore, we assume that s_1 is the initial state of $\mathcal{T}$ and that $\downarrow$ denotes its termination relation. Since $\rightarrow$ is finite, there are only finitely many actions occurring as the label of a transition of $\mathcal{T}$; we suppose that $A = \{a_1, \ldots, a_m\}$ is the set of actions occurring on transitions in $\mathcal{T}$.

We proceed to associate an $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expression $p_{\mathcal{T}}$, which has precisely one parallel component p_i for every state s_i in S. To allow these parallel components to gain and release control, we use a collection of *control actions* C, assumed to be disjoint from A, and defined as

$$C = \{enter_i \mid 1 \le i \le n\} \cup \{leave_{k,i} \mid 1 \le i \le n,\ 1 \le k \le m\} \ .$$

Gaining and releasing control is modelled by the communication function γ satisfying:

$$\gamma(enter_i, leave_{k,j}) = \begin{cases} a_k & \text{if } i = j\text{; and} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

For the specification of the $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expressions p_i we need one more definition: for $1 \le i, j \le n$ we denote by $K_{i,j}$ the set of indices of actions occurring as the label on a transition from s_i to s_j, i.e.,

$$K_{i,j} = \{k \mid s_i \xrightarrow{a_k} s_j\} \ .$$

Now we can specify the $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expressions p_i ($1 \le i \le n$) by

$$p_i = \mathbf{1} \cdot \Big(enter_i \cdot (\sum_{k \in K_{i,i}} a_k)^* \cdot (\sum_{\substack{1 \le j \le n \\ j \ne i}} \sum_{k \in K_{i,j}} leave_{k,i}\ (+\,\mathbf{1})_{s_i\downarrow})\Big)^* \ .$$

By $(+\,\mathbf{1})_{s_i\downarrow}$ we mean that the summand $+\,\mathbf{1}$ is optional; it is only included if $s_i\downarrow$. The empty summation denotes $\mathbf{0}$. (We let p_i start with $\mathbf{1}$ to get that the transition system associated with $p_{\mathcal{T}}$ is isomorphic and not just bisimilar with $\mathcal{T}$.)

Note that, in $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$, every p_i has a unique outgoing transition; specifically $p_i \xrightarrow{enter_i} p'_i$, where p'_i denotes:

$$p'_i = (\mathbf{1} \cdot (\sum_{k \in K_{i,i}} a_k)^* \cdot (\sum_{\substack{0 \le j \le n \\ j \ne i}} \sum_{k \in K_{i,j}} leave_{k,i}\ (+\,\mathbf{1})_{s_i\downarrow})) \cdot p_i \ .$$

We now define $p_{\mathcal{T}}$ by

$$p_{\mathcal{T}} = \partial_C(p'_0 \parallel p_1 \parallel \cdots \parallel p_n) \ .$$

Clearly, the construction of $p_{\mathcal{T}}$ works for every finite transition system $\mathcal{T}$. The bijection defined by $s_i \mapsto \partial_C(p_0 \parallel \cdots \parallel p_{i-1} \parallel p'_i \parallel p_{i+1} \parallel \cdots \parallel p_n)$ is an isomorphism from $\mathcal{T}$ to the transition system of $p_{\mathcal{T}}$. We shall refer to $p_{\mathcal{T}}$ as the $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expression associated with $\mathcal{T}$.

Theorem 5.2. Let $\mathcal{T}$ be a finite transition system, and let $p_{\mathcal{T}}$ be its associated $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expression. The transition system associated with $p_{\mathcal{T}}$ by the operational rules for $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ is isomorphic to $\mathcal{T}$.

Proof. The bijection defined by $s_i \mapsto \partial_C(p_0 \parallel \cdots \parallel p_{i-1} \parallel p'_i \parallel p_{i+1} \parallel \cdots \parallel p_n)$ is an isomorphism from $\mathcal{T}$ to the transition system of $p_{\mathcal{T}}$ in $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$. □

Corollary 5.3. For every finite transition system $\mathcal{T}$ there exists an instance of $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ with a suitable finite set of actions $\mathcal{A}$ and a handshaking communication function γ such that $\mathcal{T}$ is $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$-expressible up to isomorphism.

6 Conclusion

In this paper we have investigated the effect on the expressiveness of regular expressions modulo bisimilarity if different forms of parallel composition are added. We have established an expressiveness hierarchy that can be briefly summarised as:

$$\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A}) \prec \mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A}) \prec \bigcup_{\gamma} \mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma) \enspace .$$

Furthermore, while not every NFA can be expressed modulo bisimilarity with a regular expression, it suffices to add a form of $\mathrm{ACP}(\mathcal{A},\gamma)$-style parallel composition, with handshaking communication and encapsulation, to get a language that is sufficiently expressive to express all NFAs modulo bisimilarity. This result should be contrasted with the well-known result from automata theory that every non-deterministic finite automaton can be expressed with a regular expression modulo language equivalence.

As an important tool in our proof, we have characterised the strongly connected components in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ and $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$. An interesting open question is whether the two given characterisations are complete, in the sense that an NFA is expressible in $\mathrm{BPA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ or $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ iff all its strongly connected components satisfy our characterisation. If so, then our characterisation would constitute a useful complement to the characterisation of [1] and perhaps lead to a more efficient algorithm for deciding whether a non-deterministic automaton is expressible.

In [4] it is proved that every finite transition system without intermediate termination can be denoted in $\mathrm{ACP}^*_{\mathbf{0},\tau}(\mathcal{A},\gamma)$ up to *branching* bisimilarity [7], and that $\mathrm{ACP}^*_{\mathbf{0}}(\mathcal{A},\gamma)$ modulo (strong) bisimilarity is strictly less expressive than $\mathrm{ACP}^*_{\mathbf{0},\tau}(\mathcal{A},\gamma)$. In contrast, we have established that every NFA (i.e., every finite transition system not excluding intermediate termination) is denoted by an $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expression. It follows that $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ and $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1},\tau}(\mathcal{A},\gamma)$ are equally expressive.

An interesting question that reamins is whether it is possible to omit constructions from $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ without losing expressiveness. For instance, it is easily verified that $\_ \mathbin{\|\!\_} \_$ and $\_ \mid \_$ do not add expressiveness. On the other hand, we conjecture that $\partial_H(\_)$ cannot be omitted without losing expressiveness: encapsulating c in the $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ expression $\mathbf{1} \cdot (a \cdot b)^* \cdot b \parallel c$, which is used in Section 4 to show that $\mathrm{PA}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A})$ is less expressive than $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$, yields a transition system that we think cannot be expressed in $\mathrm{ACP}^*_{\mathbf{0},\mathbf{1}}(\mathcal{A},\gamma)$ without encapsulation.

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