Column Generation and its applications

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TU/e Technische Universiteit Eindhoven University of Technology

Outline

- Some real-life decision problems
  - Standard formulations
- Basics of Column Generation
  - Master formulations
- Case: Shift scheduling of airport workers
  - Problem description
  - Master formulation
  - Reduced cost and pricing problem
- Column Generation overview
- Towards an integer solution
- Using data science in the CG approach

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Different industry sectors, different problems

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- Telecommunication: Designing FTTH network.
- Airport ground operations: Shift scheduling of workers.
- Logistics: Planning routes of vehicles.
- Machine learning: Constructing max-accuracy decision trees.

Instance sizes

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- Decision trees: 5K data instances, 50 attributes.

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- Many local optimal points.

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- When no promising column exists: certificate for optimality.

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  - night shifts: longer resting times.

Basic notation

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- Service demand $R_{i,d}$ at time i in skill type d,

Master IP formulation

$$
\begin{aligned}
\text{Min} \quad & \sum_{w \in W} \sum_{s \in \mathcal{S}} c_{sw} x_{sw} \\
\text{subject to} \quad & \\
& \sum_{w \in W} Sk_w^d s^i x_{sw} \geq R_{i,d}, \quad d \in D, i \in N \\
& \sum_{s \in \mathcal{S}} x_{sw} = 1, \quad w \in W \\
& x_{sw} \in \{0,1\}, \; s \in \mathcal{S}, w \in W
\end{aligned}
$$

LP Relaxation of Master formulation

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Note: Restricted set $\mathcal{S}' \subset \mathcal{S}$

LP Relaxation of Master formulation

$$
\begin{array}{llll}
\text{Min} & \displaystyle\sum_{w\in W}\sum_{s\in\mathcal{S}'} c_{sw}x_{sw} & & \text{Duals} \\
\text{subject to} & & & \\
& \displaystyle\sum_{w\in W} Sk_w^d s^i x_{sw} \geq R_{i,d}, & d\in D, i\in N & \pi_{i,d} \\
& \displaystyle\sum_{s\in\mathcal{S}'} x_{sw} = 1, & w\in W & \theta_w \\
& 0 \leq x_{sw} \leq 1, & s\in\mathcal{S}', w\in W &
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$$

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Reduced cost of column x_{sw}

Dual constraint of column x_{sw}:

$$\sum_{d \in D} \sum_{i \in N} Sk_w^d s^i \pi_{i,d}^* + \theta_w^* \leq c_{sw} \tag{1}$$

Reduced cost of column x_{sw}:

$$\bar{c}_{sw} = c_{sw} - \sum_{d \in D} \sum_{i \in N} Sk_w^d s^i \pi_{i,d}^* - \theta_w^* \tag{2}$$

Case $\bar{c}_{sw} < 0$: (1) Estimated objective decrease (why?), (2) dual feasibility violation.

Pricing problem: Objective

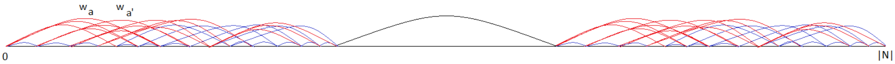
Pricing problem: Find the most promising column (schedule) with the objective

$$\min_{s\in\mathcal{S},w\in W}\left\{c_{sw}-\sum_{d\in D}\sum_{i\in N}Sk_w^d s^i \pi_{i,d}^* - \theta_w^*\right\} \tag{3}$$

$$= \max_{s\in\mathcal{S},w\in W}\left\{\sum_{i\in N}s^i\left(\sum_{d\in D}Sk_w^d \pi_{i,d}^* - c_{iw}\right)\right\} - \theta_w^* \tag{4}$$

Pricing problem: Modeling

Define a graph,



- reds: shifts, blues: resting times, blacks: unavailability.

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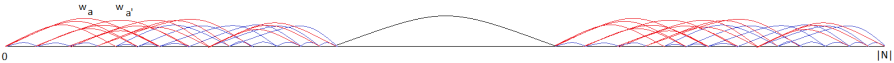

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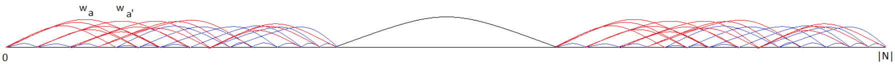

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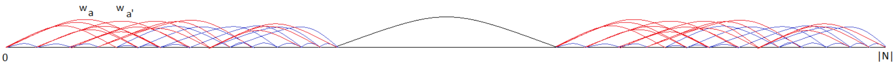

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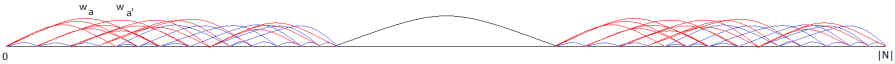

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- Pricing: find the constrained $0 - |N|$ "Longest Path"!

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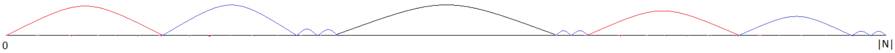
A schedule s on the graph looks like:



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- Step 3: Output the RMP solution.

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Towards integer solutions

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  - to obtain "optimal" integer sol'n.
  - to output "best-found" integer sol'n with quality measure in a time limit.

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Using data science: Advantage or waste?

Consider a data of instance history and their solutions.

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- Simply find "similar" previous solutions to the current fractional solution and round accordingly.
- Cluster instances and analyze the commonalities (patterns) in the cluster solutions.

THANKS