Integral Mumford--Tate groups and the Mumford--Tate conjecture

Milan Lopuhaä-Zwakenberg

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1 Introduction

Let X be a smooth proper scheme over a finitely generated subfield k of characteristic C. On one hand, one can consider its Betti cohomology $H^i := \mathrm{H}^i(X^{\mathrm{an}}, \mathbb{Q})$. This is a Hodge structure of weight i; let $M_\sigma \subset \mathrm{GL}(H^i)$ be its *Mumford--Tate group*, i.e. the Zariski closure of the image of the Deligne torus $\mathscr{S}$. On the other hand, one can choose a prime number ℓ, and consider the representation of $\mathrm{Gal}(\bar{k}/k)$ on $H^i_\ell := \mathrm{H}^i_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$. Define the *Galois monodromy group* G_ℓ be the identity component of the Zariski closure of the image of $\mathrm{Gal}(\bar{k}/k)$ in $\mathrm{GL}(H^i_\ell)$. We have canonical isomorphisms $H^i \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H^i_\ell$.

Conjecture 1.1 (Mumford--Tate)**.** *There is an equality $G_\ell = M_{\mathbb{Q}_\ell}$ as subgroups of $GL(H^i \otimes \mathbb{Q}_\ell)$.*

The Mumford--Tate conjecture relates the Hodge conjecture (which states that (after twisting) M_σ-invariant cohomology classes are algebraic) to the Tate conjecture (which states that (after twisting) G_ℓ-invariant cohomology classes are algebraic). In general we have no idea how to prove this conjecture. In the case that X is an abelian variety we know a little bit more. First of all, both cohomology theories are generated by their H^1, so we only need to consider $i = 1$ (which we will now omit from the notation).

- We have an equality $G_\ell \subset M_{\mathbb{Q}_\ell}$, see Deligne [1982].
- Both groups are reductive (Tannakian abstract nonsense).
- If $D := \mathrm{End}_{\mathbb{Q}}(X)$ is the endomorphism algebra of X, and ψ is a polarisation of X, then ψ induces a symplectic form on H (still denoted ψ), and D acts on (H, ψ). In this case we have $M \subset \mathrm{GSp}_D(H, \psi)$, $G_\ell \subset \mathrm{GSp}_D(H_\ell, \psi_\ell)$, and D (respectively D_ℓ) is the commutant of M (respectively G_ℓ). For a given dimension and D this allows us to classify all possible Mumford--Tate groups and Galois monodromy groups.
- The Mumford--Tate conjecture is known for CM abelian varieties, in which case both groups are tori, see Pohlmann [1968].

One way to approach the Mumford--Tate conjecture is to prove that in specific cases we actually get an equality $G_\ell = \mathrm{GSp}_D(H_\ell, \psi_\ell)$; this is done in e.g. Pink [1998]. However, we cannot expect to solve the Mumford--Tate conjecture in its entirety in this way, since we already know that there are Mumford--Tate groups that occur that are not either tori or of the form $\mathrm{GSp}_D(H, \psi)$ (see Mumford [1969]). The smallest such example occurs in dimension $g = 4$.

We can describe this example as follows. Let F be a cubic field over $\mathbb{Q}$, and let D be a quaternion algebra over F. From D we can make a central simple algebra $\mathrm{Cor}(D)$ over $\mathbb{Q}$, called the *corestriction of* D, defined by

$$\mathrm{cor}(D) := \left(\bigotimes_{F \overset{\sigma}{\hookrightarrow} \bar{\mathbb{Q}}} (D \otimes_{F,\sigma} \bar{\mathbb{Q}}) \right)^{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})},$$

where the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is given by

$$\tau \left(\bigotimes_{F \overset{\sigma}{\hookrightarrow} \bar{\mathbb{Q}}} (d_\sigma \otimes x_\sigma) \right) = \bigotimes_{F \overset{\sigma}{\hookrightarrow} \bar{\mathbb{Q}}} (d_{\tau^{-1}\sigma} \otimes \tau(x_{\tau^{-1}\sigma})).$$

there is a natural norm map $D^\times \to \mathrm{cor}(D)^\times$ given by $d \mapsto \bigotimes_\sigma (d \otimes 1)$. Now suppose that D is such that $\mathrm{cor}(D)$ is the trivial element in $\mathrm{Br}(\mathbb{Q})$, i.e. $\mathrm{cor}(D) \cong \mathrm{M}_8(\mathbb{Q})$; then we get a morphism $D^\times \to \mathrm{GL}_8(\mathbb{Q})$, which we may consider as a map of algebraic groups over $\mathbb{Q}$. The image G_D, considered as an algebraic group, is said to be of *Mumford's type*. There is a unique symplectic form on $\mathbb{Q}^8$ invariant (up to scalars) under G_D , and G_D occurs as a Mumford--Tate group if and only if F is totally real and D splits over exactly one of the real places of F, i.e. $D \otimes_\mathbb{Q} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R}) \oplus \mathbb{H} \oplus \mathbb{H}$. Furthermore, every simple abelian fourfold with endomorphism ring $\mathbb{Z}$ has a Mumford--Tate group isomorphic either to one of the G_Ds, or to GSp_8.

2 Moduli spaces

Consider the moduli space $\mathscr{A}_{4,n}$ of principally polarised abelian fourfolds with full level n structure (for some $n > 2$). The groups $G_D \subset \mathrm{GSp}_8$ (after the choice of some embedding) correspond to special subvarieties of $\mathscr{A}_{4,n}$; these are called *Mumford curve*. To prove the Mumford--Tate conjecture for abelian fourfolds we need to prove the following:

Conjecture 2.1. *Let X be an abelian fourfold with endomorphism ring $\mathbb{Z}$ over a finitely generated subfield $K \subset \mathbb{C}$; let $x \in \mathscr{A}_{4,n}(K)$ be a point representing x. Suppose X has endomorphism ring $\mathbb{Z}$. If its Galois monodromy groups are of Mumford's type for some ℓ (or equivalently for all ℓ, by Noot [2000]), then x lies on a Mumford curve.*

In fact, this is true if the Mumford--Tate conjecture is true. So how do we prove that x lies on a Mumford curve? One way to tackle this problem is to look at the reductions x_v of x at finite places v of K. If there is a Mumford curve S such that infinitely many x_v lie on its reductions S_v, then x has to lie on S, as it is the generic point of the Zariski closure of the x_v. Describing the S_v is difficult, however.

Instead we look at lifts of the x_v. It turns out that for a collection of finite places of Dirichlet density 1 the reductions x_v are ordinary; hence we can consider their canonical lifts x_v^{can}; these are abelian varieties over $\kappa(\mathrm{W}(k_v))$. If x_v^{can} lies on a Mumford curve S, then x_v lies on S_v; hence we need to prove the following claim.

Claim 2.2. *There is a Mumford curve S such that x_v^{can} lies on S for infinitely many v.*

If the Mumford--Tate conjecture is true, then this claim should be true as well, so this is a sensible strategy. If the claim is true, then the generic Mumford--Tate group H of S satisfies $H_{\mathbb{Q}_\ell} \cong G_\ell$. So we have the following two preliminary questions:

- Does there exist an algebraic group H over $\mathbb{Q}$ such that $H_{\mathbb{Q}_\ell} \cong G_\ell$ for all primes ℓ?
- If v is a finite place of K of ordinary reduction for X, does x_v^{can} lie on a Mumford curve with generic Mumford--Tate group H?

These questions turn out to be related. For the second question, we need to show that the Mumford--Tate group T_v of x_v^{can} can be embedded into H. We can describe T_v as follows: there is a set of finite places v of K of Dirichlet density 1 such that the Galois representation on H_ℓ is unramified at v for all $\ell \neq \mathrm{char}(v)$. Furthermore, if $\mathrm{Frob}_v \in \mathrm{Gal}(\bar{K}/K)$ is a Frobenius element, then its image $r_{v,\ell}$ in $\mathrm{GL}(H_\ell)$ is semisimple and well-defined up to conjugation. Furthermore, its characteristic polynomial f_v has coefficients in $\mathbb{Z}$ and does not depend on ℓ. Now let r_v be a semisimple element of $\mathrm{GL}_{8,\mathbb{Q}}$ with characteristic polynomial f_v. Then T_v is isomorphic to the Zariski closure of $\langle r_v \rangle$ in $\mathrm{GL}_{8,Q}$; this is a torus. Since $\langle r_{v,\ell} \rangle \subset G_\ell$, we get an embedding $T_{v,\mathbb{Q}_\ell} \hookrightarrow G_\ell$, and for v in a set of Dirichlet density 1 this is a maximal torus.

To answer the first question, isomorphism classes of groups of Mumford's type over $\mathbb{Q}$ correspond to pairs (F, D), where F is a rank 3 étale algebra over $\mathbb{Q}$, and D is a quaternion algebra over F with trivial corestriction to $\mathbb{Q}$. Similarly, the groups G_ℓ give us pairs (F_ℓ, D_ℓ), and we need to find a (F, D) such that $(F \otimes \mathbb{Q}_\ell, D \otimes \mathbb{Q}_\ell) \cong (F_\ell, D_\ell)$ for all ℓ. Let us start by finding D; as an étale $\mathbb{Q}$-algebra of rank 3 it should correspond to a set of cardinality 3 with a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action. However, we can recover this set from a maximal torus in H: the characters of such a maximal torus present in the representation of H form a 'cube' in its character space, and the set we are looking for is the set of pairs of opposite sides of this cube. Since the G_ℓ share maximal tori defined over $\mathbb{Q}$, we find our D; it turns out that this is indeed a real number field.

The quaternion algebra F_ℓ corresponds to a set of invariants $(x_\lambda)_\lambda \in \bigoplus_\lambda \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, where λ ranges over the $\mathbb{Q}_\ell$-algebra homomorphisms $\lambda\colon D_\ell \to \bar{\mathbb{Q}}_\ell$. The fact that $\mathrm{cor}(F_\ell) \cong \mathrm{M}_8(\mathbb{Q}_\ell)$ is equivalent to stating $\sum_\lambda x_\lambda = 0$. By choosing for every ℓ an isomorphism $D \otimes \mathbb{Q}_\ell \cong D_\ell$, we can transfer these invariants to invariants of D, and we can use these to define a quaternion algebra F over D (we choose the invariants at infinite places in such a way that we get $F \otimes \mathbb{R} \cong \mathrm{M}_2(\mathbb{R}) \oplus \mathbb{H} \oplus \mathbb{H}$); this works because the invariants will be trivial for almost all ℓ (see **?**). Then F is generally not unique, because it depends on the choices of isomorphisms, but only finitely many of these choices matter. Thus we get finitely many $\mathbb{Q}$-groups H

such that $H_{\mathbb{Q}_\ell} \cong G_\ell$ for all ℓ. If we fix a finite place v, we can choose the isomorphisms $D \otimes \mathbb{Q}_\ell \cong D_\ell$ in such a way that we actually get an embedding $T_v \hookrightarrow H$.

3 Integral data

So the good news is that for every v (in some collection of Dirichlet density 1) we have found a Mumford curve on which x_v^{can} lies. However, this is not good enough for us: we need one Mumford curve on which infinitely many x_v^{can} lie. Our goal is now to prove the following claim:

Claim 3.1. *There exists a finite set of Mumford curves S such that infinitely many x_v^{can} lie on one of these curves.*

So we need a way to 'bound' the set of allowable Mumford curves; in general there exist infinitely many Mumford curves with the same generic Mumford--Tate group. One way to do this is to look at *integral Mumford-Tate groups*. The comparison isomorphism we talked about before actually exists on an integral level, i.e. we have an isomorphism $\mathrm{H}^1(X^{\mathrm{an}}, \mathbb{Z}) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell)$. If we take Zariski closures in their general linear groups, rather than the generic fibres, we get the *integral Mumford--Tate group $\mathscr{M}$* over $\mathbb{Z}$ and the *integral Galois monodromy groups $\mathscr{G}_\ell$* over $\mathbb{Z}_\ell$. The Mumford--Tate conjecture is now isomorphic to its analogous integral statement. Hence if the Mumford--Tate conjecture is true, the point x should lie on a Mumford curve whose generic integral Mumford--Tate group $\mathscr{H}$ satisfies $\mathscr{H}_{\mathbb{Z}_\ell} \cong \mathscr{G}_\ell$ for all primes ℓ. In my thesis, I prove that only finitely many such Mumford curves exist:

Theorem 3.2. *Let g and n be integers with $n > 2$. For every prime ℓ, let $\mathscr{A}_\ell$ be a group scheme of finite type over $\mathbb{Z}_\ell$. Then there exist at most finitely many special subvarieties of $\mathscr{A}_{g,n}$ whose generic integral Mumford--Tate group $\mathscr{H}$ satisfies $\mathscr{H}_{\mathbb{Z}_\ell} \cong \mathscr{A}_\ell$ for all ℓ.*

Sketch of proof. Suppose the generic fibres of the $\mathscr{A}_\ell$ all come from one reductive $\mathbb{Q}$-group H (if not then we are done); then there are only finitely many choices for this $\mathbb{Q}$-group (see [Serre, 1997, III.4.6]), and for the isomorphism class representation $H \hookrightarrow \mathrm{GSp}(V)$. The integral structure on $\mathscr{H}$ now comes from a lattice in V, and changing a special subvariety by a Hecke correspondence is the same as taking another lattice. Thus the question we need to answer is how many lattices in V give the same integral model of H. If $\mathscr{H}$ is a model of H coming from a lattice Λ, than we can regard $\mathrm{Lie}(\mathscr{H})$ as a lattice in the vector space $\mathrm{Lie}(H)$; it is the lattice that sends Λ to itself. If we let $\mathscr{H}_0$ be a given model of H, and $\Lambda_0 \subset V$ a lattice that gives the model $\mathscr{H}_0$, then we can bound the `distance' between the lattices $\Lambda_0$ and $\Lambda$ in terms of the `distance' between the lattices $\mathrm{Lie}(\mathscr{H}_0)$ and $\mathrm{Lie}(\mathscr{H})$ in $\mathrm{Lie}(H)$. Hence if we know the model $\mathscr{H}$, then we know that there are only finitely many lattices that can yield that model. Similar methods show that it is enough to know the models $\mathscr{H}_\ell$. ☐

After all this we are stuck with proving the following statement:

Conjecture 3.3. *For infinitely many v x_v^{can} lies on a Mumford curve whose generic integral Mumford--Tate group $\mathscr{H}$ satisfies $\mathscr{H}_{\mathbb{Z}_\ell} \cong \mathscr{G}_\ell$ for all ℓ.*

Unfortunately I am not able to prove this. The main difference between the $\mathbb{Q}$-case and the $\mathbb{Z}$-case, is that the existence of an embedding $T_v \hookrightarrow H$ implies that x_v^{can} lies on a Mumford curve with generic

Mumford--Tate group H. This is because the induced representation $T_v \hookrightarrow H \hookrightarrow \mathrm{GSp}_{8,\mathbb{Q}}$ is the same as the representation $T_v \hookrightarrow \mathrm{GSp}_{8,\mathbb{Q}}$ coming from the fact that T_v is a Mumford--Tate group. However, over $\mathbb{Z}$ this is no longer the case: one integral group scheme can have multiple representations that are isomorphic over $\mathbb{Q}$. For example, if $K/\mathbb{Q}$ is a number field, then the group $G := \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_\mathrm{m})$ has a natural representation on the $\mathbb{Q}$-vector space K. The group G also has a model $\mathscr{G} := \mathrm{Res}_{\mathscr{O}_K/\mathbb{Z}}(\mathbb{G}_\mathrm{m})$. The representations of $\mathscr{G}$ on $\mathbb{Z}^{[K:\mathbb{Q}]}$ that correspond to the rational representation above now correspond bijectively to $\mathrm{Cl}(K)$. Thus in this case we can prove that we get an embedding $\mathscr{T}_v \hookrightarrow \mathscr{H}$, but unfortunately we cannot show that x_v^{can} lies on a nice Mumford curve, because of a class group-like obstruction.

We see that looking at integral Mumford--Tate groups does not directly give us the desired result. However, the theorem does suggest that integral Mumford--Tate groups can be helpful in looking at the Mumford--Tate conjecture. Although they are more difficult to describe, they carry more information that might be crucial.

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