Regression Analysis and Analysis of Variance

Exercises 5

Exercise 5.1
Calculate the Fisher information for the following distributions:
(a) Binomial $b(n, \lambda)$,
(b) Poisson $p(\lambda)$,
(c) Exponential
(d) Negative binomial $nb(\lambda)$,

Fisher information

$$ I(\theta) = -\mathbb{E} \left( \frac{\partial^2}{\partial \theta^2} \log(f(X, \theta)) \right) $$

(a) $f(x, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$, $\log(f(x, \theta)) = \log \left( \binom{n}{x} \right) + x \log(\theta) + (n-x) \log(1-\theta)$

$$ \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \log(f(X, \theta)) = \frac{\partial}{\partial \theta} x/\theta - (n-x)/(1-\theta) = -x/\theta^2 - (n-x)/(1-\theta)^2 $$

$$ -\mathbb{E}(X/\theta^2 + (n-X)/(1-\theta)^2) = n/\theta + n/(1-\theta) = n/(\theta(1-\theta)). $$

(b) $f(x, \theta) = \frac{\theta^x}{x!} \exp(-\theta)$, $\log(f(x, \theta)) = -\log(x!) + x \log(\theta) - \theta$

$$ \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \log(f(X, \theta)) = \frac{\partial}{\partial \theta} (x/\theta - 1) = -x/\theta^2 $$

$$ -\mathbb{E}(X/\theta^2) = 1/\theta. $$

There was a mistake in the formulation of the last two parts. I forgot to re-parameterize the models. We parameterize the exponential distribution as $e(1/\lambda)$.

(c) $f(x, \theta) = \theta^{-1} \exp(-\theta^{-1} x)$, $\log(f(x, \theta)) = -\log(\theta) - \theta^{-1} x$

$$ \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \log(f(X, \theta)) = \frac{\partial}{\partial \theta} (-1/\theta + x/\theta^2) = 1/\theta^2 - 2x/\theta^3 $$

$$ -\mathbb{E}(1/\theta^2 - 2X/\theta^3) = -1/\theta^2 + 2/\theta^2 = 1/\theta^2. $$

(d) We re-parameterize the negative binomial distribution as $nb(1/\lambda)$.

$$ f(x, \theta) = \theta^{-1} (1-\theta^{-1})^{x-1}$, $\log(f(x, \theta)) = -\log(\theta) + (x-1) \log(1-\theta^{-1})$
$$\frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \log(f(X, \theta)) = \frac{\partial}{\partial \theta} (-1/\theta + (x - 1)(1/(\theta - 1) - 1/\theta))$$
$$= 1/\theta^2 + (x - 1)(-1/(\theta - 1)^2 + 1/\theta^2)$$

$$-\mathbb{E}(1/\theta^2 + (x - 1)(-1/(\theta - 1)^2 + 1/\theta^2))$$
$$= -1/\theta^2 + (\theta - 1)(-1/(\theta - 1)^2 + 1/\theta^2) = 1/(\theta(\theta - 1)).$$

Exercise 5.2
Calculate the variance of the maximum likelihood estimators for the relevant parameter for the distributions of the previous exercise.

(a) MLE=$\bar{X}_n$, \( \mathbb{V}(\bar{X}_n) = \Theta(1 - \Theta)/n = 1/I(\Theta) \)
(b) MLE=$X$, \( \mathbb{V}(X) = \Theta = 1/I(\Theta) \)
(c) MLE=$X$, \( \mathbb{V}(X) = \Theta^2 = 1/I(\Theta) \)
(d) MLE=$X$, \( \mathbb{E}(X(X - 1)) = 2\Theta(\Theta - 1), \mathbb{V}(X) = \Theta(\Theta - 1) = 1/I(\Theta) \)

Exercise 5.3
Consider the mixture distribution \((1 - \epsilon)N(0, \sigma^2) + \epsilon N(0, 9\sigma^2)\). Determine the value of \( \sigma = \sigma(\epsilon) \) so that the distribution has variance 1.
The mean is zero, need only consider second moment.

\[(1 - \epsilon)\sigma^2 + 9\epsilon\sigma^2 = (1 + 8\epsilon)\sigma^2 = 1, \quad \sigma(\epsilon) = 1/\sqrt{1 + 8\epsilon} \]

Exercise 5.4
For \( \epsilon = 0(0.05)0.2 \) generate samples of size \( n = 20 \) of the distribution of the previous exercise with \( \sigma = \sigma(\epsilon) \). Use the mean as an estimate of the location parameter and calculate the usual 95% confidence interval based on the \( t \)-distribution. By simulating such samples 10000 times derive the mean length of the confidence interval and its coverage probability and plot these against the values of \( \epsilon \).

```r
simex54<-function(n,nsim){
  tmpx<-double(n)
  reps1<-double(5)
  reps2<-double(5)
  qtt<-qt(0.975,n-1)
  ieps<-0
  while(ieps<=4){
    eps<-ieps*0.05
    lint<-0
    ic<-0
    isim<-1
    while(isim<=nsim){
```
Exercise 5.5

Let $q_1(0.95, n)$ and $q_2(0.95, n)$ be the 0.95–quantiles of the statistics $OR_1$ and $OR_2$ of Exercise 3.1 For a sample $x_n = (x_1, \ldots, x_n)$ define possible outliers $x_j$ by

\[
|x_j - \text{med}(x_n)| \geq q_1(0.95, n) \text{MAD}(x_n)
\]

\[
|x_j - \text{shorth}(x_n)| \geq q_2(0.95, n) \text{lshorth}(x_n).
\]

By generating normal samples with outliers try and determine which is the better outlier identifier.