Kinetic 2-Centers in the Black-Box Model

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ABSTRACT
We study two versions of the 2-center problem for moving points in the plane. Given a set \( P \) of \( n \) points, the Euclidean 2-center problem asks for two congruent disks of minimum size that together cover \( P \); the rectilinear 2-center problem correspondingly asks for two congruent axis-aligned squares of minimum size that together cover \( P \). Our methods work in the black-box KDS model, where we receive the locations of the points at regular time steps and we know an upper bound \( d_{\text{max}} \) on the maximum displacement of any point within one time step.

We show how to maintain the rectilinear 2-center in amortized sub-linear time per time step, under certain assumptions on the distribution of the point set \( P \). For the Euclidean 2-center we give a similar result: we can maintain in amortized sub-linear time, under certain assumptions on the point distribution, a \((1 + \varepsilon)\)-approximation of the optimal 2-center. In many cases—namely when the distance between the centers of the disks is relatively large or relatively small—the solution we maintain is actually optimal.

We also present results for the case where the maximum speed of the centers is bounded. We describe a simple scheme to maintain a \( 2\)-approximation of the rectilinear 2-center, and we provide a scheme which gives a better approximation factor depending on several parameters of the point set and the maximum allowed displacement of the centers. The latter result can be used to obtain a \( 2.29\)-approximation for the Euclidean 2-center; this is an improvement over the previously best known bound of \( 8/\pi \approx 2.55 \). These algorithms run in amortized sub-linear time per time step, as before under certain assumptions on the distribution.

Categories and Subject Descriptors: F.2.2 [Analysis of Algorithms and Problem Complexity]: Nonnumerical Algorithms and Problems—Geometrical problems and computations

General Terms: Algorithms, Theory

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1. INTRODUCTION

Background. Clustering [22] partitions a given set of objects into clusters, that is, into subsets consisting of similar objects. These objects are often (represented by) points in some 2- or higher dimensional space, and the similarity between points corresponds to the distance between them. We are interested in a setting with two clusters: partition a given planar point set into two clusters such that the quality of the clustering is optimized. We consider the quality of a cluster via its smallest enclosing disk: the smaller the radius of the disk, the better the quality of the cluster. This is the planar Euclidean 2-center problem: given a set \( P \) of \( n \) points in the plane, partition \( P \) into two subsets \( P_1, P_2 \) that minimize \( \max(\text{radius}(P_1), \text{radius}(P_2)) \), where \( \text{radius}(P_i) \) denotes the radius of the smallest enclosing disk of \( P_i \). Stated differently, the 2-center problem asks for two congruent disks of minimum size that together cover \( P \). The 2-center problem can hence also be interpreted as a facility-location problem: we want to place two facilities (the disk centers) such that the maximum distance of any of the clients (the points in \( P \)) to their nearest facility is minimized.

Of interest is also the rectilinear 2-center problem, which asks for two congruent axis-aligned squares of minimum size that together cover \( P \). This 2-center variant corresponds to a facility-location problem where distances are measured in the \( L_\infty \) metric. Note that we consider the continuous 2-center problem, that is, the disk (or, square) centers can be located anywhere in the plane; they are not restricted to a given discrete set of locations.

The 2-center problem and the more general \( k\)-center problem—which asks for \( k \) disks (or squares) to cover \( P \)—have been studied extensively since their introduction by Sylvester [21] in 1857. Both the Euclidean and the rectilinear \( k\)-center problem are \#P-hard [15] when \( k \) is part of the input, but polynomial-time solutions are possible when \( k \) is a constant. The rectilinear \( k\)-center problem can be solved quite efficiently for small \( k \). Drezner [8] gave an \( O(n) \) time solution for \( k = 2 \). He described an \( O(n \log n) \) time algorithm for \( k = 3 \), which was later improved to \( O(n) \) time [14, 20]. For \( k = 4, 5 \) Segal [17], and independently Nussbaum [16], gave \( O(n \log n) \) algorithms, which are optimal [20]. In contrast, no sub-quadratic algorithm was known for the Euclidean 2-center for many years, until Sharir [19] developed an \( O(n \log^6 n) \) time algorithm. The currently best solution takes \( O(n \log^2 n (\log \log n)^2) \) time [6]. Other results include an \( \Omega(n \log n) \) lower bound for \( k = 2 \) [18] and algorithms that compute a \((1 + \varepsilon)\)-approximation of the \( k\)-center [1, 2]. For the 2-center problem in \( \mathbb{R}^2 \) such an \((1 + \varepsilon)\)-approximation can be computed in \( O(n) + (2/\varepsilon)^{O(1)} \) time.

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The kinetic 2-center problem. The results mentioned so far are for static point sets, but also kinetic versions of the 2-center problem have been studied. Here we want to maintain the optimal 2-centers as the points in \( P \) move. Unfortunately, even under the restriction that the speed of the points in \( P \) is bounded by a given value \( v_{\text{max}} \), the speed of the centers cannot be bounded if one maintains the optimal 2-center. This is not problematic for an abstract clustering problem, but undesirable for mobile facility location. Hence, Durocher and Kirkpatrick [9] describe a general strategy for maintaining an approximate 2-center in such a way that the speed of the disk centers is bounded. One variant of their strategy achieves an approximation ratio of \( 8/\pi \approx 2.55 \) while the maximum speed of the disk centers is bounded by \( (8/\pi + 1) v_{\text{max}} \approx 3.55 v_{\text{max}} \). They also prove that it is not possible to obtain a better approximation ratio than \( \sqrt{2} \). In other words, with bounded speed for the disk centers one cannot guarantee that the ratio of the radius of the disks that are maintained and the minimum possible radius of two congruent disks covering \( P \) is always less than \( \sqrt{2} \). Durocher and Kirkpatrick also show how to maintain their approximate 2-center efficiently in the kinetic data structures (KDS) framework introduced by Basch et al. [3]. Efficient KDSs for the discrete version of the 2-center problem, where the disk centers must be chosen from the input set \( P \), have been given as well [7, 11]. Har-Peled [13] computes a static clustering such that during the motion this static clustering is competitive with the optimal \( k \)-center clustering.

**Problem statement.** Durocher and Kirkpatrick [9] study the 2-center problem in the standard KDS model, where the trajectories of the points are known in advance. However, in many applications the trajectories are not known and the standard KDS framework cannot be used. Hence, we study the kinetic 2-center problem in the so-called black-box model [4, 5, 12]. In the black-box model the locations of the points are reported at regular time steps \( t_0, t_1, \ldots, t \), and we are given a value \( d_{\text{max}} \) such that any point can move at most \( d_{\text{max}} \) from one time step to the next. Thus, when \( p(t) \) denotes the location of point \( p \) at time \( t \), we have \( \text{dist}(p(t), p(t+1)) \leq d_{\text{max}} \) for all \( p \in P \) and every time step \( t \). We want to maintain the 2-center of the set \( P(t) := \{ p(t) : p \in P \} \) at every time step while using coherence to speed up the computations. This is not possible without restricting the relation of the maximum displacement \( d_{\text{max}} \) and the distribution of the point set \( P \): if all points lie within distance \( d_{\text{max}} \) from each other then the distribution at time \( t + 1 \) need not have any relation to the distribution at time \( t \), and so there is no coherence that can be used. Following our previous papers [4, 5] we therefore make the following assumption.

**Displacement Assumption:** There is a maximum displacement \( d_{\text{max}} \) and constant \( k \) such that for each point \( p \in P \) and any time step \( t \), we have

- \( \text{dist}(p(t), p(t+1)) \leq d_{\text{max}} \), and
- there are at most \( k \) other points from \( P \) within distance \( d_{\text{max}} \) from \( p(t) \).

Under this assumption, we formulate our bounds in terms of the so-called \( k \)-spread [10] of \( P \), which is defined as follows. Let \( \text{mindist}_k(P) := \min_{p \in P} \text{dist}(p, 
N_k(p, P)) \) denote the smallest distance from any point \( p \in P \) to its \( k \)-nearest neighbor \( 
N_k(p, P) \). Then the \( k \)-spread \( \Delta_k \) of \( P \) is defined as

\[
\Delta_k(P) := \text{diam}(P)/\text{mindist}_k(P).
\]

The \( 1 \)-spread is simply the regular spread of a point set. We use the \( k \)-spread instead of regular spread, since two points may pass by each other at a very close distance, causing a small value for \( \text{mindist}_1(P) \) and, consequently, a high spread. It is much less likely that \( k \) points are very close simultaneously, so \( \text{mindist}_k(P) \) tends to not be very small. For a good \( k \)-spread we also need the diameter not to be too large. This is somewhat unnatural for the 2-center problem: when the two clusters are far apart, the \( k \)-spread may become very large even though within each cluster, the points are very evenly distributed. Hence we introduce the so-called \((2, k)\)-spread \( \Delta_{2,k}(P) \):

\[
\Delta_{2,k}(P) := \min_{P_1, P_2} \max(\Delta_k(P_1), \Delta_k(P_2)),
\]

where the minimum is taken over all possible partitions of \( P \) into two subsets \( P_1, P_2 \). (The partition defining \( \Delta_{2,k}(P) \) does not need to be the same as the partition defining the optimal clustering in the 2-center problem, but since \( \Delta_{2,k}(P) \) is the minimum over all partitions this can lead only to a better \((2, k)\)-spread.) We express our results using the \((2, k)\)-spread of \( P \), that is, using the maximum value of \( \Delta_{2,k}(P(t)) \) over all time steps \( t \), which we abbreviate as \( \Delta_{2,k} := \max_t \Delta_{2,k}(P(t)) \).

**Results and Organization.** We first study the kinetic 2-center problem from a clustering point of view: without restrictions on the speed of the centers. In Section 2 we show that the rectilinear 2-center can be maintained in \( O(k\Delta_{2,k} \log n) \) amortized time per time step \( O(n) \) space. In Section 3 we investigate the Euclidean 2-center problem. We show how to maintain a \((1 + \epsilon)\)-approximation, for any \( 0 < \epsilon \leq \pi/4 \) in \( O((k/\epsilon^2)^2 \Delta_{2,k} \log^3(\log \log n)^2) \) amortized time.

In many cases—when the distance between the centers of the disks is relatively large or small—the solution we maintain is optimal.

In Section 4 we investigate the setting where the maximum speed of the centers is bounded. We first show that a \( 2 \)-approximation of the rectilinear 2-center can be maintained in \( O(k\Delta_{2,k} \log n) \) amortized time per time step, such that the centers themselves move at most \( (2\sqrt{2} + 1)d_{\text{max}} \) per time step. We also provide a different approximation scheme which gives a better approximation factor depending on several parameters of the point set and the maximum allowed displacement of the centers; this maximum should be at least \( 4\sqrt{2}d_{\text{max}} \). Then we describe how our results for the rectilinear 2-center with bounded movement speed can be adapted to provide a \( 2.29 \)-approximation for the Euclidean 2-center. This is an improvement over the previously best known bound of \( 8/\pi \approx 2.55 \) by Durocher and Kirkpatrick [9]. Note however that our centers move with a speed of \( 4\sqrt{2}d_{\text{max}} \approx 5.66d_{\text{max}} \), which is faster than in the approximation of Durocher and Kirkpatrick, where the centers move at most \( (8/\pi + 1)d_{\text{max}} \approx 3.55d_{\text{max}} \). However, in the conclusions we discuss why there is not a clear trade-off between the speed of the centers and the approximation ratio.

2. THE RECTILINEAR 2-CENTER

Let \( P \) be a set of \( n \) moving points in the plane that adheres to the Displacement Assumption with parameters \( k \) and \( d_{\text{max}} \), and let \( \Delta_{2,k} \) denote the maximum \((2, k)\)-spread of \( P(t) \) at any time \( t \). Our goal is to compute at regular time steps \( t = t_0, t_1, \ldots \) two axis-aligned squares of minimum edge length that together cover \( P(t) \). Let \( \text{bb}(P) \) denote the bounding box of \( P \). We use \( \text{bb}(P) \) as a shorthand for \( \text{bb}(P(t)) \). Drezner [8] showed that there is always an optimal rectilinear 2-center for \( P \) consisting of two squares that have corners at opposite corners of \( \text{bb}(P) \). To make this precise, we define the \( \text{nwse}\)-squares of \( P \) as the two congruent squares \( \sigma_1, \sigma_2 \) of minimum size that together cover \( P \) and such that the north-west corner of \( \sigma_1 \) coincides with the north-west corner of \( \text{bb}(P) \) and...
the south-east corner of $\sigma$ coincides with the south-east corner of $\text{bb}(P)$. The NE|SW-squares of $P$ are defined correspondingly.

Lemma 1. (Dreuzner [8]) For any set $P$ of points in the plane, either the NW|SE-squares of $P$ or the NE|SW-squares of $P$ (or both) are an optimal solution for the 2-center problem on $P$.

By the lemma above, it suffices to maintain the NW|SE-squares and the NE|SW-squares of $P(t)$. Since the two solution types are symmetric we focus on the NW|SE-squares. We use $\sigma_{nw}(t)$ and $\sigma_{se}(t)$ to denote the NW|SE-squares for $P(t)$, where $\sigma_{nw}(t)$ has its north-west corner on the north-west corner of $\text{bb}(t)$ and $\sigma_{se}(t)$ has its south-east corner on the south-east corner of $\text{bb}(t)$. Let $q_{nw}(t)$ and $q_{se}(t)$ denote the north-west and south-east corners of $\text{bb}(t)$, and define

$$d_{nw|se}(p(t)) := \min(\text{dist}_{L_{\infty}}(p(t), q_{nw}(t)), \text{dist}_{L_{\infty}}(p(t), q_{se}(t))),$$

where $\text{dist}_{L_{\infty}}(p, q)$ denotes the distance between $p$ and $q$ in the $L_{\infty}$ metric. The edge length of the NW|SE-squares is then

$$r_{nw|se}(t) := \max_{p(t) \in P(t)} d_{nw|se}(p(t)).$$

The bounding box and the edge length are sufficient to compute the NW|SE-squares, so we show how to maintain these two. We give each point $p$ a time stamp $t_{bh}(p)$ based on its shortest distance to the boundary of $\text{bb}(P)$ at the time step where the time stamp is given. The time stamp indicates how many time steps it takes before $p$ can be on the boundary of $\text{bb}(t)$ and thus for how long we can ignore $p$ (as far as the computation of the bounding box is concerned). More precisely, if we give $p$ a time stamp at time $t$, then we set the time stamp to

$$t_{bh}(p) := \min\{\text{dist}(p(t), \partial \text{bb}(t))/2d_{\text{max}} \} + 1, n).$$

(Notice that $\text{dist}(\cdot)$ refers to the Euclidean distance.) It follows that points currently on the boundary of the bounding-box receive a time stamp of 1 and are considered again in the next time step. Lemma 2 follows from the fact that the distance between a point $p$ and the boundary $\partial \text{bb}(P)$ of $\text{bb}(P)$ can decrease by at most $2d_{\text{max}}$ at every time step.

Lemma 2. If a point $p$ receives a time stamp $t_{bh}(p)$ at time $t_i$, then $p$ cannot be on the boundary of $\text{bb}(t_j)$ for $t_i < t_j < t_i + t_{bh}(p)$.

Our kinetic algorithm for maintaining $\text{bb}(P)$ now works as follows. For $t = t_0$ we compute $\text{bb}(t_0)$ from scratch and we compute a time stamp for each point. After that, at every time step $t = t_i$, we take the set $Q_{bh}(t_i)$ of points whose time stamps expire at time $t$ and we compute $\text{bb}(Q_{bh}(t_i))$. It follows from Lemma 2 that $\text{bb}(Q_{bh}(t_i)) = \text{bb}(P(t_i))$. Our approach is made more explicit in Algorithm 1, where we assume without loss of generality that $t_i = i$ and $A[0..n-1] = [0].$]

Algorithm 1: UPDATE-Bounding-Box

1. $Q_{bh}(t) \leftarrow$ set of points stored in $A[t].$
2. Compute $\text{bb}(Q_{bh}(t))$ and set $\text{bb}(t) \leftarrow \text{bb}(Q_{bh}(t)).$
3. foreach $p(t) \in Q_{bh}(t)$ do
   4. $t_{bh}(p) \leftarrow \min\{\text{dist}(p(t), \partial \text{bb}(t))/2d_{\text{max}} \} + 1, n).$
   5. $\text{Add } p \text{ to } A[(t + t_{bh}(p)) \mod n].$

The time to compute $\text{bb}(t)$ depends on the number of points in $Q_{bh}(t)$. In the worst case all points may expire in a single time step, but when the $(2, k)$-spread of $P$ is low we can show that on average only a small number of points expire. The proof of the following lemma is similar to the proof in our previous paper [4, Lemma 6], where we bounded the number of points expiring in our kinetic black-box algorithm for maintaining the convex hull of a planar point set.

Lemma 3. The number of points in $Q_{bh}(t)$ is $O(k \Delta_{2,k} \log n)$ amortized.

Proof. We prove this using an accounting scheme in which at each time step $t$ each point $p(t)$ receives

$$\min \left(1, \max \left(1 - \frac{1}{n}, \frac{6d_{\text{max}}}{d_p(t)} \right) \right)$$

euros,$

where $d_p(t) = \text{dist}(p(t), \partial \text{bb}(t)).$ Whenever a point expires, it must pay 1 euro. We show that (i) each point has at least 1 euro in its account when it expires, and (ii) that in each time step we give out $O(k \Delta_{2,k} \log n)$ euros. To prove (i) let $p$ be a point whose time stamp expires at time $t_i$ and let $t_j < t_i$ denote the previous time when $p$’s time stamp expired. We assume that $(i - j) < n$, if this were not the case, then $p$ would already have gathered 1 euro in the first $n$ time steps. Similarly we assume that at any time step $t_i$ for $j < m < i$ we have $d_{\text{max}} / d_p(t_i) < 1$, otherwise $p$ would already receive 1 euro at time $t_m$. The amount of money that $p$ receives between time $t_j$ and $t_i$ is then

$$\sum_{j \leq m < i} \frac{6d_{\text{max}}}{d_p(t_m)} \leq (i - j) \frac{6d_{\text{max}}}{\max_{j \leq m < i} d_p(t_m)}.$$

Since $p$ did not expire between time $t_j$ and $t_i$ we know that at time $t_j$ it received a time stamp $t_{bh}(p) = t_i - t_j = i - j$ and so

$$(i - j) \frac{6d_{\text{max}}}{\max_{j \leq m < i} d_p(t_m)} \leq d_p(t_j) + (i - j)2d_{\text{max}} \leq 3d_p(t_j).$$

Recall that $d_p$ can increase or decrease by at most $2d_{\text{max}}$ per time step. We use this to bound the maximum distance between $p$ and $\text{bb}(P)$ at any time $t_m$ with $j \leq m < i$ as

$$\max_{j \leq m < i} d_p(t_m) \leq d_p(t_j) + (i - j)2d_{\text{max}} \leq 3d_p(t_j).$$

In the last inequality we use our assumption that $6d_{\text{max}} / d_p(t_j) < 1$. Filling in this maximum for the total amount of money that $p$ has gathered since $t_j$ we get

$$(i - j) \frac{6d_{\text{max}}}{d_p(t_j)} \geq (i - j) \frac{6d_{\text{max}}}{3d_p(t_j)} \geq 1 \text{ euro}.$$

To prove (ii) we divide the points of $P(t)$ into several groups $G_0, G_1, G_2, \ldots$ based on their distance to $\partial \text{bb}(t)$. Group $G_i$ contains all points $p(t)$ for which $2d_{\text{max}} / \text{dist}(p(t), \partial \text{bb}(t)) < (i + 1)2d_{\text{max}}$. Points in groups $G_{n+1}, G_{n+2}, \ldots$ get $O(1/n)$ euros each, so we spend at most $O(1)$ euro on these points. Points in group $G_0$ receive 1 euro each, and for $1 \leq i \leq n$ each point in group $G_i$ receives $O(1/i)$ euro. Next we look at how many points a group $G_i$ contains.

All points in group $G_i$ are inside a band $B$ defined as the difference of two rectangles, centered at the same point, with a difference of $4d_{\text{max}}$ in edge length. However, if the clusters of points are far apart these rectangles can be very large. By the definition of $\Delta_{2,k}(P(t))$ there is a division of $P(t)$ into $Q_1(t)$ and $Q_2(t)$ such that $\Delta_1(Q_1(t)) \leq \Delta_{2,k}(P(t))$ and $\Delta_k(Q_2(t)) \leq \Delta_{2,k}(P(t))$.

1 If $\text{dist}(p(t), \partial \text{bb}(t)) = 0$ we simply give $p(t)$ one euro.
We divide $R$ into four rectangles $R_{nw}$, $R_r$, $R_w$ and $R_e$ as illustrated in Fig. 1a. We then divide the plane using a grid with cells of size $\min\text{dist}_k(Q(t))/2 \times \min\text{dist}_k(Q(t))/2$. From the definition of $\Delta_k(Q(t))$ and $d_{\max} \leq \min\text{dist}_k(Q(t))$ it follows that $R_e$ intersects at most 5 rows of this grid. In each row at most $O(\Delta_k(Q(t)))$ cells contain points of $Q(t)$. Since each cell contains at most $k$ points, $R_e$ contains $O(k\Delta_k(Q(t)))$ points of $Q(t)$. A similar argument shows that $R_r$, $R_w$ and $R_e$ contain $O(k\Delta_k(Q(t)))$ points of $Q(t)$. For $Q(t)$ we can use a grid with cells of size $\min\text{dist}_k(Q(t))/2 \times \min\text{dist}_k(Q(t))/2$ to show that $R$ contains $O(k\Delta_k(Q(t)))$ points of $Q(t)$. Since $\Delta_k(Q(t))$ and $\Delta_k(Q(t))$ are upper bounded by $\Delta_{2,k}$ it follows that $G_i$ contains at most $O(k\Delta_{2,k})$ points of $P(t)$.

We can now sum up the money spent on each group and find that each round we spend

$$O(k\Delta_{2,k} + \sum_{i=1}^n ((1/i)k\Delta_{2,k}) + 1) = O(k\Delta_{2,k}\log n) \text{ euros.} \quad \Box$$

Next we look at maintaining $r_{\text{nw}}[\text{set}](t)$ over time. Again we give each point $p$ a time stamp $t_p(p)$, which indicates how many time steps must pass before $p$ can be a defining point for $r_{\text{nw}}[\text{set}](t)$, that is, before $d_{\text{nw}}[\text{set}](p(t)) = r_{\text{nw}}[\text{set}](t)$. The time stamp for $p$ (when given at time $t$) is defined as

$$t_r(p) := \min(\{r_{\text{nw}}[\text{set}](t) - d_{\text{nw}}[\text{set}](p(t)))/(4d_{\max}) + 1, n).$$

The next lemma proves the correctness of our approach.

**Lemma 4.** If a point $p$ receives a time stamp $t_r(p)$ at time step $t_i$, then $p$ cannot be a defining point for $r_{\text{nw}}[\text{set}](t_j)$ for $t_i < t_j < t_i + t_r(p)$.

**Proof.** Each edge of $bb(P)$ and any point $p \in P$ moves at most $d_{\max}$ within one time step. Hence, $d_{\text{nw}}[\text{set}](p)$ and $r_{\text{nw}}[\text{set}](t)$ change at most $2d_{\max}$ and $r_{\text{nw}}[\text{set}](t) - d_{\text{nw}}[\text{set}](p(t))$ changes at most $4d_{\max}$, so at least $r_{\text{nw}}[\text{set}](t_i) - d_{\text{nw}}[\text{set}](p(t_i))/4d_{\max}$ time steps are needed before possibly $d_{\text{nw}}[\text{set}](p(t)) = r_{\text{nw}}[\text{set}](t)$. \Box

We maintain $r_{\text{nw}}[\text{set}](t)$ in the same way as the bounding box. We compute time stamps for all points in $P$ and at every time step $t$ we find the set $Q(t)$ of points that expire at time $t$ and compute $r_{\text{nw}}[\text{set}](t)$ from $Q(t)$. We then compute new time stamps for these points and continue to the next time step. The following lemma gives a bound on the number of points that expire per time step; its proof is again similar to that in our previous paper [4, Lemma 6].

**Lemma 5.** The number of points in $Q(t)$ is $O(k\Delta_{2,k}\log n)$ amortized.

**Proof.** Proving this can be done in the same way as we did for the bounding box in Lemma 3. We use an accounting scheme where in each time step $t$ every point $p$ gets

$$\min\left(1, \max\left(1, \frac{12d_{\max}}{r_{\text{nw}}[\text{set}](t) - d_{\text{nw}}[\text{set}](p(t))}\right)\right) \text{ euros.}$$

Whenever the time stamp of a point expires, it must pay 1 euro. The lemma is the proven by showing that (i) each point has at least 1 euro in its account when it expires, and (ii) in each time step we spend only $O(k\Delta_{2,k}\log n)$ euros. The proof for (ii) is similar to part (i) in the proof of Lemma 3, with $t_r(p) := r_{\text{nw}}[\text{set}](t) - d_{\text{nw}}[\text{set}](p(t))$ and the maximum change in $d_{\text{nw}}[\text{set}](p(t))$ between two time steps is $4d_{\max}$ instead of $2d_{\max}$.

We prove (ii) in a similar way as in Lemma 3. We divide the points $p(t) \in P(t)$ into groups based on $r_{\text{nw}}[\text{set}](t) - d_{\text{nw}}[\text{set}](p(t))$. Group $G_i$ contains points with $i4d_{\max} \leq r_{\text{nw}}[\text{set}](t) - d_{\text{nw}}[\text{set}](p(t)) < (i + 1)4d_{\max}$.

Points in groups $G_{i+1}$, $G_{i+2}, \ldots$ receive $O(1/n)$ euro each, hence, we spend at most $O(1)$ euro on points in these groups. The points in group $G_0$ receive 1 euro each and points in group $G_i$ for $1 \leq i \leq n$ receive $O(1/i)$ euro each. Points in $G_i$ are contained in a region $R = R_{nw} \cup R_{se}$, where $R_{nw}$ is the difference between two squares with edge lengths of $r_{nw}[\text{set}] - 4d_{\max}$ and $r_{nw}[\text{set}] - (i + 1)4d_{\max}$ that both have their north-west corner on $q_{nw}(t)$ and $R_{se}$ defined symmetrically (see also Fig. 1b). We divide the plane using a grid with cells of size $\min\text{dist}_k(Q(t))/2 \times \min\text{dist}_k(Q(t))/2$. Each edge of $bb(P)$ we know there is a division of $P(t)$ into $Q_1(t)$ and $Q_2(t)$ such that $\Delta_k(Q_1(t)) \leq \Delta_{2,k}(P(t))$ and $\Delta_k(Q_2(t)) \leq \Delta_{2,k}(P(t))$. We split $R_{nw}$ into two rectangles $R_1$ and $R_2$, such that $R_1$ has a height of $4d_{\max}$ and $R_2$ has a width of $4d_{\max}$, see also Fig. 1b. We cover the plane using a grid with cells of size $\min\text{dist}_k(Q(t))/2 \times \min\text{dist}_k(Q(t))/2$. From the Displacement assumption we know that $d_{\text{nw}}[\text{set}] \leq \min\text{dist}_k(Q(t))$, so $R_1$ intersects at most 9 rows of the grid. From the definition of $\Delta_k(Q_1(t))$ it follows that in each row only $O(\Delta_k(Q_1(t)))$ cells can contain points of $Q_1(t)$. Each of these cells contains at most $k$ points and, hence, $R_1$ contains $O(k\Delta_k(Q_1(t)))$ points of $Q_1(t)$ for $R_2$ and $R_{se}$ we can give similar proofs and conclude that $R_2$ contains $O(k\Delta_k(Q_1(t)))$ points of $Q_1(t)$. A similar proof shows that $R$ contains at most $O(k\Delta_k(Q_2(t)))$ points of $Q_2(t)$. For $R_2$ and $R_{se}$ we can maintain the optimal rectilinear 2-center for $P$ in the black-box KDS model in $O(k\Delta_{2,k}\log n)$ amortized time per time step and using $O(n)$ space.

**Figure 1:** a) The region $R$ is divided into four rectangles and b) points in group $G_i$ are in the union of two differences of squares.
we consider only those points whose time stamps expire. The challenge is to define the time stamps such that the solution is correct—the disks contain all points—and the number of expiring points is small on average. In the rectilinear case we made the time stamp of a point dependent on its distance to the boundary of the current solution. This works, since in the rectilinear case there are only two types of solutions, which can be maintained separately. In the Euclidean case similar flips can also occur (see Fig. 2) so we cannot base our time stamps solely on the current 2-center disks. In contrast to the rectilinear case we cannot classify a small number of solution types as there may be many. Hence, we proceed differently.

Define a point \( p \in P \) to be \( \varepsilon \)-interesting if there is a wedge \( W_\varepsilon(p) \) with apex \( p \) and opening angle \( \varepsilon \) such that \( W_\varepsilon(p) \) does not contain any other points of \( P \). Let \( P_\varepsilon \) denote the set of \( \varepsilon \)-interesting points in \( P \). We show that it suffices to consider the points in \( P_\varepsilon \) to get an approximation of an optimal solution to the 2-center problem on \( P \). In the following we use \( \text{disk}(q,r) \) to denote the disk of radius \( r \) centered at \( q \).

**Lemma 7.** Let \( D_1 = \text{disk}(d_1, r) \) and \( D_2 = \text{disk}(d_2, r) \) be the two disks of an optimal solution for the Euclidean 2-center problem on \( P_\varepsilon \), for some \( \varepsilon < \pi/4 \). If \( \text{dist}(d_1, d_2) \leq 2r - \varepsilon r/2 \) or \( \text{dist}(d_1, d_2) \geq 2r + \varepsilon r \) then \( \text{disk}(d_1, r) \) and \( \text{disk}(d_2, r) \) are an optimal solution for the 2-center problem on \( P \); otherwise \( \text{disk}(d_1, r + \varepsilon r) \) and \( \text{disk}(d_2, r + \varepsilon r) \) are a \((1 + \varepsilon)\)-approximation for the 2-center problem on \( P \).

**Proof.** First consider the case \( \text{dist}(d_1, d_2) \leq 2r - \varepsilon r/2 \). Since \( P_\varepsilon \subseteq P \), the disks in an optimal solution for \( P \) cannot have radius smaller than \( r \). Hence, it suffices to prove that \( \text{disk}(d_1, r) \cup \text{disk}(d_2, r) \) covers \( P \). Suppose for a contradiction there is an uncovered point in \( P \). Assume without loss of generality that the line \( \ell \) through \( d_1 \) and \( d_2 \) is horizontal, and let \( p \) be the highest uncovered point above \( \ell \). (If all uncovered points lie below \( \ell \) we can apply a similar argument to the lowest uncovered point.) We show that a wedge directed upward with apex \( p \) and angle \( \varepsilon \) does not contain any points of \( P \). Clearly the wedge cannot contain any points of \( P \setminus P_\varepsilon \), so we focus on showing that the wedge does not intersect the optimal 2-center disks of \( P_\varepsilon \). Let \( q \) be the tangent point of \( \ell \) through \( d_1 \) and \( d_2 \) in horizontal. We know that \( \text{dist}(q, d_1) \) and \( \text{dist}(q, d_2) \) makes with a vertical line. This is the same angle as the angle between the line segment \( d_1 q \) and a horizontal line. We know that the horizontal distance between \( q \) and \( d_1 \) is at most \( r - r\varepsilon/4 \). With these we can conclude that

\[
\cos(\beta) \leq \frac{r - r\varepsilon/4}{r} = 1 - \varepsilon/4 \leq \cos(\varepsilon/2).
\]

Since we have \( 0 \leq \beta \leq \pi/2 \) and \( 0 \leq \varepsilon \leq \pi/2 \) this implies that

**Figure 2:** A small movement causes a large jump, or flip, in the optimal centers.

For the other points outside of \( D_1 \cup D_2 \), we can use translated versions of wedges of points on \( d_1 d_2 \), so for any point \( p \) not in \( D_1 \cup D_2 \) there is an \( \varepsilon \)-wedge that does not intersect \( D_1 \) or \( D_2 \). For an arbitrary point \( p \) such a wedge may still contain points of \( P \) that are not in \( P_\varepsilon \). Let \( q \) be such a point that is furthest away from \( p \). If we translate the wedge from \( p \) to \( q \) it must still cannot intersect \( D_1 \) and \( D_2 \), and cannot contain any points of \( P \setminus P_\varepsilon \). Hence, \( q \in P_\varepsilon \), contradicting that \( \text{disk}(q_1, r), \text{disk}(q_2, r) \) is a solution for \( P_\varepsilon \).

When \( 2r - \varepsilon r/2 \leq \text{dist}(q_1, q_2) \leq 2r + \varepsilon r \) we cannot guarantee this, unfortunately. However, if we blow up the disks by a factor \((1 + \varepsilon)\) we are essentially back in the first case, see Fig. 3b. Let \( D_1^+ = \text{disk}(q_1, r^+) \) and \( D_2^+ = \text{disk}(q_2, r^+) \) with \( r^+ = (1 + \varepsilon) r \). It then holds that

\[
\text{dist}(q_1, q_2) \leq 2r + \varepsilon r = 2r + \varepsilon r = 2r^+ - \frac{\varepsilon}{1 + \varepsilon} r^+ \leq 2r^+ - \varepsilon r^+/2.
\]

We can apply the same argument as earlier to show that for any point outside \( D_1^+ \cup D_2^+ \) there is a wedge that doesn’t intersect the disks \( D_1^+ \cup D_2^+ \) and, hence, that at last one such point is in \( P_\varepsilon \), contradicting that \( \text{dist}(q_1, r), \text{disk}(q_2, r) \) is a solution for \( P_\varepsilon \).

By Lemma 7 we obtain a \((1 + \varepsilon)\)-approximation for the Euclidean 2-center problem if we can maintain the set \( P_\varepsilon \). This seems

**Figure 3:** a) The angle \( \beta \) between the a vertical line and a tangent line to \( D_1 \) through \( p \) is at least \( \varepsilon \) when \( \text{dist}(d_1, d_2) \leq 2r - \varepsilon r/2 \). b) From a point \( p \) outside of \( D_1^+ \cup D_2^+ \) there is a free wedge with angle \( \beta \geq \varepsilon \).
difficult, so instead we maintain a superset $P' \supseteq P_t$ defined as follows. Let $W_{\epsilon/2}$ be a wedge of opening angle $\epsilon/2$. We say that $W_{\epsilon/2}$ is a canonical $(\epsilon/2)$-wedge if the counter-clockwise angle that its angular bisector makes with the positive x-axis is $\epsilon/2$, for some integer $0 < i < 4\lceil \epsilon/\pi \rceil$. We now define $P'_t$ as the set of points in $P$ that have an empty canonical $(\epsilon/2)$-wedge (that is, a wedge not containing points from $P_t$ with apex $p$). The following observation implies that Lemma 7 is still true if we replace $P_t$ by $P'_t$.

**Observation 8.** Any point $p \in P$ that is the apex of an empty $\epsilon$-wedge is also the apex of an empty canonical $(\epsilon/2)$-wedge, so $P_t \subseteq P'_t$.

We are left with the problem of maintaining $P'_t$. More precisely, we want to define time stamps that guarantee that in each time step $t = t_i$ the set $Q(t)$ of expired points contains all points in $P'_t(t)$; here $P'_t(t)$ denotes the set of points in $P(t)$ that have an empty canonical $(\epsilon/2)$-wedge at time $t$. Recall that there are $4\lceil \epsilon/\pi \rceil$ different classes of canonical wedges, corresponding to the orientation of their angular bisector. We treat each of these classes separately. Consider one such class, and assume without loss of generality that its angular bisector is pointing vertically upward. We wish to maintain the set $P'^{\text{up}}_t$ of points whose upward canonical wedge is empty. Define $W^{\text{down}}_t(p)$ to be the wedge with apex $p$ that is the mirrored image of the upward canonical wedge of $p$, and let $W^{\text{down}}_t := \{W^{\text{down}}_t(p) : p \in P_t\}$ be the set of all such downward wedges. Then a point $q$ lies in the upward canonical wedge of $p$ if and only if $p \in W^{\text{down}}_t(q)$. This implies the following lemma.

**Lemma 9.** Let $E(W^{\text{down}}_t)$ denote the upper envelope of the downward wedges at time $t$. Then $p \in P'^{\text{up}}_t(t)$ if and only if $p(t)$ is a vertex of $E(W^{\text{down}}_t)$.

Because of the bounded speed of the points, we know that points far from the upper envelope $E(W^{\text{down}}_t)$ need a lot of time before they can appear on the envelope. Hence, we can use the distance from $p$ to $E(W^{\text{down}}_t)$ to define its time stamp. To be able to compute time stamps quickly, we will not use the Euclidean distance from $p$ to $E(W^{\text{down}}_t)$ but an approximation of it. Let $p'(t)$ be the vertical projection of $p(t)$ onto $E(W^{\text{down}}_t)$; see Fig. 4. Then our approximated distance is defined as $\text{dist}^{\ast}(p(t)) := \text{dist}(p(t), p'(t)) \cdot \sin(\epsilon/4)$. Note that $\text{dist}^{\ast}(p(t))$ is equal to the distance from $p(t)$ to the boundary of the downward wedge $W^{\text{down}}_t(p'(t))$. Because $W^{\text{down}}_t(p'(t))$ is completely below (or on) $E(W^{\text{down}}_t)$, the actual Euclidean distance from $p(t)$ to $E(W^{\text{down}}_t)$ is at least the approximate distance $\text{dist}^{\ast}(p(t))$. Hence, we can safely use $\text{dist}^{\ast}(p(t))$ to define the time stamps. Thus, when we compute the time stamp of a point $p$ at time $t$ we set

\[ t^{\text{up}}(p) := \min(\lfloor \text{dist}^{\ast}(p(t))/2d_{\text{max}} \rfloor + 1, n). \]

**Lemma 10.** If a point $p$ receives time stamp $t^{\text{up}}(p)$ at time $t$, then $p$ cannot be on $E(W^{\text{down}}(t_j))$ for $t_j < t < t_i + t^{\text{up}}(p)$.

The final time stamp of a point $p$ is defined as the minimum over all time stamps computed for $p$ for the $4\lceil \epsilon/\pi \rceil$ different wedge orientations. The algorithm for maintaining the Euclidean 2-center can now be summarized as follows. Initially (at time $t = t_0$) we compute a time stamp $t(p)$ for every point $p$, which is the minimum over the time stamps for the $4\lceil \epsilon/\pi \rceil$ canonical wedge orientations. Then at each time step $t = t_i$, we take the set $Q(t)$ of points whose time stamps expire at time $t$. For each canonical orientation we use a simple sweep-line algorithm to compute in $O(1/\epsilon)\log |Q(t)|$ time the envelope of the mirrored wedges of the points in $Q(t)$. Since there are $4\lceil \epsilon/\pi \rceil$ different orientations this takes $O(1/\epsilon)\log |Q(t)|$ time in total. The collection of all points $p \in Q(t)$ that are a vertex of at least one of the envelopes is the set $P'_t(t)$. We then solve the Euclidean 2-center problem on $P'_t(t)$ using an algorithm for static points, giving us two disks $\text{disk}(q_1, r)$ and $\text{disk}(q_2, r)$. (To get the best running time, we use Chan’s algorithm [6] for this.) If $2r - \epsilon/2 \leq \text{dist}(q_1, q_2) \leq 2r + \epsilon$ we report $\text{disk}(q_1, r)$ and $\text{disk}(q_2, r)$, otherwise we report $\text{disk}(q_1, r)$ and $\text{disk}(q_2, r)$. Finally, we compute new time stamps for the points in $Q(t)$. Since we already have all the envelopes this can be done in $O(1/\epsilon)\log |Q(t)|$ time in total.

Next we analyze the running time of our algorithm. We start with a bound on the size of $P'_t$.

**Lemma 11.** $|P'_t(t)| = O((k/\epsilon^2)\Delta_{2, \alpha})$.

**Proof.** Since there are $4\lceil \epsilon/\pi \rceil$ envelopes it suffices to prove that the number of vertices on any envelope is $O((k/\epsilon^2)\Delta_{2, \alpha})$. Without loss of generality consider the envelope $E(W^{\text{down}}_t)$. From the definition of $\Delta_{2, \alpha}(P(t))$ we know we can partition $P(t)$ into subsets $Q_1(t)$ and $Q_2(t)$ such that $\Delta_{2, \epsilon}(Q_1(t)) \leq \Delta_{2, \epsilon}(P(t))$ and $\Delta_{2, \epsilon}(Q_2(t)) \leq \Delta_{2, \epsilon}(P(t))$. Recall that by the definition of the $\ell$-spread $\Delta_{2, \epsilon}(P) = \max_{k \in [1, 2, \ldots, \ell]} \Delta_{2, \epsilon}(P)$. If we divide the plane into columns of width $\sin(\epsilon/4)$, then at most $\Delta_{2, \epsilon}(Q_1(t))$ columns contain points of $Q_1(t)$. The total length of the envelope within one column is $\sin(\epsilon/4)$, hence, each envelope piece within a single column contains $O(k/\sin(\epsilon/4))$ points of $Q_1(t)$. It follows that $E(W^{\text{down}}_t)$ contains at most $O(k\Delta_{2, \epsilon}(Q_1(t))/\sin(\epsilon/4))$ points of $Q_1(t)$. Similarly, we find that $E(W^{\text{down}}_t)$ contains $O(k\Delta_{2, \epsilon}(Q_2(t))/\sin(\epsilon/4))$ points of $Q_2(t)$. For $0 \leq \epsilon \leq \pi/4$ we have $\sin(\epsilon/4) \geq \epsilon/8$, so we conclude that $E(W^{\text{down}}_t)$ contains $O(k/\epsilon^2)\Delta_{2, \alpha}(P(t))$.

It remains to analyze the number of points expiring at each time step. The proof of the following lemma is similar to that of Lemma 3.

**Lemma 12.** The number of points in $Q(t)$ at any time step $t$ is $O((1/\epsilon^2)k\Delta_{2, \alpha}\log n)$ amortized.

**Proof.** To prove this we again give an accounting scheme where each point receives a certain amount of money in each time step based on its smallest approximate distance to any of the envelopes. Let $W_{\epsilon/2}'(p)$ denote a wedge whose bisector makes an angle $\epsilon/2$ with the positive x-axis and with an opening angle of $\epsilon/2$, let $W_{\epsilon/2}'(p(t)) := \{W_{\epsilon/2}'(p(t)) : p \in P_t\}$ and let $E(W_{\epsilon/2}'(t))$ denote the envelope of these wedges. Recall that we have canonical wedges and corresponding envelope for $0 \leq \epsilon \leq 4\lceil \epsilon/\pi \rceil$ different angles. We define our approximate distance to one such envelope as $\text{dist}^{\ast}(p(t), i) := \text{dist}(p(t), p'(t))/\sin(\epsilon/4)$, where $p'(t)$ is the projection of $p(t)$ onto $E(W_{\epsilon/2}'(t))$ in direction $\epsilon/2$. We base our time stamp and money scheme on the minimum of these distances.
to all envelopes and define \( d_p(t) = \min_{0 \leq i < \lfloor 4\pi / \epsilon \rfloor} \text{dist}^*(p(t), i) \). At every time step point \( p(t) \) receives

\[
\max\left(1, \min\left(1, \frac{1}{n} \cdot \frac{6d_{\text{max}}}{d_p(t)}\right)\right) \text{ euro.}
\]

As before we prove that (i) each point receives at least one euro before its time stamp expires and (ii) we spend \( O((1/\epsilon)^2k\Delta_m \log n) \) euros per time step. Proving (i) is again the same as in the proof of Lemma 3, where \( d_p \) can change at most \( 2d_{\text{max}} \) per time step. To prove (ii) overestimate the amount of money spent by looking at how much money a point receives based on its distance to each of the envelopes separately. Without loss of generality let \( \epsilon/2 \) denote the upward direction. We divide the points of \( P(t) \) into group \( G_0, G_1, G_2, \ldots \), such that for any point \( p(t) \in G_j \) we have \( jd_{\text{max}} \leq d_p(t) \leq (j + 1)2d_{\text{max}} \). Points in groups \( G_{n+1}, G_{n+2}, \ldots \) receive \( O(1/n) \) euro each, so we spend at most \( O(1) \) euro on points in those groups. Each point in \( G_0 \) receives 1 euro and each point in group \( G_i \) for \( 1 \leq j \leq n \) receives \( O(1/j) \) euro. Points in group \( G_j \) are contained in region \( R \) which is the band between two vertically shifted versions of \( \mathcal{E}(W_{\epsilon/2}(t)) \) with a vertical distance \( d_{\text{max}} \sin(\epsilon/4) \). We again split \( P(t) \) into \( Q_1(t) \) and \( Q_2(t) \) such that \( \Delta_1(Q_1(t)) \leq \Delta_2(P(t)) \) and \( \Delta_1(Q_2(t)) \leq \Delta_2(P(t)) \). To find how many points of \( Q_2(t) \) are contained in \( R \) we divide the plane using a grid with cells of size \( \min_{\text{dist}}(Q_1(t)) / 2 \) and \( \min_{\text{dist}}(Q_2(t)) / 2 \). The points of \( Q_2(t) \) are then contained in \( \Delta_2(Q_1(t)) \) columns of this grid. The difference between the highest and lowest point in \( R \) within one such column is then

\[
\frac{2d_{\text{max}}}{\sin(\epsilon/4)} + \min_{\text{dist}}(Q_1(t)) / 2 = 20\min_{\text{dist}}(Q_1(t)) / \epsilon.
\]

Since each grid cell contains \( O(k) \) points of \( Q_1(t) \) it follows that \( R \) contains \( O(k\Delta_1(Q_1(t))/\epsilon) \) points of \( Q_2(t) \). A similar proof shows that \( R \) contains \( O(k\Delta_2(Q_2(t))/\epsilon) \) points of \( Q_2(t) \) and, hence, \( R \) contains \( O(k\Delta_2 / \epsilon) \) points of \( P(t) \).

Summing up the money spent on each group and find that each round we spend

\[
O(k\Delta_2 / \epsilon) + \sum_{j=1}^{n} \left((1/j)k\Delta_2 / \epsilon + 1\right)
\]

\[
= O(1/\epsilon)k\Delta_2 \log n \text{ euro.}
\]

We may spend at most this amount of money for each direction and since there are \( 4\pi/\epsilon \) directions we spend \( O((1/\epsilon)^2k\Delta_2 \log n) \) euros in total per time step.

Putting everything together, and using that the static 2-center algorithm by Chan [6] on \( n \) points runs in \( O(m \log m \log(\log m))^2 \) time, we obtain the following theorem.

**Theorem 13.** Let \( P \) be a set of \( n \) moving points that adheres to the Displacement Assumption with parameters \( k \) and \( d_{\text{max}} \), let \( \Delta_2 \) denote the maximum \( (2, k) \)-spread of \( P \) at any time \( t \), and let \( 0 < \epsilon \leq \pi/4 \). Then we can maintain a \((1 + \epsilon)\)-approximation of the Euclidean 2-center for \( P \) in the black-box KDS model in \( O((k/\epsilon)^2\Delta_2 \log n \log(\log n)) \) amortized time per time step and using \( O(n) \) space.

Note that when the point distribution is very bad and \( \Theta(n) \) point expire at every time step our algorithm is still not much worse than computing the 2-center from scratch. This is due to the fact that the number of expiring points is bounded by \( n \), which gives an update time of \( O(\min(n \log^2 n \log(\log n)^2), (1/\epsilon) \log n) \) time.

4. **Kinetic 2-Centers with Bounded Speed**

For applications such as mobile facility location, it is desirable that the centers move with bounded speed. However, even when the speed of the points is small, flips can occur, where the centers jump over a long distance. In the rectilinear case this happens when the optimal solution goes from \( \text{NW}[\text{SE}-\text{centers}] \) to \( \text{NE}[\text{SW}-\text{centers}] \) or vice versa. In this section we investigate maintaining approximations of rectilinear and Euclidean 2-centers such that the centers move at most a certain distance \( d_{\text{max}} \) within one time step.

**The rectilinear case.** A \( 2 \)-approximation for the rectilinear 2-center problem is not difficult to construct using the so-called reflective 2-centers of Durocher and Kirkpatrick [9]. For such a reflective 2-center one maintains a central point \( q \), of the point set; in our case the center of the bounding box. The centers are then an arbitrary point \( p \in P \) and its reflection \( \overline{p} \) across \( q \). The minimum-radius congruent squares centered at \( p, \overline{p} \) that together cover \( P \) provide a \( 2 \)-approximation of the optimal solution. (We use the term radius to denote half the edge length of a square.) Using the machinery of Section 2 we can maintain \( q, p, \overline{p} \) and the minimum radius in the black-box model.

**Theorem 14.** A \( 2 \)-approximation of the rectilinear 2-center in which the centers move at most \((2\sqrt{2} + 1)d_{\text{max}} \) per time step can be maintained in \( O(k\Delta_2 \log n) \) amortized time per time step.

We also show that in the worst case we cannot achieve anything better than a \( 2 \)-approximation.

**Lemma 15.** For any approximate 2-center with an approximation factor of \( 2 - \epsilon \) for \( \epsilon > 0 \) and maximum displacement \( d_{\text{max}} \), there is a set of moving points such that the centers must move more than \( d_{\text{max}} \) to maintain the \( 2 - \epsilon \) approximation.

**Proof.** Let \( r = 4d_{\text{max}} \) and let \( P \) be a set of four points \( p_w = (-r, 0), p_n = (0, 0), p_e = (r, 0), \) and \( p_w = (0, -r) \). Let \( \sigma_1 \) and \( \sigma_2 \) denote the approximate squares and without loss of generality assume that \( p_w, p_n \in \sigma_1 \) and \( p_e, p_w \in \sigma_2 \). The point \( p_n \) and \( p_w \) then move with a speed of \( d_{\text{max}} \) per time step towards \( q_w = (r, r) \) and \( p_e \) and \( p_w \) move towards \( q_w = (-r, r) \) with a speed of \( d_{\text{max}} \) per time step. If \( \sigma_1 \) keeps containing \( p_n \) and \( p_w \), then at some point it will become \( \Theta(d_{\text{max}}) \) large, since \( p_n \) and \( p_w \) are moving faster apart. However \( \sigma_2 \) cannot contain \( p_e \) or \( p_w \) as that would cause a displacement of its center by more than \( d_{\text{max}} \).

However, when the optimal solution is far from making a flip we can obtain a better approximation ratio. A solution is far from a flip if the difference in radius between the two different types of solutions is large. Let \( \ell_0 \) and \( \ell_\epsilon \) denote the horizontal and vertical edge length of \( \text{bb}(P) \), respectively. Let \( r_{\text{nw}[\text{se}]} \) and \( r_{\text{ne}[\text{sw}]} \) denote the radius of the optimal \( \text{NW}[\text{SE}-\text{squares}] \) and \( \text{NE}[\text{SW}-\text{squares}] \) and let \( r_{\text{min}} := \min(r_{\text{nw}[\text{se}]}, r_{\text{ne}[\text{sw}]}) \) and \( r_{\text{max}} := \max(r_{\text{nw}[\text{se}]}, r_{\text{ne}[\text{sw}]}) \). Consider the situation in Fig. (5)(i). Here we can place the centers in our approximation close to the centers of the \( \text{NW}[\text{SE}-\text{squares}] \), so that we are close to optimal. In Fig. (5)(ii), however, \( r_{\text{min}} \) is almost equal to \( r_{\text{max}} \), and the optimal solution will flip from the \( \text{NW}[\text{SE}-\text{squares}] \) to the \( \text{NE}[\text{SW}-\text{squares}] \) when \( p \) moves down. To avoid a high speed for the centers in such a case, we intuitively want to pick the centers “in the middle” between the two optimal solutions—in Fig. (5)(ii) this would be at the crosses—so that it is equally easy to shift towards either type of solution. The idea is now to place the centers in our approximate solution—we denote these centers by \( c_1 \) and \( c_2 \)—as far from the middle configuration as possible towards
the current optimal solution such that we can still reach the middle before a flip occurs. Next we make this idea precise.

We first determine the radius \( r_{\text{app}} \) that we want to achieve. It interpolates between \( \min(\ell_v, \ell_h)/2 \) (the radius in our middle configuration) and \( r_{\text{min}} \), where we want to stay as close to \( r_{\text{min}} \) as possible. When a flip occurs \( r_{\text{min}} = r_{\text{max}} \) and since each can change at most \( d_{\text{max}} \) per time step, the minimum number of time steps before a flip can occur is \( d_t := (r_{\text{max}} - r_{\text{min}})/2d_{\text{max}} \). Within one time step we can reduce the distance to our middle configuration by \( d_{\text{max}}/\sqrt{2} - 2d_{\text{max}} \). Combining these two, and taking care that the radius does not become smaller than \( r_{\text{min}} \) in a situation such as Fig. 5(iii), we set

\[
    r_{\text{app}} := \max(r_{\text{min}}, r), \quad r := \frac{\min(\ell_h, \ell_v)}{2} - \frac{r_{\text{max}} - r_{\text{min}}}{2d_{\text{max}}} \left( \frac{d_{\text{max}}}{\sqrt{2}} - 2d_{\text{max}} \right).
\]

Note that we omitted the time parameter to improve readability and will continue to do so during this section for our definitions. We now define the position of the approximate centers \( c_1 \) and \( c_2 \). This is done relative to the north-west and south-east corner \( q_{\text{nw}} \) and \( q_{\text{sw}} \) of \( \text{bb}(P) \) when the optimal solution is the \( \text{NW}\text{SE}-\text{solution} \) and relative to the north-east and south-west corners \( q_{\text{ne}} \) and \( q_{\text{sw}} \) otherwise.

Let

\[
    d_2 := \min(\ell_h/2, r_{\text{app}}) \quad \text{and} \quad d_2 := \min(\ell_v/2, r_{\text{app}})
\]

denote the horizontal and vertical distance from \( c_1 \) and \( c_2 \) to the corners of the bounding box. Thus, if the optimal solution consists of \( \text{NE}\text{SW}-\text{centers} \) then \( c_1 \) and \( c_2 \) are defined as

\[
    c_1 := (x(q_{\text{nw}}) + d_2, y(q_{\text{nw}}) + d_2) \quad \text{and} \quad c_2 := (x(q_{\text{sw}}) - d_2, y(q_{\text{sw}}) - d_2).
\]

For \( i = 1, 2 \), define \( \sigma_i \) to be the square with radius \( r_{\text{app}} \) and center \( c_i \). Thus, \( \sigma_1 \cup \sigma_2 \) covers \( P \). Indeed, if \( d_x = d_y = r_{\text{app}} \) this follows from \( r_{\text{app}} \geq r_{\text{min}} \); if, for instance, \( d_x = \ell_x/2 \) instead of \( d_x = r_{\text{app}} \) then we effectively shift our square vertically without losing any points. Recall that \( r_{\text{app}} \) was defined in such a way that we can always reach the middle position in time for a flip, given the maximum displacement \( d_{\text{max}} \) of the centers. We also need to show that the centers move at most \( d_{\text{max}} \) when a flip occurs.

**Lemma 16.** The centers \( c_1 \) and \( c_2 \) as defined above move at most \( d_{\text{max}} \) in a single time step.

The coordinates of the centers \( c_1 \) and \( c_2 \) can be found in \( O(1) \) time when the \( \text{NW}\text{SE}-\text{squares} \) and \( \text{NE}\text{SW}-\text{squares} \) are known, so maintaining them takes \( O(k\Delta_{x,k}\log n) \) amortized time by Theorem 6. The approximation ratio is \( r_{\text{app}}/r_{\text{min}} \) which, after rearranging the terms, gives the following theorem.

**Theorem 17.** Let \( d_{\text{max}}^* \geq 4\sqrt{2}d_{\text{max}} \). A \( \rho \)-approximation of the rectilinear 2-center in which the centers move at most \( d_{\text{max}}^* \) per time step can be maintained in \( O(k\Delta_{x,k}\log n) \) amortized time per time step, where

\[
    \rho = \max(1, 1 + \rho'), \quad \text{with} \quad \rho' = \frac{\min(\ell_v, \ell_h)/2 - r_{\text{max}} - r_{\text{min}}}{r_{\text{min}}} - \left( \frac{d_{\text{max}}}{2\sqrt{2}d_{\text{max}} - 2} \right).
\]

Since \( r_{\text{max}} + r_{\text{min}} \geq \min(\ell_v, \ell_h)/2 \) the second term, which becomes smaller as the overlap between the \( \text{NE}\text{SW} \) squares and \( \text{NW}\text{SE} \) squares increases, is at most 1. The condition \( d_{\text{max}}^* \geq 4\sqrt{2}d_{\text{max}} \) then ensures that \( \rho \leq 2 \). Note that \( \rho \) gets smaller when \( d_{\text{max}}^* \) increases (and we are thus further from a flip). Theorem 17 is also useful because we can use it to get a good approximation ratio for the Euclidean problem, as shown next.

**The Euclidean case.** We can use our bounded-speed rectilinear 2-center approximation to obtain a bounded-speed Euclidean 2-center approximation: we simply use the minimum enclosing disks of the two squares from the solution to the rectilinear problem. A straightforward approximation ratio is then \( 2\sqrt{2} \), since the rectilinear 2-center gives an approximation factor of at most 2 and we may lose an additional \( \sqrt{2} \) by placing a disk to cover the entire square as opposed to just the points contained in it. This would be worse than the reflective 2-center by Durocher and Kirkpatrick [9], who can guarantee a \( 8/\pi \) approximation. However, we show that when the approximation factor of the rectilinear 2-center is close to 2, then lower bound on radius \( r_{\text{opt}} \) of the optimal euclidean 2-center disks is larger than \( r_{\text{min}} \). By doing a more careful analysis we find an approximation factor of 2.29, with \( d_{\text{max}} = 4\sqrt{2}d_{\text{max}} \).

Let \( \sigma_{\text{nw}} \) and \( \sigma_{\text{sw}} \) be the \( \text{NE}\text{SW}-\text{squares} \), let \( \sigma_{\text{ne}} \) and \( \sigma_{\text{sw}} \) be the \( \text{NW}\text{SE}-\text{squares} \), and assume without loss of generality that an optimal solution to the rectilinear problem is given by the \( \text{NE}\text{SW}-\text{squares} \). Define \( F := (\min(\ell_v, \ell_h)/2 - r_{\text{max}})/r_{\text{min}} \). According to Theorem 17 the approximation ratio is at most \( 1 + F \), so \( \sigma_{\text{ne}} \) and \( \sigma_{\text{sw}} \) have radius \( (1 + F)r_{\text{min}} \). Thus the radius of the bounding disks of these squares is \( r_{\text{ne}}^{\text{opt}} := (1 + F)\sqrt{2}r_{\text{min}} \). Note that the optimal radius \( r_{\text{ne}}^{\text{opt}} \) for the Euclidean 2-center problem is at least \( r_{\text{min}} \). Indeed, if there would be a solution with smaller radius then we could also get a better solution to the rectilinear problem by taking the bounding boxes of the disks. For \( F \leq 1/2 \) that means we get an approximation factor of \((1 + F)^2/2 \leq 3/2 \approx 2.12 \). For \( F > 1/2 \) we will provide a lower bound on \( r_{\text{ne}}^{\text{opt}} \) that depends on \( F \) to get our final approximation ratio of 2.29.

Let \( p_1, p_2, p_3, \) and \( P \) denote points on the top, right, bottom and left edge of \( \text{bb}(P) \), respectively. Note that \( F = (\min(\ell_v, \ell_h)/2 - r_{\text{max}})/r_{\text{min}} > 1/2 \) implies \( r_{\text{min}} + 2r_{\text{max}} < \min(\ell_v, \ell_h) \), so \( r_{\text{min}} < \min(\ell_v, \ell_h)/3 \). This means that \( \sigma_{\text{nw}} \) does not intersect

**Figure 5:** Three situations for the rectilinear 2-center. The NW|SE-squares are gray (solid squares as centers) and the NE|SW-squares dashed (drawn slightly bigger than their actual size, non-filled squares as centers). The points from \( P \) are not shown in (i) and (iii), and in (ii) as small disks.
the bottom edge of $bh(P)$ and $\sigma_{sw}$ does not intersect its top edge, and so $p_a, p_b \in \sigma_{sw}$ and $p_c, p_d \in \sigma_{sw}$. We also have $r_{\text{max}} < \min(\ell_e, \ell_b)/2$, so $p_a, p_b \in \sigma_{sw}$ and $p_c, p_d \in \sigma_{sw}$. We now distinguish three cases, depending on how the disks in an optimal solution to the Euclidean 2-center problem cover $p_a, p_b, p_c, \text{ and } p_d$.

Case (i): one of the disks in an optimal solution contains $p_a$ and $p_b$, or $p_c$ and $p_d$. The radius of the optimal disks is at least $\min(\ell_e, \ell_b)/2$ which (as explained earlier) is at least $(3/2) r_{\text{min}}$ due to our assumption that $F > 1/2$. So we get an approximation ratio of $2\sqrt{2}/(3/2) < 2$.

Case (ii): one of the disks in an optimal solution contains $p_a$ and $p_b$, and the other disk contains $p_c$ and $p_d$. Let $D_{sw}$ denote the disk that covers $p_a$ and $p_b$, and let $D_{ne}$ denote the disk that covers $p_c$ and $p_d$. We claim that $D_{sw}$ intersects the top or right edge of $\sigma_{sw}$ or that $D_{ne}$ intersects the bottom or left edge of $\sigma_{sw}$. Indeed, if this were not the case the NE|SW-squares could be made smaller. Without loss of generality we assume that $D_{sw}$ intersects the right edge of $\sigma_{sw}$, which we denote by $e_w$, and that the lower-left corner of $\sigma_{sw}$ is located at $(0, 0)$. Since $F \cdot 2r_{\text{min}} = \min(\ell_e, \ell_b) - r_{\text{max}}$, the rectangle $[0, F \cdot 2r_{\text{min}}] \times [0, F \cdot 2r_{\text{min}}]$ is not covered by the NW|SE-squares, which means it cannot contain points from $P$. In particular, $p_a \in e^*_w$ and $p_b \in e^*_w$, where $e^*_w := [F \cdot 2r_{\text{min}}, 2r_{\text{max}}] \times 0$ and $e^*_w := 0 \times [F \cdot 2r_{\text{min}}, 2r_{\text{max}}]$. To get a lower bound on the radius of $D_{sw}$, we thus need to find the smallest radius of any disk that intersects the three segments $e^*_w$, $e^*_w$, and $e_w$.

It is easy to see that the smallest disk $D$ that intersects $e^*_w$, $e^*_w$, and $e_w$ must have its center $c$ on the bisector of $e^*_w$ and $e^*_w$, which is the diagonal of $\sigma_{sw}$ going from the lower-left to the upper-right corner. Since $F > 1/2$, we can also argue that $c$ cannot have an $x$-coordinate greater than $F \cdot 2r_{\text{min}}$, since in this case shifting $c$ along the diagonal downwards would give a smaller radius. We can conclude that the points nearest to $c$ on $e^*_w$ and $e^*_w$ are the left and bottom endpoints of $e^*_w$ and $e^*_w$, respectively. Thus we wish to determine the radius of the smallest circle passing through these two points and intersecting $e_w$. When $F$ is large enough a disk with the left and bottom endpoints of $e^*_w$ and $e^*_w$ as diametrical points also intersects $e_w$. Let $r$ denote the radius of such a disk. The horizontal distance from $c$ to $e_w$ should then be at most $r$. It follows that $2r_{\text{min}} - F_{\text{min}} \leq \sqrt{2}Fr_{\text{min}}$ which gives us $F \geq 2\sqrt{2} - 2$. So for $2\sqrt{2} - 2 \leq F \leq 1$ a disk with radius $r = \sqrt{2}F_{\text{min}}$ intersects $e^*_w, e^*_w$, and $e_w$. The maximum approximation ratio for $2\sqrt{2} - 2 \leq F \leq 1$ then occurs when $F = 2\sqrt{2} - 2$ and is

$$\frac{(1 + \sqrt{2})r_{\text{min}}}{\sqrt{2}F_{\text{min}}} = \frac{2\sqrt{2} - 1}{2\sqrt{2} - 2} \approx 2.21.$$  

When $1/2 < F < 2\sqrt{2} - 2$ the smallest disk intersecting $e^*_w$, $e^*_w$, and $e_w$ is tangent to $e_w$. The radius $r$ of such a disk satisfies

$$r^2 = (F \cdot 2r_{\text{min}} - 2r_{\text{min}} + r)^2 + (2r_{\text{min}} - r)^2,$$

which can be seen by considering the gray triangle in Fig. 6. Solving this for $r$ we get $r = 2r_{\text{min}}(2 - \sqrt{2}\sqrt{1 - F})$. We argued that $r_{\text{opt}} \geq r$, hence we obtain an approximation ratio of at least

$$\frac{(F + 1)r_{\text{min}}\sqrt{2}}{r} = \frac{F + 1}{\sqrt{2}(2 - \sqrt{2}\sqrt{1 - F})}.$$

Finding the maximum of this for $1/2 < F \leq 1$ then gives an approximation ratio of

$$\frac{3\sqrt{5} - 5}{2\sqrt{2}(8 - 3\sqrt{5} - \sqrt{2}(7 - 3\sqrt{5}))} \leq 2.29.$$

Case (iii): one of the disks in an optimal solution contains $p_a$ and $p_b$, and the other disk contains $p_c$ and $p_d$. Let $D_{sw}$ and $D_{ne}$ denote the disks that cover $p_a, p_b$ and $p_c, p_d$ respectively. Then either $D_{sw}$ intersects the bottom or right edge of $\sigma_{sw}$, or $D_{ne}$ intersects the top or left edge of $\sigma_{sw}$. Without loss of generality we assume that $D_{sw}$ intersects the right edge of $\sigma_{sw}$, which we again denote by $e^*_w$. (Note that the situation is now slightly different from Case (ii), because $\sigma_{sw}$ is one of the (non-optimal) NW|SE-squares.) Now assume without loss of generality that the top-left corner of $\sigma_{sw}$ is located at $(0, 0)$. Because $F \cdot 2r_{\text{min}} = \min(\ell_e, \ell_b) - r_{\text{max}}$ there is a square with edge length $2r_{\text{max}} - (2 - 2F)r_{\text{min}}$ with its north-west corner at $(0, 0)$ that does not contain any points. To get a lower bound on the radius of $D_{sw}$, we thus need to find the smallest radius of any disk intersecting the three segments $e^*_w := [2r_{\text{max}} - (2 - 2F)r_{\text{min}}, 2r_{\text{max}}] \times 0, e^*_x := 0 \times [2r_{\text{max}} - (2 - 2F)r_{\text{min}}, 2r_{\text{max}}]$, and $e_w$.

It is easy to see that the smallest disk $D$ that intersects $e^*_w$, $e^*_w$, and $e_w$ must have its center $c$ on the bisector of $e^*_w$ and $e^*_w$, which is the diagonal of $\sigma_{sw}$ going from the upper-left to the lower-right corner. Since $F > 1/2$ and $2r_{\text{max}} - (2 - 2F)r_{\text{min}} \geq r_{\text{max}}$, we can also argue that $c$ cannot have an $x$-coordinate greater than $2r_{\text{max}} - (2 - 2F)r_{\text{min}}$, since in this case shifting $c$ along the diagonal upwards would give a smaller radius. We can conclude that the points nearest to $c$ on $e^*_w$ and $e^*_w$ are the left and top endpoints of $e^*_w$ and $e^*_w$, respectively. Thus we wish to find the radius of the smallest circle passing through these two points and intersecting $e_w$. We define two sub-cases based on whether or not a disk with the top and left endpoints of $e^*_w$ and $e^*_w$ as diametrical points intersects $e_w$. This is the case when

$$2r_{\text{max}} - (r_{\text{max}} - (1 - F)r_{\text{min}}) \leq \sqrt{2}(r_{\text{max}} - (1 - F)r_{\text{min}}),$$

which gives us $F \geq 1 - a/(3 + 2\sqrt{2})$, where $a = r_{\text{max}}/r_{\text{min}}$. It is safe to assume that $1 < a < 2$, otherwise any disk that intersects $e^*_x$ and $e_w$ must have radius at least $2r_{\text{min}}$. Hence, the approximation ratio we get is

$$\frac{F + 1}{\sqrt{2}(F \cdot r_{\text{min}} + r_{\text{max}} - r_{\text{min}})} \leq \frac{F + 1}{F + a - 1} \leq \frac{2 - a/(3 + 2\sqrt{2})}{1 - a/(3 + 2\sqrt{2})} \leq \frac{2 - 1/(3 + 2\sqrt{2})}{1 - 1/(3 + 2\sqrt{2})} \leq 2.21.$$
When $1/2 < F < 1 - a/(3 + 2\sqrt{2})$ the smallest disk intersecting $e_w$, $e_v$, and $e_v$ is tangent to $e_v$. The radius $r$ of such a disk satisfies $r^2 = (2r_{\text{max}} - (2 - 2F) \cdot r_{\text{min}} - 2r_{\text{max}} + r)^2 + (2r_{\text{max}} - r)^2$, which solves to $r = 2r_{\text{min}}(1 + a - F - \sqrt{2a - aF})$. The approximation ratio then becomes $\frac{(F + 1)r_{\text{min}}\sqrt{2}}{2r_{\text{min}}(1 + a - F - \sqrt{2a - aF})}$.

To find the maximum we should maximize over $F$ and $a$. Since $F > 1/2$ and $a \geq 1$, we know that $a - \sqrt{a}\sqrt{2(1 - F)}$ increases with $a$, so the approximation ratio decreases with $a$. Hence, the worst approximation ratio is achieved for $a = 1$. This reduces the approximation ratio to $\frac{(F + 1)}{\sqrt{2(2 - F - \sqrt{2(1 - F)}}}$, which is equal to the approximation ratio in case (ii) and yields a maximum of 2.29. (This is the maximum for all $1/2 < F \leq 1$, so the fact that our upper bound on $F$ depends on $a$ does not change the maximum approximation ratio.)

As for the running time, we observe that the disks and corresponding centers can be computed in $O(1)$ time from the approximate rectilinear 2-centers, which takes $O(k\Delta_{2,2}\log n)$ amortized time per time step by Theorem 17.

**Theorem 18.** A $2.29$-approximation of the Euclidean 2-center in which the centers move at most $d_{\text{max}} = 4\sqrt{2}d_{\text{max}}$ per time step can be maintained in $O(k\Delta_{2,2}\log n)$ amortized time per time step.

**5. CONCLUSIONS**

We presented a new algorithm for maintaining the optimal rectilinear 2-center and a $1 + \varepsilon$-approximation for the Euclidean 2-center in the black-box model. Both algorithms use that points move only a small distance at each time step to avoid inspecting all points in every time step. This allows for sublinear update time when the distribution of points is good, that is, when the $(2, k)$-spread $\Delta_{2,k}$ is small.

An interesting question is what can be done when a small number of points move faster than predicted. The difficulty in dealing with these outliers arises when they are interesting points. In that case the time stamps of many other points may become invalid as the structure itself changes more than anticipated. It is still unknown how to efficiently deal with such outliers.

We also presented algorithms for maintaining constant-factor approximations of the rectilinear and Euclidean 2-center that guarantee a small movement speed of the centers. For the Euclidean 2-center we obtain a $2.29$-approximation, which improves a previous algorithm by Durocher and Kirkpatrick [9] that has an approximation ratio of 2.55. However, in the solution by Durocher and Kirkpatrick, the centers move at most $(8/\pi + 1)d_{\text{max}} \approx 3.55d_{\text{max}}$ per time step. One may thus wonder if there is a fundamental trade-off between the approximation ratio and the speed of the centers. The main problem with bounded speed, however, is that a sudden change between two solutions can occur when two very different solutions have nearly the same size. This seems to be what bounds the approximation ratio and a higher (but constant) movement speed for the centers does not help avoid such a flip. The main open question in this case is what is the best possible approximation ratio for centers with constant speed as the best known lower bound is only $\sqrt{2}$.

**6. REFERENCES**


