Remarks on Thatte's Transformation of Term Rewriting Systems

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Abstract

We carry out a detailed analysis of Thatte's transformation of term rewriting systems. We refute an earlier claim that this transformation preserves confluence for weakly persistent systems. We prove the preservation of weak normalization, and of confluence in weakly normalizing systems and in nonoverlapping systems with linear subtemplates. We conclude by proving that weak persistence is an undecidable property of term rewriting systems.

1 Introduction

The use of reduction systems as a semantical basis of programming paradigms leads naturally to the consideration of methods of transforming such systems. Some types of systems are more suitable for implementation than others; for instance because they have an easily decidable normalizing reduction strategy, or because the rules have a particular format. We want to transform arbitrary systems into systems of suitable type. Evidently the transformed system must be able to serve the purpose of its original in some sense. If it is, we can say the transformation is *correct*. Thus the concrete content of the notion of correctness depends on the particular application.

S. R. Thatte [11] proposed a transformation of arbitrary rewrite systems into *constructor-based systems*. To make a case for the correctness of his transformation when applied to an orthogonal system, Thatte adduces that his transformation preserves orthogonality and that there is a close correspondence between the reduction graphs of the original and its transform. Then, in [12], Thatte claims that if the orthogonality requirement is weakened by omitting left-linearity, his transformation is still correct in the sense that it preserves confluence. This claim, however, was refuted in [14], where the notion of weak persistence was proposed as the appropriate weakening of orthogonality.

In this paper we discuss the correctness of Thatte's transformation by investigating what properties it preserves. As a useful tool in our investigation, we propose, for arbitrary abstract reduction systems, a notion of ω-simulation, based on a simulation concept of Kamperman and Walters [5, 6]. From the close correspondence between the reduction graphs of

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original and transform exhibited by ω-simulation, preservation of confluence and preservation of weak normalization follow. We generalize a proof of Fokkink and Van de Pol [3] to show that ω-simulation also preserves the normal form relation.

Contrary to what is stated in [14], weak persistence is not a sufficient condition for the preservation of confluence. It does preserve confluence under a condition intermediate between weak persistence and orthogonality. Also, it preserves semi-completeness for weakly persistent systems. For terminating rewrite systems a simpler transformation method that preserves confluence is given in [13]. We conclude with a discussion of the computable content of the notion of weak persistence, proving that it is undecidable in general.

The remainder of the paper is organized as follows. In Section 2 we fix our terminology and notation with respect to abstract reduction systems. In Section 3 we discuss some correctness criteria for transformations of abstract reduction systems and introduce the notion of ω-simulation. In Section 4 we fix our terminology and notation with respect to term rewriting systems, and in Section 5 we present Thatte's transformation. In Section 6 we give the two counterexamples that refute claims made about Thatte's transformation in the literature. In Section 7 we prove that there is an ω-simulation from a sublinear system to its Thatte transform, and that Thatte's transformation preserves semi-completeness. In Section 8 we establish that it is in general undecidable whether a term rewriting system is weakly persistent. We end the paper with a few concluding remarks.

2 Definitions

We fix our terminology and notation generally in line with [2] and [10].

An abstract reduction system is a pair consisting of a set and a family of binary relations on this set; we shall only consider abstract reduction systems with just one binary relation. We adopt the convention that the base set of an abstract reduction system $\mathfrak{A}$ is A, $\mathfrak{B}$ is based on B, and so on. If $\mathfrak{A} = \langle A, \to \rangle$, we write $a \to b$ (sometimes $a \to_{\mathfrak{A}} b$) instead of $\langle a, b \rangle \in \to$.

The transitive closure of a binary relation R we denote by R^+, the transitive-reflexive closure by R^*. Instead of $\to^*$ we usually write $\twoheadrightarrow$. The symmetric closure of a binary relation $\to$ we denote by $\leftrightarrow$. A reduction is a (finite or infinite) sequence $a_0 \to a_1 \to \cdots$.

Let $\mathfrak{A} = \langle A, \to \rangle$ be an abstract reduction system. An element a of $\mathfrak{A}$ is a *normal form* if $\mathfrak{A} \models \neg\, \exists x.\ a \to x$. We write $a \to^! b$ for $a \twoheadrightarrow b \wedge \neg\, \exists x.\ b \to x$; and call b a *normal form of a in* $\mathfrak{A}$ if $\mathfrak{A} \models a \to^! b$. The set of all normal forms of a in $\mathfrak{A}$ we denote by $\mathcal{N}_{\mathfrak{A}}(a)$. For $X \subseteq A$, $\mathcal{N}_{\mathfrak{A}}(X) = \bigcup_{x \in X} \mathcal{N}_{\mathfrak{A}}(x)$. Generally, for any binary relation R, we put $R(X) = \{y \mid \exists x \in X. xRy\}$, and $R(x) = R(\{x\})$.

We abbreviate $\exists y.x \to^! y$ to $\mathsf{WN}(x)$; if $\mathfrak{A} \models \mathsf{WN}(a)$, a is said to be *weakly normalizing*. An element a is *strongly normalizing*, or *terminating*, notation $\mathfrak{A} \models \mathsf{SN}(a)$, if there is no infinite reduction sequence beginning with a. We abbreviate $\forall yz(x \to y \wedge x \to z \implies \exists u(y \twoheadrightarrow u \wedge z \twoheadrightarrow u))$ to $\mathsf{WCR}(x)$, and say a is *weakly confluent* or *weakly Church-Rosser* if $\mathfrak{A} \models \mathsf{WCR}(a)$. *Confluence* or the *Church-Rosser property* is expressed by $\forall yz(x \twoheadrightarrow y \wedge x \twoheadrightarrow z \implies \exists u(y \twoheadrightarrow u \wedge z \twoheadrightarrow u))$, abbreviated $\mathsf{CR}(x)$. Without parameter WCR stands for $\forall x. \mathsf{WCR}(x)$, and likewise CR, SN, etc.; $\mathfrak{A}$ is confluent, or Church-Rosser, if $\mathfrak{A} \models \mathsf{CR}$, and weakly so if $\mathfrak{A} \models \mathsf{WCR}$. We abbreviate $\forall yz(x \to^! y \wedge x \to^! z \implies y = z)$ to $\mathsf{UN}^{\to}(x)$, and $\forall x. \mathsf{UN}^{\to}(x)$ to $\mathsf{UN}^{\to}$; an element a of $\mathfrak{A}$ is *uniquely normalizing* if $\mathfrak{A} \models \mathsf{UN}^{\to}(a)$, and $\mathfrak{A}$ is uniquely normalizing if $\mathfrak{A} \models \mathsf{UN}^{\to}$. An abstract reduction system is *complete* if it is both SN and CR; *semi-complete* if it is both WN and CR.

The domain of a relation R we denote by $\mathsf{dom}(R)$ and the range by $\mathsf{ran}(R)$. The *closure* of a set X of elements of an abstract reduction system $\mathfrak{A} = \langle A, \to \rangle$ is $\twoheadrightarrow(X)$. If ϕ is a function into the base set of $\mathfrak{A}$, then an element of $\mathfrak{A}$ is *ϕ-reachable* if it belongs to the closure of $\mathsf{ran}(\phi)$; and $\mathfrak{A}$ is ϕ-reachable if every element of $\mathfrak{A}$ is ϕ-reachable.

3 Simulations and correctness criteria

We introduce the concepts of simulation and ω-simulation and prove some general properties of ω-simulation.

If an abstract reduction system $\mathfrak{B}$ simulates an abstract reduction system $\mathfrak{A}$, we expect that there is a mapping ϕ from objects of $\mathfrak{A}$ to objects of $\mathfrak{B}$; we want to simulate reductions of a by reductions of $\phi(a)$. Moreover, we require a backward translation ψ with $\psi(\phi(a)) = a$. The backward translation need not be defined for every object of $\mathfrak{B}$.

Definition 3.1 Let $\mathfrak{A}$ and $\mathfrak{B}$ be abstract reduction systems. A *simulation* of $\mathfrak{A}$ by $\mathfrak{B}$ is a pair $\langle\phi,\psi\rangle$ of a function $\phi : A \to B$ and a partial function $\psi : B \to A$ such that $\psi\phi = \mathsf{id}(A)$. A simulation is *functional* if it is of the form $\langle\phi,\phi^{-1}\rangle$; since a functional simulation $\langle\phi,\phi^{-1}\rangle$ is completely determined by its forward component ϕ, we may use ϕ to refer to it.

We call $\mathfrak{B}$ the *transform of* $\mathfrak{A}$, *by* $\langle\phi,\psi\rangle$; $\mathfrak{B}$, or an element of $\mathfrak{B}$, is *reachable* if it is ϕ-reachable. If $\langle\phi_1,\psi_1\rangle$ is a simulation of $\mathfrak{A}$ by $\mathfrak{B}$, and $\langle\phi_2,\psi_2\rangle$ of $\mathfrak{B}$ by $\mathfrak{C}$, then $\langle\phi_2\phi_1,\psi_1\psi_2\rangle$ is a simulation of $\mathfrak{A}$ by $\mathfrak{C}$.

The following definition has been adapted from [3].

Definition 3.2 Let $\langle\phi,\psi\rangle$ be a simulation of $\mathfrak{A}$ by $\mathfrak{B}$ such that $\mathcal{N}_{\mathfrak{B}}(\mathsf{ran}(\phi)) \subseteq \mathsf{dom}(\psi)$. Then $\langle\phi,\psi\rangle$ *preserves the normal form relation* if for all $a \in A$, $\psi(\mathcal{N}_{\mathfrak{B}}(\phi(a))) = \mathcal{N}_{\mathfrak{A}}(a)$.

It is easy to see that these requirements do not guarantee that $\mathfrak{B}$ inherits SN or CR from $\mathfrak{A}$, even if $\mathfrak{B}$ is ϕ-reachable. Accordingly, Fokkink and Van de Pol consider a further requirement on termination behavior in $\mathsf{ran}(\phi)$. Let us say, given a simulation $\langle\phi,\psi\rangle$ of $\mathfrak{A}$ by $\mathfrak{B}$ and a property X defined for elements of abstract reduction systems (in particular, X may be one of SN, WN, CR, WCR, $\mathsf{UN}^{\to}$), that $\langle\phi,\psi\rangle$ *preserves* X if for all $a \in A$, $\mathfrak{A} \models \mathrm{X}(a)$ implies $\mathfrak{B} \models \mathrm{X}(\phi(a))$. A reachable transform of a weakly normalizing system by a weak normalization preserving simulation need not be weakly normalizing. But:

Proposition 3.3 Let $\mathfrak{B}$ be a reachable transform of $\mathfrak{A}$ by $\langle\phi,\psi\rangle$. If $\mathfrak{A} \models \mathsf{SN}$ and $\langle\phi,\psi\rangle$ preserves termination, then $\mathfrak{B} \models \mathsf{SN}$.

Proposition 3.4 Let $\mathfrak{B}$ be a reachable transform of $\mathfrak{A}$ by $\langle\phi,\psi\rangle$. If $\mathfrak{A} \models \mathsf{CR}$ and $\langle\phi,\psi\rangle$ preserves confluence, then $\mathfrak{B} \models \mathsf{CR}$.

Proof. If $\mathfrak{B} \not\models \mathsf{CR}$, then $\mathfrak{B} \not\models \mathsf{CR}(b)$ for some $b \in B$, and since $\mathfrak{B}$ is reachable there exists $a \in A$ such that $\phi(a) \twoheadrightarrow b$. Then $\mathfrak{B} \not\models \mathsf{CR}(\phi(a))$, hence $\mathfrak{A} \not\models \mathsf{CR}(a)$. ☐

The analogous result for weak confluence does not hold.

Example 3.5 Consider the abstract reduction systems $\mathfrak{A} = \langle\{a,b\},\{a \to b\}\rangle$ and $\mathfrak{B} = \langle\{a,b,c,d\},\{a \to c,\ c \to b,\ c \to d\}\rangle$ with the simulation $\langle\mathsf{id}(A),\mathsf{id}(A)\rangle$. Note that $\mathfrak{A} \models \mathsf{WCR}$ and $\mathfrak{B} \models \mathsf{WCR}(a)$, $\mathfrak{B} \models \mathsf{WCR}(b)$, so $\langle\mathsf{id}(A),\mathsf{id}(A)\rangle$ preserves weak confluence. Also $\mathfrak{B}$ is reachable; but $\mathfrak{B} \not\models \mathsf{WCR}(c)$.

Fokkink and Van de Pol call $\mathfrak{B}$ a correct transformation of $\mathfrak{A}$ if there is a simulation $\langle\phi,\psi\rangle$ that preserves the normal form relation and termination.

Kamperman and Walters (see [5]) propose another correctness criterion. Roughly speaking, they call a simulation *sound* if each reduction in $\mathfrak{B}$ simulates a reduction in $\mathfrak{A}$, and *complete* if each reduction in $\mathfrak{A}$ is simulated by one in $\mathfrak{B}$. These properties also play a part in preservation proofs in [11] and [14].

Definition 3.6 A simulation $\langle\phi,\psi\rangle$ of $\mathfrak{A}$ by $\mathfrak{B}$ *preserves reduction graphs* if

1. $a \to_{\mathfrak{A}} a'$ implies $\phi(a) \to^{+}_{\mathfrak{B}} \phi(a')$;

2. if $b \to^{+}_{\mathfrak{B}} b'$ for $b \in \mathsf{dom}(\psi)$, then there exists $c \in \mathsf{dom}(\psi)$ such that $\psi(b) \to^{+}_{\mathfrak{A}} \psi(c)$ and $b' \twoheadrightarrow_{\mathfrak{B}} c$.

Condition 1 generalizes the notion of completeness of [5]; condition 2 is a form of soundness adapted with a view to Proposition 3.8 below.

We study the consequences, in a special case, of reduction graph preservation for the normal form relation and confluence.

Definition 3.7 An *ω-simulation* is a functional reduction graph preserving simulation.

The existence of an ω-simulation implies a close connection between normal form relations.

Proposition 3.8 Suppose ϕ is an ω-simulation of $\mathfrak{A}$ by $\mathfrak{B}$. Then for $a \in A$ and $b \in \mathsf{ran}(\phi)$,

(i) a is a normal form of $\mathfrak{A}$ iff $\phi(a)$ is a normal form of $\mathfrak{B}$;

(ii) if b' is a normal form of b, then $b' \in \mathsf{ran}(\phi)$, and $\phi^{-1}(b) \to^{!}_{\mathfrak{A}} \phi^{-1}(b')$.

Proof.

(i) If a is not a normal form, then neither is $\phi(a)$, by condition 1 of Definition 3.6. Conversely, if $\phi(a)$ is not a normal form of $\mathfrak{B}$, then by condition 2 of Definition 3.6, there exists $c \in \mathsf{ran}(\phi)$ such that $a \to^{+}_{\mathfrak{A}} \phi^{-1}(c)$, so a is not a normal form.

(ii) Suppose $b \to^{!} b'$ in $\mathfrak{B}$. If $b = b'$, then clearly $\phi^{-1}(b) \twoheadrightarrow_{\mathfrak{A}} \phi^{-1}(b')$, $b' \in \mathsf{ran}(\phi)$ and, by (i), $\phi^{-1}(b')$ is a normal form; so $\phi^{-1}(b) \to^{!}_{\mathfrak{A}} \phi^{-1}(b')$. On the other hand, if $b \neq b'$, then, by condition 2 of Definition 3.6, there exists $c \in \mathsf{ran}(\phi)$ such that $\phi^{-1}(b) \to^{+}_{\mathfrak{A}} \phi^{-1}(c)$ and $b' \twoheadrightarrow_{\mathfrak{B}} c$. Since b' is a normal form, it follows that $b' = c$. □

Corollary 3.9 An ω-simulation preserves the normal form relation.

Remark 3.10 Kennaway *et al.* [7] propose a notion quite similar to ω-simulation. They call a mapping $\phi : A \to B$ from the base set of $\mathfrak{A}$ into the base set of $\mathfrak{B}$ *adequate* if

1. ϕ is surjective;

2. $a \in A$ is a normal form of $\mathfrak{A}$ iff $\phi(a)$ is a normal form of $\mathfrak{B}$;

3. if $a \twoheadrightarrow_{\mathfrak{A}} a'$, then $\phi(a) \twoheadrightarrow_{\mathfrak{B}} \phi(a')$; and

4. for all $a \in A$ and $b \in B$, if $\phi(a) \twoheadrightarrow b$, then there is a $a' \in A$ such that $a \twoheadrightarrow a'$ and $b \twoheadrightarrow \phi(a')$.

According to Proposition 3.8(i), every ω-simulation satisfies condition 2 of adequacy. Conditions 3 and 4 are also satisfied by ω-simulations, since they are directly implied by conditions 1 and 2 of Definition 3.6, respectively. It follows that every surjective ω-simulation is *adequate*.

A useful property for correctness criteria is preservation of the correctness property under composition: that every composition of correct transformations is a correct transformation. It allows us to divide a complex transformation into steps that are more easily seen to be correct ('stepwise refinement' in [9]). The criterion of Fokkink and Van de Pol has this property, trivially. Preservation of reduction graphs does not survive composition, but ω-simulation does.

Theorem 3.11 Suppose ϕ and ψ are ω-simulations, of $\mathfrak{A}$ by $\mathfrak{B}$ and of $\mathfrak{B}$ by $\mathfrak{C}$ respectively. Then the composition $\psi\phi$ is an ω-simulation of $\mathfrak{A}$ by $\mathfrak{C}$.

Proof. Trivially, $\psi\phi$ satisfies the first condition of Definition 3.6.

To check condition 2, suppose $c \to^+_{\mathfrak{C}} c'$ with $c \in \mathsf{ran}(\psi\phi)$. We must find c^* such that $c' \twoheadrightarrow_{\mathfrak{C}} c^*$, $c^* \in \mathsf{ran}(\psi\phi)$, and $\phi^{-1}\psi^{-1}(c) \to^+_{\mathfrak{A}} \phi^{-1}\psi^{-1}(c^*)$.

Since $c \in \mathsf{ran}(\psi)$, there exists c'' such that $c' \twoheadrightarrow_{\mathfrak{C}} c''$, $c'' \in \mathsf{ran}(\psi)$ and $\psi^{-1}(c) \to^+_{\mathfrak{B}} \psi^{-1}(c'')$. Since $\psi^{-1}(c) \in \mathsf{ran}(\phi)$, there exists $b \in B$ such that $\psi^{-1}(c'') \twoheadrightarrow_{\mathfrak{B}} b$, $b \in \mathsf{ran}(\phi)$ and $\phi^{-1}\psi^{-1}(c) \to^+_{\mathfrak{A}} \phi^{-1}(b)$. Now $\psi(b)$ is the element c^* we are looking for. □

It is easy to see that a simulation that preserves reduction graphs does not necessarily preserve confluence, even if the transform is reachable. We do get this preservation property for ω-simulations.

Theorem 3.12 Let ϕ be an ω-simulation of $\mathfrak{A}$ by $\mathfrak{B}$.

i. For all $a \in A$, if $\mathfrak{A} \models \mathsf{CR}(a)$, then $\mathfrak{B} \models \mathsf{CR}(\phi(a))$.

ii. If $\mathfrak{B}$ is reachable and $\mathfrak{A} \models \mathsf{CR}$, then $\mathfrak{B} \models \mathsf{CR}$.

Proof.

(i) Fix $a \in A$, assume $\mathsf{CR}(a)$, and suppose $c \twoheadleftarrow_{\mathfrak{B}} \phi(a) \twoheadrightarrow_{\mathfrak{B}} b$. Since ϕ is an ω-simulation, there exist b', c' such that $b \twoheadrightarrow_{\mathfrak{B}} b'$, $c \twoheadrightarrow_{\mathfrak{B}} c'$, $b', c' \in \mathsf{ran}(\phi)$, $a \twoheadrightarrow_{\mathfrak{A}} \phi^{-1}(b')$ and $a \twoheadrightarrow_{\mathfrak{A}} \phi^{-1}(c')$. Since $\mathfrak{A} \models \mathsf{CR}(a)$, there exists a' such that $\phi^{-1}(c') \twoheadrightarrow_{\mathfrak{A}} a' \twoheadleftarrow_{\mathfrak{A}} \phi^{-1}(b')$. But then $c' \twoheadrightarrow_{\mathfrak{B}} \phi(a') \twoheadleftarrow_{\mathfrak{B}} b'$, hence $c \twoheadrightarrow_{\mathfrak{B}} \phi(a') \twoheadleftarrow_{\mathfrak{B}} b$. □

An ω-simulation does not necessarily preserve termination. For example, consider the ω-simulation $\mathsf{id}(A)$ of $\mathfrak{A}$ by $\mathfrak{B}$ where $\mathfrak{A}$ is given by $\{a \to c\}$ and $\mathfrak{B}$ by $\{a \to b,\ b \to b,\ b \to c\}$. In this case $\mathfrak{A}$ is SN, and $\mathfrak{B}$ is not, due to the cycle at b. In contrast, preservation of weak normalization is immediate by Proposition 3.8.

Theorem 3.13 Let ϕ be an ω-simulation of $\mathfrak{A}$ by $\mathfrak{B}$.

(i) For all $a \in A$, $\mathfrak{A} \models \mathsf{WN}(a)$ iff $\mathfrak{B} \models \mathsf{WN}(\phi(a))$.

(ii) If $\mathfrak{B}$ is reachable, then $\mathfrak{A} \models \mathsf{WN}$ iff $\mathfrak{B} \models \mathsf{WN}$.

By their failure to preserve termination, ω-simulations are not necessarily correct in the sense of Fokkink and Van de Pol. Conversely, a correct simulation need not be an ω-simulation. For example, if $\mathfrak{A}$ is given by $\{a \to a,\ a \to b\}$ and $\mathfrak{B}$ by $\{a \to b,\ a \to c,\ c \to c\}$, the simulation $\langle \mathsf{id}(A), \mathsf{id}(A) \rangle$ is correct, but there is no ω-simulation of $\mathfrak{A}$ by $\mathfrak{B}$.

4 Term rewriting systems

Given a signature (set of function symbols with fixed arities) Σ, we construct the universe U_Σ of terms over Σ in variables from some fixed infinite set in the usual way. For precise reference to subterm occurrences we use the notion of *position*. For a term t the set $P(t)$ of positions in t is defined as follows:

1. if t is a variable, then $P(t) = \{\lambda\}$ (where λ denotes the empty sequence);

2. if $t = F(t_1, \ldots, t_n)$, for n-ary function symbol F and terms $t_1, \ldots, t_n$, then $P(t) = \{\lambda\} \cup \{i.u \mid 1 \le i \le n \text{ and } u \in P(t_i)\}$ (where $i.u$ is the sequence obtained from u by prefixing i).

The subterm of t at position p we denote by t/p; the symbol (function or variable) at p by $t\tilde{~}p$. A position $p \in P(t)$ is a *variable position* if t/p is a variable, otherwise it is a *function symbol position*. We denote the set of function symbol positions in t by $P^{\mathcal{F}}(t)$, and the set of variable positions by $P^{\mathcal{V}}(t)$. We write $p \leq q$ if p is an initial segment of q, and $p < q$ if it is a proper initial segment. The concatenation of p and q will be denoted by $p.q$. Two positions are *disjoint* if neither is an initial segment of the other.

Suppose s and t are terms, and $p \in P^{\mathcal{F}}(t)$. We say *s overlaps t at p* if there exist substitutions σ, τ such that $s^\sigma = (t/p)^\tau$. If $p = \lambda$, then *s root overlaps t*; otherwise *s nonroot overlaps t*. If a set X of terms has no nonroot overlapping elements, we call X *nonoverlapping*.

We write $l \to r$ for a rewrite rule with left-hand side l and right-hand side r; the usual restrictions on l and r apply, so l must not be a variable and all variables occurring in r must also occur in l. A set $\mathcal{R}$ of rewrite rules induces a reduction relation $\to_{\mathcal{R}}$ on U_Σ. A reduction step from an instance l^σ of a left-hand side of a rewrite rule to the corresponding instance r^σ of its right-hand side we call a *contraction*. We write $s \xrightarrow{p}_{\mathcal{R}} t$ to express that $s \to_{\mathcal{R}} t$ by contraction of a redex at position p, i.e. for some rule $l \to r$ and substitution σ, $s = s[l^\sigma]_p$ and $t = s[r^\sigma]_p$. We say that a position q in s is *contracted* in such a reduction step if $q = p.o$ with $o \in P^{\mathcal{F}}(l)$. The *base* of $\mathcal{R}$ is the set $\mathcal{L}^{\mathcal{R}}$ of left-hand sides of elements of $\mathcal{R}$. A *subtemplate* of $\mathcal{R}$ is a nonvariable proper subterm of an element of $\mathcal{L}^{\mathcal{R}}$. Instances of subtemplates will be called *pseudoredexes*.

A *term rewriting system* is a triple $\langle \Sigma, T, \mathcal{R} \rangle$ in which $\mathcal{R}$ is a set of rewrite rules over Σ, and $T \subseteq U_\Sigma$ is closed under the induced reduction relation. We write $\langle \Sigma, \mathcal{R} \rangle$ instead of $\langle \Sigma, U_\Sigma, \mathcal{R} \rangle$, and sometimes $\mathcal{R}$ for $\langle \Sigma, \mathcal{R} \rangle$ where Σ is the least signature over which $\mathcal{R}$ can be constructed. All definitions and results about abstract reduction systems in the previous sections carry over to term rewriting systems via the abstract reduction system naturally associated with every term rewriting system: the abstract reduction system associated with $\langle \Sigma, T, \mathcal{R} \rangle$ has T as base set and the restriction of $\to_{\mathcal{R}}$ to $T \times T$ as binary relation.

Remark 4.1 With our definition of term rewriting system we deviate from [10] by taking as the set of terms T a *subset* of U_Σ. Thus, in a transformation, Σ can be expanded without adding all the new terms generated by the added symbols; our definition of Thatte's transformation in the next section makes use of this.

A term is *linear* if no variable occurs in it more than once. A term rewriting system is *left-linear* if all the terms in its base are linear, and *right-linear* if all the right-hand sides of its rules are linear. It is *linear* if it is both left-linear and right-linear.

A term rewriting system is *nonoverlapping* if its base is nonoverlapping (terminology of Thatte [12] at variance with Terese [10]). Note that a term rewriting system is nonoverlapping if and only if none of its pseudoredexes is a redex. A nonoverlapping term rewriting system is *orthogonal* if it is left-linear and there are no root overlaps between left-hand sides of distinct rules. As is well known, orthogonal term rewriting systems are confluent.

Let p be a position in a term s, and $s \xrightarrow{o}_{\mathcal{R}} t$ a reduction. A position p' in t is a *descendant* of p over this reduction if either $p' = p$ and $o \not\leq p$, or $\mathcal{R}$ contains a rule $l \to r$ such that for some substitution σ, $s/o = l^\sigma$ and $t/o = r^\sigma$, and there exist $q \in P^{\mathcal{V}}(l)$, $q' \in P^{\mathcal{V}}(r)$ such that $l/q = r/q'$ and for some u, $p = o.q.u$, and $p' = o.q'.u$. We say that p' is a descendant of $p \in P(s_0)$ over a longer reduction

$$s_0 \to \cdots \to s_{n-1} \to s_n$$

if p' is a descendant over $s_{n-1} \to s_n$ of a descendant of p over $s_0 \to \cdots \to s_{n-1}$. (We note that adding rewrite rules may increase the offspring of a position over a given reduction.)

5 Thatte's transformation

In [11], S. R. Thatte introduced a transformation for term rewriting systems that turns an arbitrary orthogonal system into a constructor-based system. The characteristic property of such a system is that its signature may be divided into a set of *defined symbols* (appearing only as outermost function symbols of left-hand sides of rules) and a set of *constructor symbols* (not appearing as outermost symbols).

Definition 5.1 Let $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ be a term rewriting system. A function symbol is *defined* in $\mathfrak{R}$ if it occurs as leading symbol in an element of $\mathcal{L}^{\mathcal{R}}$; it is an *argument symbol* in $\mathfrak{R}$ if it occurs otherwise in $\mathcal{L}^{\mathcal{R}}$. The *constructors* of $\mathfrak{R}$ are the function symbols that are not defined. The system $\mathfrak{R}$ is *constructor-based* if all argument symbols are constructors.

Systems that are not constructor-based have a symbol that is both defined and appears in some argument of a left-hand side, i.e. such a symbol has a dual rôle. The idea of Thatte's transformation is to remove this duality by adding a fresh symbol C_F for each F that is both defined and an argument symbol, which takes over its argument rôle.

From a system $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ we construct the Thatte-transform $\mathfrak{R}^\sharp = \langle \Sigma^\sharp, T^\sharp, \mathcal{R}^\sharp \rangle$ as follows. The signature $\Sigma^\sharp$ is Σ extended with fresh n-ary function symbols C_F for all n-ary F that are both defined and argument symbols in $\mathfrak{R}$. We define a function h that takes $\Sigma^\sharp$-terms to Σ-terms, and a function c vice versa: the first replaces the constructor variants C_F by the original F; the second replaces the F that are both defined and argument symbols by C_F. Besides c, we have another function c' that only replaces inner occurrences:

$$c'(F(t_1, \ldots, t_n)) = F(c(t_1), \ldots, c(t_n)).$$

The rule set $\mathcal{R}^\sharp$ is the union of sets $\mathcal{R}_1^\sharp$ and $\mathcal{R}_2^\sharp$, separately inducing reduction relations that we shall denote by $\rightarrow_1$ and $\rightarrow_2$. The set $\mathcal{R}_1^\sharp$ consists of transformed versions of the rules in $\mathcal{R}$: for every rule $l \rightarrow r$ in $\mathcal{R}$, $\mathcal{R}_1^\sharp$ contains $c'(l) \rightarrow r$. The rules in $\mathcal{R}_2^\sharp$ replace function symbols by their constructor variants: if u is a subtemplate, and the leading symbol of u has a new constructor variant in $\Sigma^\sharp$, then $c'(u) \rightarrow c(u)$ belongs to $\mathcal{R}_2^\sharp$. The set $T^\sharp$ is the closure of T under the reduction relation $\rightarrow_{\mathcal{R}^\sharp}$. We put $\mathfrak{R}_2^\sharp = \langle \Sigma^\sharp, T^\sharp, \mathcal{R}_2^\sharp \rangle$.

Clearly $\mathfrak{R}^\sharp$ is constructor-based; and $\mathfrak{R}_2^\sharp$ is complete. If $\mathfrak{R}$ is nonoverlapping and left-linear, then so is $\mathfrak{R}^\sharp$. In particular, a left-hand side of an $\mathcal{R}_2^\sharp$-rule cannot overlap a left-hand side of $\mathcal{R}_1^\sharp$: nonroot overlaps are impossible because $\mathfrak{R}^\sharp$ is constructor-based, and root overlap would imply overlap in $\mathfrak{R}$. Further note that since the variable occurrences in the right-hand sides in $\mathcal{R}_2^\sharp$ are the same as in the left-hand sides, $\mathfrak{R}_2^\sharp$ is also right-linear.

Thatte proved [11]:

Lemma 5.2 Let $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ be a term rewriting system, with transform $\mathfrak{R}^\sharp$. Then

(i) If $t_1 \rightarrow_{\mathcal{R}^\sharp} t_2$, then $h(t_1) \twoheadrightarrow_{\mathcal{R}} h(t_2)$.

(ii) If $t_1 \rightarrow_{\mathcal{R}} t_2$, then $t_1 \rightarrow^+_{\mathcal{R}^\sharp} t_2$.

For a term rewriting system $\mathfrak{R}$, let $\phi_{\mathfrak{R}}$ be the function that maps terms of $\mathfrak{R}$ to their normal forms in $\mathfrak{R}_2^\sharp$. The restriction of h to $\mathsf{ran}(\phi_{\mathfrak{R}})$ is the inverse of $\phi_{\mathfrak{R}}$.

Lemma 5.3 Let $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ be a nonoverlapping left-linear term rewriting system. If $t \in T^\sharp$, then $h(t) \twoheadrightarrow_2 t$.

Proof. Induction on the length of a given $\mathcal{R}^\sharp$-reduction from a Σ-term s to t.

If $s = t$, then $h(t) = t$.

Suppose $s \twoheadrightarrow_{\mathcal{R}^\sharp} t' \rightarrow_{\mathcal{R}^\sharp} t$, and $h(t') \twoheadrightarrow_2 t'$. If $t' \rightarrow_2 t$, then $h(t') = h(t)$, so we are done. Otherwise $t' \rightarrow t$ by some $\mathcal{R}^\sharp_1$-rule $l \rightarrow r$, say $t' = u[l^\sigma]$ and $t = u[r^\sigma]$. Then since $\mathfrak{R}^\sharp$ is left-linear and constructor-based, $h(u[y]) \twoheadrightarrow_2 u[y]$ for arbitrary y. On the other hand, for any variable x in l, $h(t') \twoheadrightarrow_2 t'$ implies $h(x^\sigma) \twoheadrightarrow_2 x^\sigma$, because the rules of $\mathfrak{R}^\sharp_2$ only change the leading function symbol. Since r is a Σ-term, this adds up to $h(t) \twoheadrightarrow_2 t$. □

Lemma 5.4 Let $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ be a nonoverlapping left-linear term rewriting system. Then for all s, s' and t, if $s' \twoheadleftarrow_2 s \rightarrow_1 t$, there exists t' such that $s' \rightarrow_1 t' \twoheadleftarrow_2 t$.

Proof. It suffices to show that if $s' \xleftarrow{p}_2 s \xrightarrow{q}_1 t$, there exists t' such that $s' \rightarrow_1 t' \twoheadleftarrow_2 t$. We distinguish cases according to the relation between p and q.

1. $p = q$ is impossible, since $h(s/p)$ would be both a redex and a pseudoredex.

2. If $q < p$, or p and q are disjoint, then t' may be found as in the standard confluence proof for orthogonal systems.

3. The case that $p < q$ is similar, except that by the right-linearity of $\mathcal{R}^\sharp_2$ we may be sure that q has exactly one descendant. □

Theorem 5.5 If $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ is a nonoverlapping left-linear term rewriting system, then $\phi_{\mathfrak{R}}$ is an ω-simulation of $\mathfrak{R}$ by $\mathfrak{R}^\sharp$.

Proof. Let $\phi = \phi_{\mathfrak{R}}$; that ϕ^{-1} is a function is immediate from the form of $\mathcal{R}^\sharp_2$. We must check the conditions of Definition 3.6.

1. Suppose $s \rightarrow_{\mathcal{R}} t$; say $s/p = l^\sigma$ and $t = s[r^\sigma]_p$, with $l \rightarrow r$ in $\mathcal{R}$. Then

$$t \leftarrow_1 s[c'(l)^\sigma] \twoheadrightarrow_2 \phi(s),$$

so by Lemma 5.4 there exists t' such that $t \twoheadrightarrow_2 t' \leftarrow_1 \phi(s)$. Since $t' \twoheadrightarrow_2 \phi(t)$ (recall that $\mathfrak{R}^\sharp_2$ is complete), $\phi(s) \rightarrow^+_{\mathcal{R}^\sharp} \phi(t)$.

2. Suppose $s \in \mathsf{ran}(\phi)$ and $s \rightarrow^+_{\mathcal{R}^\sharp} t$. We must find a reduct t' of t in $\mathsf{ran}(\phi)$ such that $h(s) \rightarrow^+_{\mathcal{R}} h(t')$. By Lemma 5.3, $h(t) \twoheadrightarrow_2 t$. By completeness of $\mathfrak{R}^\sharp_2$ it follows that $t \twoheadrightarrow_2 \phi(h(t))$. Take $t' = \phi(h(t))$; then $h(t') = h(t)$ and $t' \in \mathsf{ran}(\phi)$. Now since s is an $\mathcal{R}^\sharp_2$-normal form, the first step of the reduction must be an application of a rule in $\mathcal{R}^\sharp_1$; we have $s \rightarrow_1 s' \twoheadrightarrow_{\mathcal{R}^\sharp} t$. So $h(s) \rightarrow_{\mathcal{R}} h(s')$; and by Lemma 5.2(i), $h(s') \twoheadrightarrow_{\mathcal{R}} h(t)$. □

Theorem 5.6 With any term rewriting system $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$, the simulations $\langle \mathsf{id}(T), h \rangle$ and $\phi_{\mathfrak{R}}$ preserve termination.

Proof. Observe that (1) $t \rightarrow_1 t'$ implies $h(t) \rightarrow_{\mathfrak{R}} h(t')$; and (2) $t \rightarrow_2 t'$ implies $h(t) = h(t')$. Let s be a term of $\mathfrak{R}$ that has an infinite reduction in $\mathfrak{R}^\sharp$. Because $\mathfrak{R}^\sharp_2$ is SN, there must be infinitely many $\mathcal{R}^\sharp_1$-steps in this reduction, so it has the form

$$s \twoheadrightarrow_2 s_0 \rightarrow_1 s_1 \twoheadrightarrow_2 s_2 \rightarrow_1 \cdots \twoheadrightarrow_2 s_{2n} \rightarrow_1 s_{2n+1} \twoheadrightarrow_2 \cdots$$

However, because of (1) and (2) together, this corresponds to

$$s = h(s_0) \rightarrow_{\mathcal{R}} h(s_1) \rightarrow_{\mathcal{R}} \cdots \rightarrow_{\mathcal{R}} h(s_{2n-1}) \rightarrow_{\mathcal{R}} h(s_{2n+1}) \rightarrow_{\mathcal{R}} \cdots,$$

an infinite reduction in $\mathfrak{R}$. So $\langle \mathsf{id}(T), h \rangle$ preserves termination; and since $t \twoheadrightarrow_2 \phi_{\mathfrak{R}}(t)$, the closure of $\mathsf{ran}(\phi_{\mathfrak{R}})$ is contained in the closure of $\mathsf{ran}(\mathsf{id}(T))$ (which is $T^\sharp$). □

From these theorems we get by Corollary 3.9 that the Thatte transformation of a nonoverlapping left-linear term rewriting system is correct in the sense of [3].

6 Weakening left-linearity: counterexamples

Thatte originally defined his transformation for orthogonal term rewriting systems [11], for which preservation of confluence is immediate. As we have seen above, the condition of orthogonality may be weakened somewhat: the system should be nonoverlapping and left-linear. Naturally, the further question arises to what extent *these* two conditions are necessary. That the term rewriting system to be transformed be nonoverlapping appears to be essential. E.g., transforming $\{F(G(x)) \to F(x),\ G(x) \to x\}$ we would introduce an irreducible divergence $x \leftarrow G(x) \to C_G(x)$. On the other hand, the part of left-linearity is less clear. Thatte claimed [12] that it may be omitted. This claim was refuted by Verma [14], who constructed a nonoverlapping confluent term rewriting system with nonconfluent Thatte transform. We repeat the example, adding a proof of confluence.

Example 6.1 ([14]) Let $\mathfrak{R}$ be the term rewriting system consisting of the rules

$$\begin{array}{lcl} H(F(x,x)) & \to & H(A), \\ F(x,G(x)) & \to & A, \\ C & \to & G(C). \end{array}$$

CLAIM. $\mathfrak{R}$ is confluent.

Proof. We declare types for the function symbols as follows:

$$\begin{array}{lll} C : \alpha & A : \beta & H : \beta \to \gamma. \\ G : \alpha \to \alpha & F : \alpha \times \alpha \to \beta & \end{array}$$

By a theorem of Aoto and Toyama [1, Theorem 27] it suffices to prove that all terms that are well-formed according to these declarations are confluent. (Note that the rules preserve well-formedness of a term.)

Consider a divergence $t_1 \twoheadleftarrow t_0 \twoheadrightarrow t_2$. We distinguish cases:

1. All contractions in the given divergence are applications of the rule $C \to G(C)$. Then since the system $\mathcal{R}^-$ with signature $\{F, G, H, A, C\}$ and single rule $C \to G(C)$ is orthogonal, there must be a common reduct.

2. The divergence is of type β, i.e. t_0 may be written as $F(s_1, s_2)$, and there is an application of the rule $F(x, G(x)) \to A$ in one of the divergent reductions. If there is such an application in both reductions, then $t_1 = t_2 = A$; and if

$$t_0 = F(s_1, s_2) \twoheadrightarrow F(t, G(t)) \to A = t_1$$

   and $t_2 = F(s_1', s_2')$, then, since $t \to G(t)$, by the previous case, s_1', s_2' and $G(t)$ have a common reduct t', so

$$t_2 \twoheadrightarrow F(t', t') \to F(t', G(t')) \to A.$$

3. The divergence is of type γ: $t_0 = H(F(s_1, s_2))$, and there is an application of the rule $H(F(x, x)) \to H(A)$ in one of the divergent reductions. Similar to case 2. □

The rules of the transform $\mathfrak{R}^\sharp$ are

$$\begin{array}{lcl} H(C_F(x,x)) & \to & H(A), \\ F(x,G(x)) & \to & A, \\ C & \to & G(C), \\ F(x,x) & \to & C_F(x,x) \end{array}$$

and this system shows the irreparable divergence

$$C_F(G(C),G(C)) \leftarrow F(G(C),G(C)) \to F(G(C),G(G(C))) \to A.$$

Apparently some semblance of left-linearity is needed. In [14] the notion of *weak persistence* is put forward. The following definition suits the original intuition of [14] somewhat better than the the definition given there:

Definition 6.2 A term rewriting system $\mathfrak{R}$ is *weakly persistent* if pseudoredexes of $\mathfrak{R}$ never reduce to redexes.

The two definitions are in fact equivalent:

Proposition 6.3 A term rewriting system $\mathfrak{R}$ is weakly persistent iff in every reduction

$$s_0 \xrightarrow{p_0} s_1 \xrightarrow{p_1} s_2 \xrightarrow{p_2} \cdots \xrightarrow{p_{n-1}} s_n \xrightarrow{p_n} s_{n+1}$$

of a substitution instance s_0 of a subtemplate s of $\mathfrak{R}$, either $\lambda \in \{p_0, \ldots, p_{n-1}\}$ or an initial segment of p_n belongs to $P^{\mathcal{V}}(s)$.

Proof.

($\Rightarrow$) Let s_0 be a substitution instance of a subtemplate s of $\mathfrak{R}$, and

$$\rho := s_0 \xrightarrow{p_0} s_1 \xrightarrow{p_1} s_2 \xrightarrow{p_2} \cdots \xrightarrow{p_{n-1}} s_n \xrightarrow{p_n} s_{n+1}$$

a reduction. If nowhere in ρ a position in $P^{\mathcal{F}}(s)$ gets contracted, then an initial segment of p_n belongs to $P^{\mathcal{V}}(s)$. Otherwise let k be minimal with $p_k \in P^{\mathcal{F}}(s)$, and let $t = s/p_k$. Then t is a subtemplate, and s_0/p_k a substitution instance of t that reduces to the redex s_k/p_k.

($\Leftarrow$) Let $\rho := (s^\sigma \twoheadrightarrow t)$ be a reduction of a substitution instance of a subtemplate s to a redex t, of minimal length. Then ρ can be extended to a reduction

$$s^\sigma \twoheadrightarrow t \xrightarrow{\lambda} u.$$

Since $\lambda \notin P^{\mathcal{V}}(s)$, some step in ρ must be a contraction, which contradicts the minimality of ρ. □

A nonoverlapping term rewriting system is trivially weakly persistent if all its subtemplates are linear. If $\langle \Sigma, T, \mathcal{R} \rangle$ is a nonoverlapping left-linear term rewriting system, then

$$\langle \Sigma \cup \{D, E\}, \mathcal{R} \cup \{D(x,x) \to E\} \rangle$$

(where $D, E \notin \Sigma$) is weakly persistent. The term rewriting system of Example 6.1 is *not* weakly persistent: because in the reduction

$$F(G(C),G(C)) \to F(G(C),G(G(C))) \to A$$

the pseudoredex $F(G(C),G(C))$ becomes a redex.

Verma claimed [14] that Thatte's transformation preserves confluence for weakly persistent systems—but this claim too is false.

Example 6.4 Let Σ consist of constant symbols A, B, C, D and E, unary function symbols G and H, and binary F; consider the rewrite systems

$$\mathcal{R}_1 = \left\{ \begin{array}{lcl} F(A,B) & \to & E, \\ C & \to & G(C), \\ G(x) & \to & F(x,G(x)) \end{array} \right\}$$

and $\mathcal{R} = \mathcal{R}_1 \cup \{H(F(x,x)) \to D\}$. The system $\mathfrak{R} = \langle \Sigma, \mathcal{R} \rangle$ is weakly persistent, since the only interesting subtemplate is $F(x,x)$, the only redex with leading F is $F(A,B)$, and there is no nontrivial reduction to either A or B. If we can show that $\mathfrak{R}$ is confluent, the claim of [14] will be refuted, for $\mathfrak{R}^\sharp$ is not confluent: the divergence

$$\begin{array}{ccccccc} G(C) & \longrightarrow & F(C,G(C)) & \longrightarrow & F(G(C),G(C)) & \longrightarrow & C_F(G(C),G(C)) \\ \downarrow & & & & & & \\ \mathrm{G(G(C))} & & & & & & \\ \twoheaddownarrow & & & & & & \\ \mathrm{G}(C_F(\mathrm{G(C),G(C)})) & & & & & & \end{array}$$

cannot be recovered. For, since all reducts of $C_F(G(C),G(C))$ are of the form $C_F(s,t)$, to recover the divergence we need a reduction $G(C_F(s,t)) \twoheadrightarrow C_F(u,v)$. Consider a shortest reduction of this kind. By the nature of $\mathfrak{R}^\sharp$, it must have the form

$$\begin{array}{ccc} G(C_F(s,t)) & & \\ \twoheaddownarrow & & \\ G(C_F(s',t')) & \longrightarrow & F(C_F(s',t'),G(C_F(s',t'))) \\ & & \twoheaddownarrow \\ & & F(C_F(s'',t''),C_F(s'',t'')) \\ & & \twoheaddownarrow \\ & & C_F(u,v) \end{array}$$

But within this reduction we have a shorter reduction of $G(C_F(s',t'))$ to $C_F(s'',t'')$, which contradicts our choice of the initial reduction.

We declare types for the function symbols as follows:

$$\begin{array}{ll} A,B,C,E : \alpha & D : \beta \\ G \quad : \alpha \to \alpha & H : \alpha \to \beta. \\ F \quad : \alpha \times \alpha \to \alpha & \end{array}$$

The system $\mathfrak{R}_1 = \langle \Sigma, \mathcal{R}_1 \rangle$ is orthogonal, hence terms of sort α are confluent. Consider a divergence $t_1 \twoheadleftarrow H(s) \twoheadrightarrow t_2$ of type β. If there is no application of the rule $H(F(x,x)) \to D$ in either of the divergent reductions, then confluence follows from the orthogonality of $\mathfrak{R}_1$; if there is such an application in both reductions, then $t_1 = D = t_2$. It remains to consider divergences of the form

$$H(s') \twoheadleftarrow H(s) \twoheadrightarrow H(F(t,t)) \to D.$$

The terms s' and $F(t,t)$ of sort α have a common reduct s''. Since $\mathfrak{R}_1$ is confluent and A and B are distinct normal forms of $\mathfrak{R}_1$, $s'' \neq E$; hence there exist t_1 and t_2, with common reduct t', such that $s'' = F(t_1,t_2) \twoheadrightarrow F(t',t')$, and we find $H(s') \twoheadrightarrow H(F(t',t')) \rightarrow D$.

7 Positive results

In this section we show that Thatte's transformation preserves confluence under a condition intermediate between weak persistence and orthogonality.

We also prove that weak persistence is sufficient to guarantee that Thatte's transformation preserves the unique normal form property and weak normalization. As a corollary of this, we obtain that Thatte's transformation is correct in the sense of [3] for uniquely normalizing, weakly persistent systems and that it preserves semi-completeness for weakly persistent systems.

Sublinear systems

Definition 7.1 A term rewriting system is *sublinear* if every subtemplate with defined leading symbol is linear.

A system $\mathfrak{R}$ is sublinear iff $\mathfrak{R}_2^\sharp$ is linear. Since in the proof of Lemma 5.3 left-linearity is needed only in the context of $\mathfrak{R}_2^\sharp$, we have:

Lemma 7.2 Let $\mathfrak{R}$ be a nonoverlapping sublinear term rewriting system. If $t \in T^\sharp$, then $h(t) \twoheadrightarrow_2 t$.

Lemma 5.4 does not generalize so easily. We need a sublemma:

Lemma 7.3 Let $\mathfrak{R}$ be a nonoverlapping sublinear term rewriting system. If $s' \leftarrow_2 s \twoheadrightarrow_2 t$, then either $s' \twoheadrightarrow_2 t$ or for some t', $s' \twoheadrightarrow_2 t' \leftarrow_2 t$.

Proof. Induction on the length of a given reduction $s \twoheadrightarrow_2 t$. Suppose

$$s' \leftarrow s \twoheadrightarrow t_0 \rightarrow t.$$

By induction hypothesis either $s' \twoheadrightarrow t_0$, hence $s' \twoheadrightarrow t$; or there exists t_1 such that $s' \twoheadrightarrow t_1 \leftarrow t_0$. Suppose we have $t_1 \xleftarrow{p} t_0 \xrightarrow{q} t$. If p and q are disjoint, we get $t_1 \xrightarrow{q} t' \xleftarrow{p} t$. If $p = q$, then $t = t_1$. If $p < q$, then since $\mathfrak{R}_2^\sharp$ is nonoverlapping and left-linear, and $\mathfrak{R}_2^\sharp$-steps consist in changing a single function symbol, t_1/q and t/p are still redexes; and $t_1 \xrightarrow{q} t' \xleftarrow{p} t$. The case that $q < p$ is similar. □

Lemma 7.4 Let $\mathfrak{R}$ be a nonoverlapping sublinear term rewriting system. Then in $\mathfrak{R}^\sharp$, if $s' \twoheadleftarrow_2 s \rightarrow_1 t$, there are t', t'' such that $s' \twoheadrightarrow_2 t' \rightarrow_1 t'' \twoheadleftarrow_2 t$.

Proof. Induction on the length of the given reduction $s \twoheadrightarrow_2 s'$. Suppose

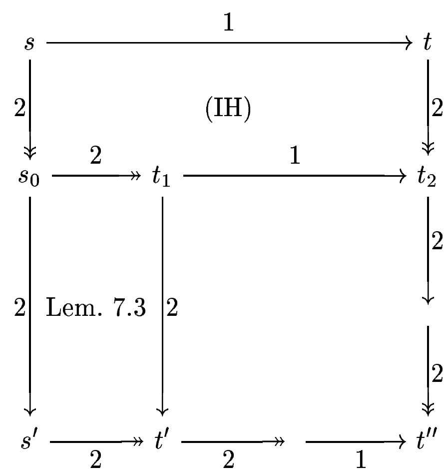
$$s' \leftarrow_2 s_0 \twoheadleftarrow_2 s \rightarrow_1 t.$$

By induction hypothesis there exist t_1, t_2 such that $s_0 \twoheadrightarrow_2 t_1 \rightarrow_1 t_2 \twoheadleftarrow_2 t$. Apply Lemma 7.3 to the divergence $s' \leftarrow_2 s_0 \twoheadrightarrow_2 t_1$. If $s' \twoheadrightarrow_2 t_1$, we are done. Otherwise we have t' such that $s' \twoheadrightarrow_2 t' \leftarrow_2 t_1$. We are left with a divergence $t' \xleftarrow{p}_2 t_1 \xrightarrow{q}_1 t_2$; we must find a common reduct t'' of t' and t_2.

If p and q are disjoint, this is easy. If $p < q$, then since $\mathfrak{R}^\sharp$ is nonoverlapping, $\mathfrak{R}_2^\sharp$ is left-linear, and $\mathfrak{R}_2^\sharp$-rewriting is just replacing function symbols, t'/q and t_2/p are redexes and we have $t' \xrightarrow{q} t'' \xleftarrow{p} t_2$. If $q < p$ there may be a complication in that t_1/q may be a redex in virtue of certain subterms being the same, and one of these is changed by rewriting t_1/p. Then to restore the redex at q, the other subterms must be rewritten as well: we get

$$t' \twoheadrightarrow_2 \xrightarrow{q}_1 t'' \twoheadleftarrow_2 \leftarrow_2 t_2.$$

The diagram below illustrates the last case.



□

Now we may reason as in the proof of Theorem 5.5, using Lemmas 7.4 and 7.2 instead of 5.4 and 5.3, to prove

Theorem 7.5 If $\mathfrak{R}$ is a nonoverlapping sublinear term rewriting system, then $\phi_{\mathfrak{R}}$ is an ω-simulation of $\mathfrak{R}$ by $\mathfrak{R}^\sharp$.

Corollary 7.6 Let $\mathfrak{R}$ be a nonoverlapping sublinear term rewriting system.

(i) If $\mathfrak{R}$ is confluent, so is $\mathfrak{R}^\sharp$.

(ii) $\mathfrak{R} \models s \rightarrow^! t$ if and only if $\mathfrak{R}^\sharp \models \phi_{\mathfrak{R}}(s) \rightarrow^! \phi_{\mathfrak{R}}(t)$.

(iii) If $\mathfrak{R} \models \mathsf{SN}(t)$, then $\mathfrak{R}^\sharp \models \mathsf{SN}(\phi_{\mathfrak{R}}(t))$.

Proof.

(i) Consider a divergence $t \twoheadleftarrow s \twoheadrightarrow u$ in $\mathfrak{R}^\sharp$. By Lemma 5.2 $h(t) \twoheadleftarrow_{\mathcal{R}} h(s) \twoheadrightarrow_{\mathcal{R}} h(u)$; hence, since $\phi_{\mathfrak{R}}$ is an ω-simulation, $\phi_{\mathfrak{R}}(h(t)) \twoheadleftarrow_{\mathcal{R}^\sharp} \phi_{\mathfrak{R}}(h(s)) \twoheadrightarrow_{\mathcal{R}^\sharp} \phi_{\mathfrak{R}}(h(u))$. By Theorem 3.12 $\phi_{\mathfrak{R}}(h(t))$ and $\phi_{\mathfrak{R}}(h(u))$ have a common reduct v in $\mathfrak{R}^\sharp$. Furthermore, by Lemma 7.2, $h(t) \twoheadrightarrow_2 t$ and $h(u) \twoheadrightarrow_2 u$; it follows that $t \twoheadrightarrow_2 \phi_{\mathfrak{R}}(h(t))$ and $u \twoheadrightarrow_2 \phi_{\mathfrak{R}}(h(u))$, so v is also a common reduct of t and u.

(ii) By Proposition 3.8(i), t is a normal form of $\mathfrak{R}$ if and only if $\phi_{\mathfrak{R}}(t)$ is a normal form of $\mathfrak{R}^\sharp$. By Definition 3.6, $\mathfrak{R} \models s \twoheadrightarrow t$ implies $\mathfrak{R}^\sharp \models \phi_{\mathfrak{R}}(s) \twoheadrightarrow \phi_{\mathfrak{R}}(t)$; and by Proposition 3.8(ii), $\mathfrak{R}^\sharp \models \phi_{\mathfrak{R}}(s) \rightarrow^! \phi_{\mathfrak{R}}(t)$ implies $\mathfrak{R} \models s \rightarrow^! t$.

(iii) By Theorem 5.6. □

In particular, for nonoverlapping sublinear term rewriting systems the Thatte transformation is correct in the sense of [3].

Weakly persistent systems

We proceed to reconsider weak persistence, and show that this condition, though insufficient for confluence preservation, nevertheless has some useful consequences. Observe that if a term rewriting system $\mathfrak{R}$ is weakly persistent, then $\mathfrak{R}$, and consequently $\mathfrak{R}^\sharp$, is nonoverlapping.

Proposition 7.7 Let $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ be a weakly persistent term rewriting system. If $s \in T^\sharp$ and there is a $t \in T$ such that $h(s) \xrightarrow{\lambda}_{\mathcal{R}} t$, then $s^\frown\lambda = h(s)^\frown\lambda$.

Proof. We argue by contraposition. Suppose $s \in T^\sharp$, $h(s) \xrightarrow{\lambda}_{\mathcal{R}} t$, and s begins with a constructor symbol C_F. Then there exists a reduction $\rho := u \twoheadrightarrow_{\mathcal{R}^\sharp} s$ with $u \in T$; at some point in ρ an $\mathcal{R}_2^\sharp$-rule $l \to r$ is applied to obtain the leading C_F of s. Then ρ has the form

$$u \twoheadrightarrow v[l^\sigma]_p \to_2 v[r^\sigma]_p \twoheadrightarrow s.$$

Since the occurrence of r^σ at p in $v[r^\sigma]_p$ can only be affected by contractions at $q > p$, we also have $r^\sigma \twoheadrightarrow s$. Then by Lemma 5.2, $h(r^\sigma) \twoheadrightarrow_{\mathcal{R}} h(s)$. But by construction of $\mathcal{R}^\sharp$, $h(r^\sigma)$ is a pseudoredex. Since $h(s)$ is a redex, this means that $\mathfrak{R}$ is not weakly persistent. □

Definition 7.8 Let $\mathfrak{R}$ be a term rewriting system. A term s of $\mathfrak{R}^\sharp$ is *balanced* if for all subterms u, v of s, $h(u) = h(v)$ implies $u = v$.

Proposition 7.9 Let $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ be a weakly persistent term rewriting system and $s \in T^\sharp$ balanced. If $h(s) \to_{\mathcal{R}} t$ by application of a rule $l \to r$ at position $p \in P(s)$, then there exist $l' \in h^{-1}(l)$ and a substitution σ such that $s/p = (l')^\sigma$ and $s \twoheadrightarrow_2 s[c'(l)^\sigma]_p \to_1 s[r^\sigma]_p$.

Proof. Let $h(s)/p = l^\tau$. Since s is balanced, there exists a substitution σ such that $\tau = h \circ \sigma$ and for some $l' \in h^{-1}(l)$, $s/p = (l')^\sigma$. By Proposition 7.7 and the construction of $\mathcal{R}_1^\sharp$, $(l')^\frown\lambda = s^\frown p = h(s)^\frown p = c'(l)^\frown\lambda$. Use the reduction $l' \twoheadrightarrow_2 c'(l) \to_1 r$. □

Corollary 7.10 Let $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ be a weakly persistent term rewriting system. If s is a balanced normal form of $\mathfrak{R}^\sharp$, then $h(s)$ is a normal form of $\mathfrak{R}$.

Let s and t be terms; if $P(t) \supseteq P(s)$ and $s^\frown p = t^\frown p$ for all $p \in P^{\mathcal{F}}(s)$ we shall call t a *replacement instance* of s.

Proposition 7.11 Let $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ be a weakly persistent and uniquely normalizing term rewriting system. If s, t are normal forms of $\mathfrak{R}^\sharp$, then $h(s) = h(t)$ implies $s = t$.

Proof. Suppose s and t are distinct normal forms of $\mathfrak{R}^\sharp$, and $h(s) = h(t)$. We derive a contradiction.

Since subterms of normal forms are normal forms, we may assume that s and t are balanced, and that they differ only at the root; say $s = C_F(u_1, \ldots, u_n)$ and $t = F(u_1, \ldots, u_n)$. Since $s \in T^\sharp$, it is a reduct of some $u \in T$. At some point in the reduction $u \twoheadrightarrow s$ an $\mathcal{R}_2^\sharp$-rule $l \to r$ is applied to obtain the leading C_F. As in the proof of Proposition 7.7 we may assume that $l(v_1, \ldots, v_m) \to_2 r(v_1, \ldots, v_m) \twoheadrightarrow s$. Since there are no defined function symbols in r, s is a replacement instance of r, and hence t is a replacement instance of l. Since t is a normal form, it is not a *substitution* instance of l, so there must be $p_1, p_2 \in P^{\mathcal{V}}(l)$ such that

$l/p_1 = l/p_2$ and $t/p_1 \neq t/p_2$. Let $l(v_1, \ldots, v_m)/p_1 = v_j$. Apparently $t/p_1 \twoheadleftarrow v_j \twoheadrightarrow t/p_2$; so by Lemma 5.2,

$$h(t)/p_1 \twoheadleftarrow_{\mathcal{R}} h(v_j) \twoheadrightarrow_{\mathcal{R}} h(t)/p_2.$$

Since t is balanced, by Corollary 7.10 $h(t)/p_1$ and $h(t)/p_2$ are normal forms of $\mathfrak{R}$, and $h(t)/p_1 \neq h(t)/p_2$, contradicting the assumption that $\mathfrak{R}$ is uniquely normalizing. □

Corollary 7.12 Let $\mathfrak{R}$ be a weakly persistent, uniquely normalizing term rewriting system.

(i) Normal forms of $\mathfrak{R}^\sharp$ are balanced.

(ii) If s, t are terms of $\mathfrak{R}^\sharp$, s is in normal form and $h(s) = h(t)$, then $t \twoheadrightarrow_2 s$.

Proof.

(ii) Let n be the $\mathcal{R}_2^\sharp$-normal form of t. Since $u \rightarrow_1 v$ implies $h(u) \rightarrow_{\mathcal{R}} h(v)$, n is a normal form of $\mathfrak{R}^\sharp$. So $n = s$. □

Theorem 7.13 If the term rewriting system $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ is weakly persistent, the simulations $\langle \mathsf{id}(T), h \rangle$ and $\phi_{\mathfrak{R}}$ preserve unique normalization.

Proof. Since $\mathsf{id}(T)(s) \twoheadrightarrow_2 \phi_{\mathfrak{R}}(s)$, it suffices to show that $\langle \mathsf{id}(T), h \rangle$ preserves unique normalization.

Suppose $\mathfrak{R} \models \mathsf{UN}^{\rightarrow}(s)$, and $\mathfrak{R}^\sharp \models n_1 \leftarrow^! s \rightarrow^! n_2$. By Lemma 5.2(i), $h(n_1) \twoheadleftarrow_{\mathcal{R}} s \twoheadrightarrow_{\mathcal{R}} h(n_2)$. By Corollary 7.12(i), n_1 and n_2 are balanced, and by Corollary 7.10, $h(n_1)$ and $h(n_2)$ are in normal form. So $h(n_1) = h(n_2)$ by unique normalization, and $n_1 = n_2$ by Proposition 7.11. □

Lemma 7.14 Let $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ be weakly persistent and uniquely normalizing. Let $n_1, \ldots, n_k$ be normal forms of $\mathfrak{R}^\sharp$, and $t(x_1, \ldots, x_k)$ a Σ-term such that $t(n_1, \ldots, n_k) \in T^\sharp$ and $t(h(n_1), \ldots, h(n_k))$ normalizes in $\mathfrak{R}$. Then $t(n_1, \ldots, n_k)$ normalizes in $\mathfrak{R}^\sharp$.

Proof. We use induction on the length of normalizations in $\mathfrak{R}$.

If $t(h(n_1), \ldots, h(n_k))$ is a normal form, then the $\mathcal{R}_2^\sharp$-normal form of $t(n_1, \ldots, n_k)$ is a normal form of $\mathfrak{R}^\sharp$. So suppose we have

$$(*) \qquad t(h(n_1), \ldots, h(n_k)) \rightarrow t' \rightarrow^! m.$$

We may assume that $t(n_1, \ldots, n_k)$ is balanced. For if it is not, let p, q be maximal positions with

$$h(t(n_1, \ldots, n_k)/p) = h(t(n_1, \ldots, n_k)/q) \quad \text{and} \quad t(n_1, \ldots, n_k)/p \neq t(n_1, \ldots, n_k)/q.$$

Say

$$t(n_1, \ldots, n_k)/p = G(t_1, \ldots, t_j) \quad \text{and} \quad t(n_1, \ldots, n_k)/q = C_G(t_1, \ldots, t_j).$$

Then $t(n_1, \ldots, n_k)/q$ must be contained in some n_i, so $t(n_1, \ldots, n_k)/q$ is a normal form. So by Corollary 7.12(ii), $t(n_1, \ldots, n_k)/p \twoheadrightarrow_2 t(n_1, \ldots, n_k)/q$. Continuing in this way we will eventually reach a balanced term $u(m_1, \ldots, m_l)$ with $u(y_1, \ldots, y_l)$ a Σ-term and $m_1, \ldots, m_l$ in normal form; since

$$u(h(m_1), \ldots, h(m_l)) = t(h(n_1), \ldots, h(n_k))$$

$u(h(m_1), \ldots, h(m_l))$ normalizes in $\mathfrak{R}$.

Let the first step of $(*)$ be an application of rule $l \to r$ of $\mathcal{R}$ at position o. By Proposition 7.9 there is a substitution σ such that

$$t(n_1, \ldots, n_k) \twoheadrightarrow_2 t(n_1, \ldots, n_k)[c'(l)^\sigma]_o \to_1 t(n_1, \ldots, n_k)[r^\sigma]_o$$

and $h(t(n_1, \ldots, n_k)[r^\sigma]_o) = t'$. Since $o \in P^{\mathcal{F}}(t(x_1, \ldots, x_k))$, $t(n_1, \ldots, n_k)[r^\sigma]_o$ still is an $\mathfrak{R}$-term with $\mathfrak{R}^\sharp$-normal forms substituted for the variables; so by induction hypothesis it has a normal form. □

The following definition gives a sufficient condition on $\mathfrak{R}$ such that $\mathfrak{R}^\sharp$ is closed under subterms (needed for the inductive proof of Theorem 7.17 below).

Definition 7.15 We call a term rewriting system $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ an *SR-system* if T is

(i) closed under subterms, i.e. if s is a subterm of $t \in T$, then $s \in T$; and

(ii) closed under right-hand sides, i.e. if $r(x_1, \ldots, x_m)$ is a right-hand side of an element of $\mathcal{R}$, and $u_1, \ldots, u_m \in T$, then $r(u_1, \ldots, u_m) \in T$.

Note that a term rewriting system in the sense of [10] is an SR-system.

Lemma 7.16 If $\mathfrak{R}$ is an SR-system, then $\mathfrak{R}^\sharp$ is closed under subterms.

Proof. Let s be a subterm of $t \in T^\sharp$; say t is a reduct of $u \in T$. We use induction on the length of a given reduction $u \twoheadrightarrow t$.

If $t = u$, then $s \in T$ since T is closed under subterms.

Now suppose $u \twoheadrightarrow t' \to t$. If s is a reduct of a subterm s' of t', then by induction hypothesis $s' \in T^\sharp$, and $s \in T^\sharp$ since $T^\sharp$ is closed under $\to_{\mathcal{R}^\sharp}$. Otherwise, $s = r_1(v_1, \ldots, v_m)$, where r_1 is a proper non-variable subterm of the right-hand side of an $\mathcal{R}_1^\sharp$-rule. Then $v_1, \ldots, v_m \in T^\sharp$ by induction hypothesis, say $u_j \twoheadrightarrow v_j$ with $u_j \in T$ $(1 \leq j \leq m)$, and s is a reduct of $r_1(u_1, \ldots, u_m)$, which belongs to T since T is closed under right-hand sides and subterms. □

Theorem 7.17 If $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ is a weakly persistent and uniquely normalizing SR-system, then $\mathfrak{R} \models \mathsf{WN}$ implies $\mathfrak{R}^\sharp \models \mathsf{WN}$.

Proof. By Theorem 7.13, $\mathsf{UN}^\to$ is preserved. We proceed by induction on $\mathfrak{R}^\sharp$-terms (cf. Lemma 7.16).

Variables are in normal form. Now suppose $s_1 \to^! n_1, \ldots, s_k \to^! n_k$ in $\mathfrak{R}^\sharp$. If $s = C_F(s_1, \ldots, s_k)$, then $C_F(n_1, \ldots, n_k)$ is a normal form of s. If $s = F(s_1, \ldots, s_k)$ with $F \in \Sigma$, then we apply Lemma 7.14 to the term $F(n_1, \ldots, n_k)$. □

Corollary 7.18 Let $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ be a weakly persistent SR-system.

(i) If $\mathfrak{R}$ is weakly normalizing, then both $\langle \mathsf{id}(T), h \rangle$ and $\phi_{\mathfrak{R}}$ preserve confluence.

(ii) If $\mathfrak{R}$ is semi-complete, then so is $\mathfrak{R}^\sharp$.

Proof.

(i) Suppose $t_1 \twoheadleftarrow_{\mathcal{R}^\sharp} s \twoheadrightarrow_{\mathcal{R}^\sharp} t_2$, and $s \in T$. By Theorem 7.17, there are $n_1, n_2 \in T^\sharp$ such that $\mathfrak{R}^\sharp \models t_1 \to^! n_1,\ t_2 \to^! n_2$; and since $\mathsf{id}(T)$ preserves $\mathsf{UN}^\to$ by Theorem 7.13, $n_1 = n_2$. Since $\mathsf{id}(T)(s) \twoheadrightarrow_2 \phi_{\mathfrak{R}}(s)$ it immediately follows that $\phi_{\mathfrak{R}}$ also preserves confluence.

(ii) If $\mathfrak{R}$ is semi-complete, then by Theorem 7.17 $\mathfrak{R}^\sharp \models \mathsf{WN}$ and by the first part of this corollary and Proposition 3.4 $\mathfrak{R}^\sharp \models \mathsf{CR}$. □

Let $\mathfrak{R} = \langle \Sigma, T, \mathcal{R} \rangle$ be a weakly persistent, uniquely normalizing SR-system. If n is an $\mathfrak{R}^\sharp$-normal form of $s \in T$, then by Corollary 7.12(i), n is balanced; so by Corollary 7.10, $h(n)$ is a normal form of $\mathfrak{R}$; and by Lemma 5.2(i), $s \twoheadrightarrow_{\mathcal{R}} h(n)$. Conversely, if $s \rightarrow^!_{\mathcal{R}} n'$, then by Lemma 5.2(ii) $s \twoheadrightarrow_{\mathcal{R}^\sharp} n'$. Let n be the $\mathfrak{R}^\sharp_2$-normal form of n': then n is a normal form of $\mathfrak{R}^\sharp$, and $n' = h(n)$. So $\langle \mathsf{id}(T), h \rangle$ preserves the normal form relation. Since by Theorem 5.6 $\langle \mathsf{id}(T), h \rangle$ also preserves termination, the transformation of $\mathfrak{R}$ into $\mathfrak{R}^\sharp$ is correct in the sense of [3].

8 Weak persistence is undecidable

We reduce the problem of deciding for an arbitrary closed term t of Combinatory Logic whether $t \twoheadrightarrow I$ to the problem of deciding whether a term rewriting system $\mathfrak{R}(t)$ is weakly persistent. The undecidability of the former problem is a consequence of a theorem of D. S. Scott (cf. the proof of Corollary 5.4.2 in [10]), so the undecidability of the latter problem follows.

The signature of Combinatory Logic ($\mathfrak{CL}$) consists of the constant symbols S, K and I and a binary function symbol for *application*. It is customary to write the application of s to t as (st), and to omit brackets according to the convention of association to the left. The rules of Combinatory Logic are

$$\begin{array}{lcl} Sxyz & \rightarrow & xz(yz), \\ Kxy & \rightarrow & x, \\ Ix & \rightarrow & x. \end{array}$$

Combinatory Logic is not weakly normalizing. We obtain a weakly normalizing term rewriting system if we add to the signature of $\mathfrak{CL}$ the constant symbols $\underline{S}$, $\underline{K}$ and $\underline{I}$ and also add the following three underlining rules:

$$\begin{array}{lcl} Sxyz & \rightarrow & \underline{S}xyz, \\ Kxy & \rightarrow & \underline{K}xy, \\ Ix & \rightarrow & \underline{I}x; \end{array}$$

the extended system is denoted $\mathfrak{CL}^{\mathrm{u}}$. In $\mathfrak{CL}^{\mathrm{u}}$ every term has a normal form, viz. its normal form with respect to the three underlining rules above. The extension is conservative with respect to conversion.

Lemma 8.1 If s and t are $\mathfrak{CL}$-terms, then $s \leftrightarrow^* t$ in $\mathfrak{CL}$ if and only if $s \leftrightarrow^* t$ in $\mathfrak{CL}^{\mathrm{u}}$.

Proof. Since $\mathfrak{CL}^{\mathrm{u}}$ contains all of $\mathfrak{CL}$, the implication from left to right is immediate. For the other implication, we consider the term rewriting system $\mathfrak{CL}^{\mathrm{r}}$ obtained from $\mathfrak{CL}^{\mathrm{u}}$ by reversing the underlining rules. Observe that $s \leftrightarrow^* t$ in $\mathfrak{CL}^{\mathrm{r}}$ if and only if $s \leftrightarrow^* t$ in $\mathfrak{CL}^{\mathrm{u}}$, so it is enough to prove that $s \leftrightarrow^* t$ in $\mathfrak{CL}^{\mathrm{r}}$ implies $s \leftrightarrow^* t$ in $\mathfrak{CL}$ for all $\mathfrak{CL}$-terms s and t. Note that $\mathfrak{CL}^{\mathrm{r}}$ is orthogonal and hence confluent. So, if $s \leftrightarrow^* t$ in $\mathfrak{CL}^{\mathrm{r}}$, then there exists u such that $s \twoheadrightarrow u \twoheadleftarrow t$ in $\mathfrak{CL}^{\mathrm{r}}$. Further note that in $\mathfrak{CL}^{\mathrm{r}}$ reducts of $\mathfrak{CL}$-terms are $\mathfrak{CL}$-terms and that the reversed underlining rules are not applicable to $\mathfrak{CL}$-terms. It follows that the reductions $s \twoheadrightarrow u$ and $t \twoheadrightarrow u$ are $\mathfrak{CL}$-reductions, so $s \leftrightarrow^* t$ in $\mathfrak{CL}$. □

Let t be a closed $\mathfrak{CL}$-term, and let $\underline{t}$ be its normal form with respect to the underlining rules of $\mathfrak{CL}^{\mathrm{u}}$. We define the term rewriting system $\mathfrak{R}(t)$ as the extension of $\mathfrak{CL}^{\mathrm{u}}$ with a binary function symbol F, a unary function symbol H, constant symbols A and B, and the rules

$$\begin{array}{lcl} H(F(x,x)) & \rightarrow & A, \\ F(\underline{t}, I) & \rightarrow & B. \end{array}$$

Let u be a term, let $p_1, \ldots, p_n$ be a sequence of disjoint positions in u and let $s_1, \ldots, s_n$ be a sequence of terms; we define $u[s_1, \ldots, s_n]_{p_1,\ldots,p_n}$ by induction on n as follows:

1. if $n = 0$, then $u[s_1, \ldots, s_n]_{p_1,\ldots,p_n} = u$;
2. if $n > 0$, then $u[s_1, \ldots, s_n]_{p_1,\ldots,p_n} = (u[s_1, \ldots, s_{n-1}]_{p_1,\ldots,p_{n-1}})[s_n]_{p_n}$.

Lemma 8.2 Let u and v be $\mathfrak{CL}^{\mathrm{u}}$-terms, let $p_1, \ldots, p_n$ be a sequence of disjoint positions in u, and let $s_1, \ldots, s_n$ be a sequence of $\mathfrak{R}(t)$-terms with $s_i {\sim} \lambda \in \{A, B, F, H\}$ for all $1 \leq i \leq n$. If $u[s_1, \ldots, s_n]_{p_1,\ldots,p_n} \twoheadrightarrow v$, then $u \twoheadrightarrow v$.

Proof. Induction on the length of the reduction $u[s_1, \ldots, s_n]_{p_1,\ldots,p_n} \twoheadrightarrow v$.

If $u[s_1, \ldots, s_n]_{p_1,\ldots,p_n} = v$, then $u[s_1, \ldots, s_n]_{p_1,\ldots,p_n}$ is a $\mathfrak{CL}^{\mathrm{u}}$-term, so $n = 0$; it follows that $u = v$.

Suppose that $u[s_1, \ldots, s_n]_{p_1\ldots,p_n} \xrightarrow{q} u' \twoheadrightarrow v$. If $q \geq p_i$ for some $1 \leq i \leq n$, then there exists s'_i such that $s_i \to s'_i$ and $u' = u[s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n]$. Since $s'_i {\sim} \lambda \in \{A, B, F, H\}$, it follows by the induction hypothesis that $u \twoheadrightarrow v$. On the other hand, if $q \not\geq p_i$ for all $1 \leq i \leq n$, then $u[s_1, \ldots, s_n]_{p_1\ldots,p_n} \xrightarrow{q} u'$ by an application of a rule in $\mathfrak{CL}^{\mathrm{u}}$. Since $\mathfrak{CL}^{\mathrm{u}}$ is left-linear and left-hand sides of $\mathfrak{CL}^{\mathrm{u}}$-rules do not contain symbols in $\{A, B, F, H\}$, it follows that there exists a $\mathfrak{CL}^{\mathrm{u}}$-term u'' such that $u \xrightarrow{q} u''$. Moreover, $u' = u''[s'_1, \ldots, s'_m]_{p'_1,\ldots,p'_m}$, with $p'_1, \ldots, p'_m$ the descendants of $p_1, \ldots, p_n$ over $u[s_1, \ldots, s_n]_{p_1\ldots,p_n} \xrightarrow{q} u'$ and $s'_1, \ldots, s'_m \in \{s_1, \ldots, s_n\}$. So by the induction hypothesis, it follows that $u'' \twoheadrightarrow v$; hence $u \twoheadrightarrow v$. □

Theorem 8.3 If t is a closed $\mathfrak{CL}$-term, then $t \twoheadrightarrow I$ in $\mathfrak{CL}$ iff $\mathfrak{R}(t)$ is not weakly persistent.

Proof. If $t \twoheadrightarrow I$ in $\mathfrak{CL}$, then $F(t, t) \twoheadrightarrow F(\underline{t}, I)$ in $\mathfrak{R}(t)$, so $\mathfrak{R}(t)$ is not weakly persistent. Conversely, suppose $\mathfrak{R}(t)$ is not weakly persistent. Note that $\mathfrak{R}(t)$ is nonoverlapping, so $F(x, x)$ is the only subtemplate that may give rise to a pseudoredex reducing to a redex (all other subtemplates are linear); it follows that there exists an $\mathfrak{R}(t)$-term u such that the pseudoredex $F(u, u)$ reduces to a redex. Since $F(\underline{t}, I)$ is the only redex with F as leading symbol, $\underline{t} \twoheadleftarrow u \twoheadrightarrow I$. We may assume by Lemma 8.2 that u is a $\mathfrak{CL}^{\mathrm{u}}$-term, so $t \twoheadrightarrow \underline{t} \twoheadleftarrow u \twoheadrightarrow I$ in $\mathfrak{CL}^{\mathrm{u}}$. By Lemma 8.1 it follows that $t \leftrightarrow^* I$ in $\mathfrak{CL}$, and since $\mathfrak{CL}$ is confluent and I is a normal form of $\mathfrak{CL}$, we conclude that $t \twoheadrightarrow I$. □

Corollary 8.4 The problem of deciding whether a term rewriting system is weakly persistent is recursively unsolvable.

9 Concluding remarks

We have proved that for weakly persistent systems Thatte's transformation preserves SN, UN and semi-completeness, but in general not CR. We have also proved that for the class of nonoverlapping sublinear systems, it does preserve CR. Weak persistence is an undecidable property of term rewriting systems, whereas both sublinearity and the property of being nonoverlapping are decidable.

As a convenient tool in our proof that Thatte's transformation preserves CR for nonoverlapping sublinear systems, we have proposed the notion of ω-simulation for abstract reduction systems. We established an ω-simulation from every nonoverlapping sublinear system to its Thatte transform, and derived the preservation of CR from a general result about ω-simulations.

It seems that our notion of ω-simulation is sufficiently general to be of use in the analysis of other transformations of term rewriting systems. For instance, *currying* and the related notion of *partial parameterization* studied in [4, 8] give rise to functional simulations *cur* and PP, respectively, that are easily seen to be ω-simulations (cf. Lemma 2.1 and Theorem 2.1 in [8]).

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