Exploring families of energy-dissipation landscapes via tilting — three types of EDP convergence *

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1 Introduction: Gradient systems and kinetic relations

This paper revolves around a subtle distinction between two concepts: *passing to the limit* in a gradient system, on one hand, and *deriving effective kinetic relations* on the other. The two concepts are strongly related, and in many examples they even appear to be the same. Our main contributions are to show that they are different, to show that well-known techniques developed for the former may give incorrect results for the latter, and to introduce new tools to remedy this.

1.1 Gradient systems

A *gradient system* is a triple $(\mathbf{Q}, \mathcal{E}, \mathcal{R})$ of a space $\mathbf{Q}$, a functional $\mathcal{E}$ on $\mathbf{Q}$, and a *dissipation potential* $\mathcal{R}$. We give exact definitions in Section 2 below; we first illustrate the concept with an example.

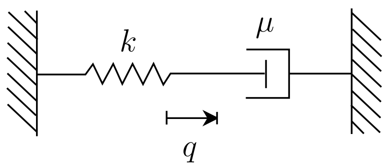

Figure 1.1: A spring-damper system. The spring has spring constant k, and the damper viscosity constant μ.

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Figure 1.1 shows a simple system, in which a spring relaxes by moving a damper (a shock absorber). The state of the system is the spring displacement $q \in \mathbb{R}$, the energy contained in the spring is $\mathcal{E}_1(q) := kq^2/2$, and the spring exerts a force ξ equal to the negative derivative $-\mathrm{D}\mathcal{E}_1(q) = -kq$ of the energy. The damper is defined by the property that its velocity v is related to the force ξ on the damper by $\mu v = \xi$, for some coefficient $\mu > 0$. By combining these two relations we find the evolution equation for the state q,

$$\mu \dot{q} = -kq. \tag{1.1}$$

We recognize equation (1.1) as a *gradient flow* when we observe that the damper relation $\mu v = \xi$ can also be written in terms of a dissipation potential $\mathcal{R}_1(v) := \mu v^2/2$ and its Legendre dual $\mathcal{R}_1^*(\xi) := \xi^2/(2\mu)$. The dissipation potential $\mathcal{R}_1$ defines the relation $\mu v = \xi$ (which we call a *kinetic relation*, which we further discuss below) through any of the three equivalent conditions

$$\mathrm{D}\mathcal{R}_1(v) = \xi, \qquad v = \mathrm{D}\mathcal{R}_1^*(\xi), \qquad \text{or} \qquad \mathcal{R}_1(v) + \mathcal{R}_1^*(\xi) = v\xi. \tag{1.2}$$

The *gradient system* $(\mathbb{R}, \mathcal{E}_1, \mathcal{R}_1)$ characterizes the equation (1.1) through the combination of the kinetic relation (1.2) with the identification of the force $\xi = -\mathrm{D}\mathcal{E}_1(q)$.

In this example, one readily recognizes a 'classical' spring energy in $\mathcal{E}_1(q) = kq^2/2$, and the quadratic form of $\mathcal{R}_1(v) = \mu v^2/2$ is a natural choice for a damper (see e.g. [Pel14, Ch. 5]). However, other gradient-flow formulations for the same evolution equation (1.1) exist, if $\mathcal{R} = \mathcal{R}(q, v)$ may depend not only on velocity v but also on the state q:

$$\begin{aligned} \mathcal{E}_2 &:= \mathcal{E}_1 & \qquad \mathcal{E}_3 &:= \mathcal{E}_1 \\ \mathcal{R}_2(q, v) &:= \frac{\mu^3}{4k^2q^2} v^4 & \qquad \mathcal{R}_3(q, v) &:= \frac{kq}{1 - \mathrm{e}^{-kq/\mu}} \big(\mathrm{e}^v - v - 1\big) \end{aligned}$$

Each of the systems $(\mathbb{R}, \mathcal{E}_i, \mathcal{R}_i)$ generates the same equation (1.1) as $\mathrm{D}_v\mathcal{R}_i(q, \dot{q}) = -\mathrm{D}\mathcal{E}_i(q)$.

One can in fact generate a wide variety of gradient systems for the same equation (1.1): take any smooth and convex $\psi : \mathbb{R} \to \mathbb{R}$ with $\min \psi = \psi(0) = 0$, define $\varphi(q) = -kq/\psi'(-kq/\mu)$ and $\mathcal{R}_\psi(q, v) := \varphi(q)\psi(v)$, and then the gradient system $(\mathbb{R}, \mathcal{E}_1, \mathcal{R}_\psi)$ will generate equation (1.1). The two examples $\mathcal{R}_2$ and $\mathcal{R}_3$ above are both of this type.

These dissipation potentials might well be considered less 'natural' than $\mathcal{R}_1$. To start with, it is not obvious which modelling arguments would lead to the kinetic relations of $\mathcal{R}_2$ and $\mathcal{R}_3$, which are

$$v^3 = \frac{k^2q^2}{\mu^3}\,\xi \quad (\text{for } \mathcal{R}_2), \qquad \text{and} \qquad \mathrm{e}^v - 1 = \frac{1 - \mathrm{e}^{-kq/\mu}}{kq}\,\xi \quad (\text{for } \mathcal{R}_3).$$

In addition, a definition like that of $\mathcal{R}_3$ is dimensionally inconsistent, since arguments of the exponential function should be dimensionless. Both these problems are related to a deeper and more troubling problem: The dissipation potentials depend not only on μ but also on k, implying that the kinetic relation generated by $\mathcal{R}_2$ or $\mathcal{R}_3$, which is supposed to characterize the damper, depends on the strength k of the spring. This is an unsatisfactory situation: we consider the spring and the damper to be two independent objects, and their mathematical characterizations should therefore also be independent.

This example points towards the problem that we aim to solve in this paper. This problem arises especially when taking limits of gradient systems in some parameter $\varepsilon \to 0$;

in such limits it is unavoidable that the limiting dissipation potential depends on the state q as well as the rate of change v. As a result, the limiting evolution equation will have many gradient-flow structures, as in the example above. It turns out that one of the most common concepts used to define limits of gradient systems, which we call 'simple EDP-convergence' in this paper and which we explain below, often selects limit dissipation potentials that are 'unhealthy' in the same way as $\mathcal{R}_2$ and $\mathcal{R}_3$ are 'unhealthy': they depend on aspects of the energy in an unsatisfactory way.

The aim of this paper is to construct alternative convergence concepts that lead to limiting gradient systems that are more 'natural' or 'healthy'. What we mean by these terms will become clear below, but first we consider an example to further illustrate the problem.

1.2 Second example: wiggly dissipation

In Section 3 we study the following example in detail. Consider a sequence of gradient systems $(\mathbb{R}, \mathcal{E}, \mathcal{R}_\varepsilon)$, indexed by $\varepsilon > 0$, where $\mathcal{E}$ is some smooth ε-independent function, and

$$\mathcal{R}_\varepsilon(q, v) := \frac{1}{2}\mu\Big(\frac{q}{\varepsilon}\Big)\, v^2.$$

Here $\mu \in C(\mathbb{R})$ is positive and 1-periodic. For this 'wiggly dissipation' system the gradient-flow equation takes the form

$$\mu\Big(\frac{q}{\varepsilon}\Big)\dot q = -\mathcal{E}'(q). \tag{1.3}$$

An example of a solution is given in Figure 1.2.

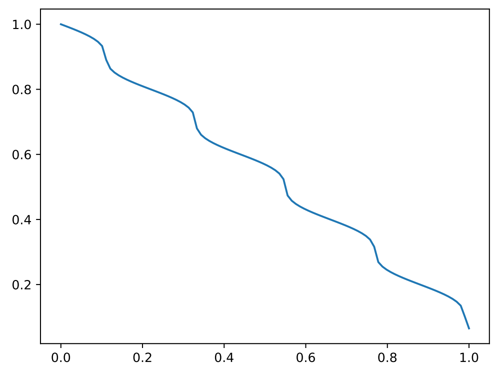

Figure 1.2: An example simulation of the gradient system of this section, with $\mathcal{E}(q) = q$, $\mu(y) = 1.1+\cos 2\pi y$, and $\varepsilon = 1/5$. For these choices equation (1.3) becomes $\dot q = -1/(1.1+\cos 2\pi y)$; note how the solution decreases sharply when μ is close to zero.

We show in Section 3 that as $\varepsilon \to 0$, the solutions q_ε of (1.3) converge to limit functions q that solve the limiting equation

$$\overline{\mu}\dot q = -\mathcal{E}'(q) \qquad \text{with} \quad \overline{\mu} = \int_0^1 \mu(y)\,\mathrm{d}y. \tag{1.4}$$

In fact, $(\mathbb{R}, \mathcal{E}, \mathcal{R}_\varepsilon)$ converges in the *simple EDP-sense* (defined in Section 2.3) to a limiting system $(\mathbb{R}, \mathcal{E}, \mathcal{R}_0)$, where

$$\mathcal{R}_0(q, v) := \mathcal{M}_0(v, -\mathcal{E}'(q)) - \mathcal{M}_0(0, -\mathcal{E}'(q)), \tag{1.5a}$$

and

$$\mathcal{M}_0(v,\xi) = \inf\Big\{ \int_{s=0}^1 \Big(\frac{\mu(q,z(s))\big(vz'(s)\big)^2}{2} + \frac{\xi^2}{2\mu(q,z(s))}\Big)\,\mathrm{d}s \;\Big|\; z\colon [0,1]\to\mathbb{R},\ z(1) = z(0) + \operatorname{sign}(v)\Big\}, \tag{1.5b}$$

We verify explicitly in Section 3 that the system $(\mathbb{R},\mathcal{E},\mathcal{R}_0)$ indeed generates equation (1.4), i.e. that

$$\overline{\mu}\dot{q} = -\mathcal{E}'(q) \qquad \Longleftrightarrow \qquad \mathrm{D}_v\mathcal{R}_0(q,\dot{q}) = -\mathcal{E}'(q).$$

This limiting dissipation potential $\mathcal{R}_0$ suffers from the same problem as $\mathcal{R}_2$ and $\mathcal{R}_3$ above: it depends explicitly on the energy function $\mathcal{E}$. If we repeat the simple EDP-convergence theorem for a perturbed energy $\mathcal{E}+\mathcal{F}$, for instance, then the perturbation $\mathcal{F}$ propagates into the formula (1.5a) for $\mathcal{R}_0$; changing the energy thus leads to a different dissipation potential $\mathcal{R}_0$. As above, we consider this unsatisfactory, since the energy driving the system is conceptually separate from the mechanism for dissipating that energy.

In contrast, if we disregard the fact that equation (1.4) arises as a limit, and consider it as an isolated system, then we might conjecture a gradient structure with dissipation potential $\widehat{\mathcal{R}}(v) := \overline{\mu}v^2/2$ instead. Indeed, combined with the energy $\mathcal{E}$ this potential $\widehat{\mathcal{R}}$ also generates equation (1.4); it is much simpler to interpret than $\mathcal{R}_0$, and most importantly, it does not depend on $\mathcal{E}$.

1.3 Towards a better convergence concept

These examples show that we have on one hand an unsatisfactory convergence result, in which $(\mathbb{R},\mathcal{E},\mathcal{R}_0)$ is proved to arise as the unique limit of the sequence $(\mathbb{R},\mathcal{E},\mathcal{R}_\varepsilon)$ in the simple EDP sense, but this limit is unsatisfactory as a description of a gradient system.

On the other hand, the alternative dissipation potential $\widehat{\mathcal{R}}$ generates the same limit equation and does not suffer from the philosophical problems that $\mathcal{R}_0$ does; its only drawback is that the system $(\mathbb{R},\mathcal{E},\widehat{\mathcal{R}})$ is not the limit of the sequence $(\mathbb{R},\mathcal{E},\mathcal{R}_\varepsilon)$ in the simple EDP-sense.

As mentioned above, these observations strongly suggest to seek alternative convergence concepts for gradient systems, which should generate limiting potentials that do not depend on the limiting energy. Specfically, we will seek convergence concepts—let us indicate them with '$\square$'—that have the following property: if

$$(\mathbf{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon) \xrightarrow{\square} (\mathbf{Q},\mathcal{E}_0,\mathcal{R}_0), \tag{1.6}$$

then for all $\widetilde{\mathcal{F}} \in C^1(\mathbf{Q})$ we have

$$(\mathbf{Q},\mathcal{E}_\varepsilon+\widetilde{\mathcal{F}},\mathcal{R}_\varepsilon) \xrightarrow{\square} (\mathbf{Q},\mathcal{E}_0+\widetilde{\mathcal{F}},\mathcal{R}_0), \tag{1.7}$$

where the dissipation potential $\mathcal{R}_0$ in (1.7) is the same as in (1.6), and therefore does not depend on $\widetilde{\mathcal{F}}$.

Indeed, the two new concepts that we introduce in Section 2.6 both have this property, and we show in Section 3 that by applying one of these we find not $\mathcal{R}_0$ but $\widehat{\mathcal{R}}$ as the limit dissipation potential.

1.4 The larger picture: effective kinetic relations

Our aim of deriving 'healthy' limiting gradient systems could also be formulated as the challenge of deriving *effective kinetic relations*. We already introduced a *kinetic relation* as a relation between a *force* ξ and a *velocity* v; such kinetic relations appear naturally in gradient systems, since the dissipation potential $\mathcal{R}$ in a gradient system exactly characterizes such a relation via any of the three equivalent relations

$$\xi = \mathrm{D}_v\mathcal{R}(q,v), \quad v = \mathrm{D}_\xi\mathcal{R}^*(q,\xi), \quad \text{or} \quad \mathcal{R}(q,v) + \mathcal{R}^*(q,\xi) = \langle v,\xi\rangle.$$

We define the *contact set* as the corresponding set of pairs (v,ξ):

$$\begin{aligned}\mathcal{C} = \mathcal{C}_{\mathcal{R}\oplus\mathcal{R}^*}(q) &:= \Big\{ (v,\xi) \in \mathbf{Q}\times\mathbf{Q}^* \ \Big|\ \mathcal{R}(q,v) + \mathcal{R}^*(q,\xi) = \langle v,\xi\rangle \Big\}\\ &= \operatorname{graph} \mathrm{D}_v\mathcal{R}(q,\cdot).\end{aligned}$$

This set $\mathcal{C}$ characterizes the pairs of velocities v and forces ξ that are admissible to the system. As was already mentioned, the equation generated by the gradient system can be viewed as the result of applying the kinetic relation $(v,\xi) \in \mathcal{C}_{\mathcal{R}\oplus\mathcal{R}^*}(q)$ to a context where the force ξ is generated by the potential $\mathcal{E}$:

$$\xi = -\mathrm{D}\mathcal{E}(q) \qquad \text{and} \qquad (\dot q,\xi) \in \mathcal{C}_{\mathcal{R}\oplus\mathcal{R}^*}(q). \tag{1.8}$$

Kinetic relations appear throughout physics and mechanics. Well-known examples are Stokes' law $\xi = 6\pi\eta R\, v$ for the drag force ξ on a sphere dragged through a viscous fluid (where η is the dynamic viscosity and R the radius of the sphere), power-law viscous relationships of the form $\xi = c|v/v_0|^{p-1}v/v_0$, and Coulomb friction $\xi \in c\operatorname{sign} v$. These examples show that the relationship may be linear or nonlinear, and single- or multi-valued. A priori, there is no reason why a kinetic relation should be the graph of the derivative of a dissipation potential, but here we are interested in the ones that do have that property.

We now turn to the challenge of deriving *effective* kinetic relations. We are given a sequence of kinetic relations parametrized by ε. The interpretation of ε as a small parameter, or a small scale, often implies that there are natural 'macroscopic,' 'averaged,' or 'effective' forces and velocities, which reflect the behaviour of the true forces and velocities in the system at scales that are large with respect to ε, while smoothing out the behaviour at smaller scales. To derive an effective kinetic relation is to derive a new relation between the limits of such macroscopic forces and velocities as $\varepsilon \to 0$, leading to a characterization of the kinetic relation for 'the limiting system'.

Again, these effective kinetic relations are very common; for instance, Stokes' law, Fourier's law, Fick's law, and many similar laws actually are effective kinetic relations, derived from more microscopic systems, often consisting of particles. Throughout science, such effective kinetic relations are the starting point for the modelling of dissipative systems at this effective scale [Ött05, Ber07, Mie11, Pel14]. A detailed understanding of the properties and assumptions that lie at the basis of such effective kinetic relations therefore is essential.

We now turn to the issue of 'healthy' and 'unhealthy' kinetic relations. The limiting dissipation potential $\mathcal{R}_0$ in the second example above depends on the energy $\mathcal{E}$, i.e. $\mathcal{R}_0(q,\xi) = \mathcal{R}_{0,\mathcal{E}}(q,\xi)$. It follows that the contact set $\mathcal{C}_{\mathcal{R}_{0,\mathcal{E}}\oplus\mathcal{R}^*_{0,\mathcal{E}}}$ also is $\mathcal{E}$-dependent; the gradient-flow equation (1.8) then takes the self-referential form

$$(\dot q, -\mathrm{D}\mathcal{E}(q)) \in \mathcal{C}_{\mathcal{R}_{0,\mathcal{E}}\oplus\mathcal{R}^*_{0,\mathcal{E}}}.$$

The set $\mathcal{C}_{\mathcal{R}_{0,\mathcal{E}}\oplus\mathcal{R}^*_{0,\mathcal{E}}}$ does not make any sense as a general kinetic relation, however, because $\mathcal{C}_{\mathcal{R}_{0,\mathcal{E}}\oplus\mathcal{R}^*_{0,\mathcal{E}}}$ does not provide us any information about admissible pairs (v,ξ) *other* than the case $\xi = -\mathrm{D}\mathcal{E}(q)$. In order to find the velocity for a force $\tilde\xi \neq -\mathrm{D}\mathcal{E}(q)$, we would need to construct a different energy $\tilde{\mathcal{E}}(q)$ such that $\tilde\xi = -\mathrm{D}\tilde{\mathcal{E}}(q)$, repeat the convergence process for this energy $\tilde{\mathcal{E}}$, obtain a different limiting dissipation potential $\mathcal{R}_{0,\tilde{\mathcal{E}}}$, and read off the admissible velocities from the resulting contact set $\mathcal{C}_{\mathcal{R}_{0,\tilde{\mathcal{E}}}\oplus\mathcal{R}^*_{0,\tilde{\mathcal{E}}}}$. Since this latter set is generically different from $\mathcal{C}_{\mathcal{R}_{0,\mathcal{E}}\oplus\mathcal{R}^*_{0,\mathcal{E}}}$, this shows how a single contact set $\mathcal{C}_{\mathcal{R}_{0,\mathcal{E}}\oplus\mathcal{R}^*_{0,\mathcal{E}}}$ can not be considered a kinetic relation.

Instead, we seek a limiting kinetic relation that is defined as *one single* set $\mathcal{C}$ of pairs (v,ξ) that provides us with all admissible combinations. The convergence concepts that we construct below are constructed with this aim in mind.

1.5 Third example: wiggly energy

In the example of Section 1.2 the 'correct' limiting dissipation potential $\widehat{\mathcal{R}}(v) = \overline{\mu}v^2/2$ only contained information from $\mathcal{R}_\varepsilon$. When considering a sequence of Γ-converging energies $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$, however, the 'correct' limiting dissipation potential may also contain information from $\mathcal{E}_\varepsilon$. This may seem to contradict our claim above that dependence of the limiting dissipation on the energy is 'unhealthy'; as we shall see below, however, 'correct' or 'healthy' will mean that the limiting dissipation potential can depend on $\mathcal{E}_\varepsilon$, but not on $\mathcal{E}_0$.

To illustrate this we revisit the classical example of a gradient flow in a 'wiggly' energy landscape [Pra28, Jam96, ACJ96, DFM17]. Again we take as state space $\mathbf{Q} = \mathbb{R}$, but now the energy $\mathcal{E}_\varepsilon$ is ε-dependent while the dissipation potential $\mathcal{R}_\varepsilon = \mathcal{R}$ does not depend on ε:

$$\mathcal{E}_\varepsilon(q) := \mathcal{E}_0(q) + \varepsilon\kappa\big(q, \tfrac{1}{\varepsilon}q\big), \qquad \mathcal{R}(v) := \frac{\mu}{2}v^2, \tag{1.9}$$

where $\mathcal{E}_0 : \mathbb{R} \to \mathbb{R}$ is smooth, $\mu > 0$ is constant, and $\kappa : \mathbb{R}^2 \to \mathbb{R}$ is smooth and 1-periodic in its second argument, i.e. $\kappa(q, y+1) = \kappa(q,y)$ for all q and y. The induced evolution equation is

$$\mu\dot q = -\mathrm{D}\mathcal{E}_0(q) - \varepsilon\mathrm{D}_q\kappa\big(q,\tfrac{1}{\varepsilon}q\big) - \mathrm{D}_y\kappa\big(q,\tfrac{1}{\varepsilon}q\big).$$

This system was studied in detail in [DFM17]. In Section 4 we describe this, and place the system in the context of this paper. We find that the system $(\mathbb{R},\mathcal{E}_\varepsilon,\mathcal{R})$ converges in the *simple EDP-sense* to a limiting system $(\mathbb{R},\mathcal{E}_0,\mathcal{R}_0)$, where $\mathcal{E}_0$ is the ε-independent part of $\mathcal{E}_\varepsilon$ as in (1.9), and $\mathcal{R}_0$ is given by

$$\mathcal{R}_0(q,v) = \mathcal{M}_0(q,v,-\mathcal{E}_0'(q)) - \mathcal{M}_0(q,0,-\mathcal{E}_0'(q)),$$

where this time $\mathcal{M}_0(q,v,\xi)$ is given by

$$\mathcal{M}_0(q,v,\xi) = \inf\Big\{\frac12\int_0^1 \big[v^2\dot z^2(s) + (\xi - D_y\kappa(q,z(s)))^2\big]\,\mathrm{d}s \;\Big|\; z\colon [0,1]\to\mathbb{R},\ z(1) = z(0) + \mathrm{sign}(v)\Big\}. \tag{1.10}$$

We show in Section 4 that, as in the previous example, $\mathcal{R}_0$ also depends on $\mathcal{E}_0'(q)$.

In Section 4 we also show that in one of the two new convergence concepts, *contact tilt-EDP-convergence*, the system $(\mathbb{R},\mathcal{E}_\varepsilon,\mathcal{R})$ converges to a limiting system $(\mathbb{R},\mathcal{E}_0,\widetilde{\mathcal{R}})$. For the limiting dissipation potential $\widetilde{\mathcal{R}}$ we only have a , but we do prove (Lemma **??**) that $\widetilde{\mathcal{R}}$ does not depend on $\mathcal{E}_0$.

1.6 Tilt- and contact tilt-EDP-convergence

The reason why gradient-flow convergence does not necessarily lead to a 'healthy' kinetic relation is *relaxation:* for a given macroscopic velocity v and force ξ, the limiting dissipation potential is found by a minimization over microscopic degrees of freedom constrained to the macroscopic imposed velocity. This can be recognized in the definitions of $\mathcal{M}_0$ in (1.5b) and (1.10), and is very similar to the *cell problems* that arise in homogenization [Hor97, CiD99, Bra02]. In the cases of this paper, the solutions of these cell problems may not be of gradient-flow type, leading to a situation where the limit problem does not describe a gradient-flow structure. We analyse this in more detail in Section 5.

To correct this, we introduce two novel aspects. The first is to consider not a single sequence $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ of gradient systems, but a full class of perturbed versions of this sequence. Given $\mathcal{F} \in C^1(\mathbf{Q})$, we perturb each energy $\mathcal{E}_\varepsilon$ with $\mathcal{F}$ (thus 'tilting it'):

$$\mathcal{E}_\varepsilon^{\mathcal{F}} := \mathcal{E}_\varepsilon + \mathcal{F}.$$

We then will require convergence of all tilted systems simultaneously. Through the freedom of choosing the tilt $\mathcal{F}$, this allows us more freedom to probe the whole space of velocities v and forces ξ for each q.

This setup leads to a first new convergence concept, that we call *EDP-convergence with tilting*, or tilt-EDP-convergence. Unfortunately, it suffers from the same problems of relaxation, and therefore it only is useful as an auxiliary concept.

The second new aspect is to weaken the definition of tilt-EDP-convergence to require only a reduced connection between the relaxed problem and the limiting dissipation potential—a connection that only holds 'at the contact set $\mathcal{C}$'. This leads to a concept *contact EDP-convergence with tilting.* We show in the examples later in this paper that this contact EDP-convergence concept for gradient systems yields kinetic relations that do not suffer from the force dependence that we observed above.

1.7 Setup of the paper

In Section 2 we define gradient systems and gradient flows, recall the existing concept of simple EDP-convergence, and introduce the two novel convergence concepts *tilt-EDP-convergence* and *contact tilt-EDP-convergence*. In Section 3 we study in detail the example of an oscillating dissipation potential that we briefly mentioned above. In Section 5 we discuss in depth the reasons why the concept of contact tilt-EDP-convergence is an improvement over the classical concept of EDP-convergence, and why it corrects the 'incorrect' kinetic relationship that we mentioned above. In Section **??** we describe a second example in detail, that of diffusion through a membrane in the limit of vanishing thickness. In Section 6 we connect the tilting of energies as described above with tilting of random variables in large-deviation principles, and show how the independence of the dissipation potential of the force arises naturally in that context. In Section 7 we give some concluding remarks.

2 Gradient systems and convergence

From now on we aim for rigour.

2.1 Basic definitions

The context for this paper is a smooth finite-dimensional Riemannian manifold $\mathbf{Q}$, which may be compact or not. A common choice is $\mathbf{Q} = \mathbb{R}^n$. We write $|\cdot|$ for the local norms on the tangent and cotangent spaces $T\mathbf{Q}$ and $T^*\mathbf{Q}$, and $T\mathbf{Q} \otimes T^*\mathbf{Q}$ for the combined tangent-cotangent bundle

$$T\mathbf{Q} \otimes T^*\mathbf{Q} := \Big\{(q, v, \xi) \;\Big|\; q \in \mathbf{Q},\ v \in T_q\mathbf{Q},\ \xi \in T_q^*\mathbf{Q}\Big\}.$$

Definition 2.1 (Gradient systems and dissipation potentials)**.** In this paper a *gradient system* is a triple $(\mathbf{Q}, \mathcal{E}, \mathcal{R})$:

- $\mathbf{Q}$ is a smooth finite-dimensional Riemannian manifold.
- $\mathcal{E} : \mathbf{Q} \to \mathbb{R}$ is a continuously differentiable functional, often called the 'energy'.
- $\mathcal{R} : T\mathbf{Q} \to \mathbb{R}$ is a *dissipation potential*, which means that for each $q \in \mathbf{Q}$,
  - $\mathcal{R}(q, \cdot) : T_q\mathbf{Q} \to [0, \infty]$ is convex, proper, and lower semicontinuous,
  - $\mathcal{R}(q, 0) = \min_{v \in T_q\mathbf{Q}} \mathcal{R}(q, v) = 0$.

The dissipation potential has a natural Legendre-Fenchel dual $\mathcal{R}^* : T^*\mathbf{Q} \to \mathbb{R}$,

$$\mathcal{R}^*(q, \xi) := \sup_{v \in T_q\mathbf{Q}} \langle \xi, v\rangle - \mathcal{R}(q, v). \tag{2.1}$$

By our assumptions on $\mathcal{R}$, the dual potential $\mathcal{R}^*$ also is convex, proper, lower semicontinuous, non-negative, and zero at zero. We denote the subdifferentials of $\mathcal{R}$ and $\mathcal{R}^*$ with respect to their second arguments as $\partial_v\mathcal{R}$ and $\partial_\xi\mathcal{R}^*$.

The following lemma gives a well-known connection between growth and subdifferentials:

Lemma 2.2. *Let $\mathcal{R} : T\mathbf{Q} \to [0, \infty]$ be a dissipation potential with dual dissipation potential $\mathcal{R}^*$. For each $q \in \mathbf{Q}$, the following are equivalent:*

1. *The map $v \mapsto \mathcal{R}(q, v)$ is superlinear, i.e. $\lim_{|v|\to\infty} |v|^{-1}\mathcal{R}(q, v) = +\infty$;*
2. *For each $\xi \in T_q^*\mathbf{Q}$, the subdifferential $\partial_\xi\mathcal{R}^*(q, \xi)$ is non-empty.*

Proof. To show the forward implication, note that the superlinearity implies that for every ξ the supremum in (2.1) is achieved, and therefore the subdifferential is not empty. For the opposite implication, note that for all ξ, $\mathcal{R}^*(q, \xi)$ is finite, and therefore the right-hand side in the inequality $\mathcal{R}(q, v) \geq \langle v, \xi\rangle - \mathcal{R}^*(q, \xi)$ grows linearly at infinity with rate ξ. By arguing by contradiction one finds that $\mathcal{R}(q, \cdot)$ is superlinear. □

Remark 2.3. The finite-dimensionality and smoothness assumptions that we make are of course stronger than necessary for the definition of gradient systems [AGS08]. We make these assumptions nonetheless to prevent technical issues from distracting from the structure of the development. We expect, however, that many of these assumptions can be relaxed while preserving the philosophy of the paper. □

2.2 The equation defined by a gradient system

The equation induced by the gradient system is, in three equivalent forms,

$$\dot q \in \partial_\xi \mathcal{R}^*(q, -\mathrm{D}\mathcal{E}(q)), \tag{2.2a}$$

$$\partial_v \mathcal{R}(q, \dot q) \ni -\mathrm{D}\mathcal{E}(q), \tag{2.2b}$$

$$\mathcal{R}(q, \dot q) + \mathcal{R}^*(q, -\mathrm{D}\mathcal{E}(q)) = \langle \dot q, -\mathrm{D}\mathcal{E}(q)\rangle. \tag{2.2c}$$

The final line can be used to generate an additional formulation. For curves $q \in AC([0,T], \mathbf{Q})$, define the *dissipation functional* as

$$\mathfrak{D}^T(q) := \int_0^T \big(\mathcal{R}(q, \dot q) + \mathcal{R}^*(q, -\mathrm{D}\mathcal{E}(q)\big)\,\mathrm{d}t. \tag{2.3}$$

By integrating the Legendre-Fenchel inequality $\mathcal{R}(q, \dot q) + \mathcal{R}^*(q, -\mathrm{D}\mathcal{E}(q)) \geq \langle \dot q, -\mathrm{D}\mathcal{E}(q)\rangle$ we find

Lemma 2.4 (Chain rule)**.** *Under the assumptions of this section,*

$$\mathcal{E}(q(T)) + \mathfrak{D}^T(q) \geq \mathcal{E}(q(0)) \qquad \textit{for any } q \in AC([0,T], \mathbf{Q}). \tag{2.4}$$

On the other hand, by integrating (2.2c) in time we find that solutions q of (2.2) achieve equality in (2.4). This leads to a further characterization of solutions:

Lemma 2.5 (Energy-Dissipation Principle)**.** *Let $q \in AC([0,T]; \mathbf{Q})$. The following are equivalent:*

1. *For almost all $t \in [0,T]$, q satisfies any of the three characterizations* (2.2)*;*
2. *The curve q satisfies*
$$\mathcal{E}(q(T)) + \mathfrak{D}^T(q) \leq \mathcal{E}(q(0)). \tag{2.5}$$

Remark 2.6. The assumption that $v \mapsto \mathcal{R}(q,v)$ is minimized at $v = 0$ can be interpreted as an expression of the 'nature' of a gradient flow: 'not moving requires no dissipation of energy', or, when $v = 0$ is the unique minimizer, 'moving requires dissipation'. Both cases can be recognized in equation (2.2):

- Since $0 \in \partial_v \mathcal{R}(q, 0)$ implies $0 \in \partial_\xi \mathcal{R}^*(q, 0)$, formulation (2.2a) implies that $\dot q = 0$ is possible when $\mathrm{D}\mathcal{E}(q) = 0$;
- If $v = 0$ is the unique minimizer of $\mathcal{R}(q, \cdot)$, then (2.2b) and $\mathrm{D}\mathcal{E}(q) = 0$ together force $\dot q = 0$.

As mentioned in the Introduction, a gradient system $(\mathbf{Q}, \mathcal{E}, \mathcal{R})$ can be considered to define a kinetic relation, at each $q \in \mathbf{Q}$, through the *contact set*

$$\mathcal{C}_{\mathcal{R}\oplus\mathcal{R}^*}(q) := \Big\{(v, \xi) \in T_q\mathbf{Q} \times T_q\mathbf{Q}^* : \mathcal{R}(q, v) + \mathcal{R}^*(q, \xi) = \langle v, \xi\rangle\Big\}.$$

The same 'nature' of a gradient flow can be recognized as the property that the kinetic relation is *dissipative*, i.e. that $\langle v, \xi\rangle \geq 0$ for all $(v, \xi) \in \mathcal{C}_{\mathcal{R}\oplus\mathcal{R}^*}(q)$; this follows immediately from the property that both $\mathcal{R}$ and $\mathcal{R}^*$ are non-negative, which itself is a consequence of the minimality of $v = 0$. □

2.3 Simple EDP-convergence

The Energy-Dissipation Principle formulation (2.5) of a gradient flow leads to a natural concept of gradient-system convergence. A first version of this concept was formulated by Sandier and Serfaty [SaS04] and generalizations have been used in a large number of proofs (see e.g. [Ser11, Mie12, AM*12, MPR14, Mie16a, Mie16b, LM*17a]).

Definition 2.7 (Simple EDP-convergence)**.** A sequence of gradient systems $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ converges *in the simple EDP-sense* to a gradient system $(\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$, written $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{EDP}} (\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$, if

1. $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ in $\mathbf{Q}$;
2. For each $T > 0$ the functional $\mathfrak{D}_\varepsilon^T$ Γ-converges in $C_b([0,T];\mathbf{Q})$ to the limit functional

$$\mathfrak{D}_0^T(q_0) := \int_0^T \big[\mathcal{R}_0(q_0, \dot q_0) + \mathcal{R}_0^*(q_0, -\mathrm{D}\mathcal{E}_0(q_0))\big]\,\mathrm{d}t. \tag{2.6}$$

The two parts of Definition 2.7 naturally combine to enable passing to the limit in the integrated formulation (2.5), as illustrated by the proof of this lemma:

Lemma 2.8 (Simple EDP-convergence implies that solutions converge to solutions)**.** *Assume that* $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{EDP}} (\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$. *Let* $q_\varepsilon \in AC([0,T],\mathbf{Q})$ *be solutions of* $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, *and assume the convergences*

$$q_\varepsilon \to q_0 \text{ in } C_b([0,T],\mathbf{Q}) \qquad \text{and} \qquad \mathcal{E}_\varepsilon(q_\varepsilon(0)) \to \mathcal{E}_0(q_0(0)).$$

Then q_0 *is a solution of* $(\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$.

Proof. From parts 1 and 2 of Definition 2.7 we find that

$$\mathcal{E}_0(q_0(T)) + \mathfrak{D}_0^T(q_0) - \mathcal{E}_0(q_0(0)) \le \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(q_\varepsilon(T)) + \mathfrak{D}_\varepsilon^T(q_\varepsilon) - \mathcal{E}_\varepsilon(q_\varepsilon(0)) = 0.$$

By Lemma 2.5 it follows that the limit q_0 is a solution of $(\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$. □

2.4 Tilting the gradient systems

As we explained in the Introduction, simple EDP-convergence may lead to 'unhealthy' limiting dissipation potentials, which violate the requirement (1.6)–(1.7). As central step towards improving the situation we embed the single sequence $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ in a family of sequences $(\mathbf{Q}, \mathcal{E}_\varepsilon + \mathcal{F}, \mathcal{R}_\varepsilon)$, parameterized by functionals $\mathcal{F} \in C^1(\mathbf{Q};\mathbb{R})$, thus 'tilting' the functionals $\mathcal{E}_\varepsilon$. Tilting $\mathcal{E}_\varepsilon$ does not change the Γ-convergence properties: we have

$$\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0 \quad \Longleftrightarrow \quad \mathcal{E}_\varepsilon + \mathcal{F} \xrightarrow{\Gamma} \mathcal{E}_0 + \mathcal{F} \quad \text{for all } \mathcal{F} \in C^1(\mathbf{Q};\mathbb{R}).$$

However, for the dissipation functional $\mathfrak{D}_\varepsilon^T$ we obtain new and nontrivial information by considering the dissipation functional for the tilted energy:

$$\mathfrak{D}_\varepsilon^T(q,\mathcal{F}) := \int_0^T \mathcal{M}_\varepsilon(q, \dot q, -\mathrm{D}\mathcal{E}_\varepsilon(q) - \mathrm{D}\mathcal{F}(q))\,\mathrm{d}t \quad \text{with} \quad \mathcal{M}_\varepsilon(q,v,\xi) = \mathcal{R}_\varepsilon(q,v) + \mathcal{R}_\varepsilon^*(q,\xi).$$

We now assume that the Γ-limits of $\mathfrak{D}_\varepsilon(\cdot,\mathcal{F})$ exist, i.e.

$$\mathfrak{D}_\varepsilon^T(\cdot,\mathcal{F}) \xrightarrow{\Gamma} \mathfrak{D}_0^T(\cdot,\mathcal{F}) : q \mapsto \int_0^T \mathcal{N}_0(q,\dot q,-\mathrm{D}\mathcal{F}(q))\,\mathrm{d}t \qquad \text{for all } \mathcal{F}\in C^1(\mathbf{Q};\mathbb{R}). \tag{2.7}$$

To recover the original structure of integrals $\mathfrak{D}_\varepsilon^T$ in terms of $\mathcal{M}_\varepsilon$, we define

$$\mathcal{M}_0(q,v,\xi) := \mathcal{N}_0\big(q,v,\xi{+}\mathrm{D}\mathcal{E}_0(q)\big),$$

such that $\mathfrak{D}_0^T$ has the desired form

$$\mathfrak{D}_0^T(q,\mathcal{F}) = \int_0^T \mathcal{M}_0\big(q,\dot q,-\mathrm{D}\mathcal{E}_0(q){-}\mathrm{D}\mathcal{F}(q)\big)\,\mathrm{d}t.$$

We capture this discussion in a definition that provides the basis for the later convergence concepts.

Assumption 2.9 (Basic assumptions). *Assume that the sequence $(\mathbf{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ satisfies*

1. $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0$ *in* $\mathbf{Q}$;
2. *For each $T>0$ here exists a functional $\mathfrak{D}_0^T : AC([0,T];\mathbf{Q})\times C^1(\mathbf{Q};\mathbb{R}) \to [0,\infty]$ such that for each $\mathcal{F}\in C^1(\mathbf{Q};\mathbb{R})$, the sequence $\mathfrak{D}_\varepsilon^T(\cdot,\mathcal{F})$ Γ-converges to $\mathfrak{D}_0^T(\cdot,\mathcal{F})$ in the topology of $C_b([0,T];\mathbf{Q})$.*
3. *There exists a function $\mathcal{N}_0 : T\mathbf{Q}\otimes T^*\mathbf{Q}\to[0,\infty]$, independent of T, such that*

$$\forall \mathcal{F}\in C^1(\mathbf{Q};\mathbb{R}) : \qquad \mathfrak{D}_0^T(q,\mathcal{F}) = \int_0^T \mathcal{N}_0(q(t),\dot q(t),-\mathrm{D}\mathcal{F}(q))\,\mathrm{d}t.$$

For each $q\in\mathbf{Q}$ and $\eta\in T_q^*\mathbf{Q}$, $v\mapsto\mathcal{N}_0(q,v,\eta)$ is convex and lower semicontinuous;

Define $\mathcal{M}_0 : T\mathbf{Q}\otimes T^*\mathbf{Q}\to\mathbb{R}$ by

$$\mathcal{M}_0(q,v,\xi) := \mathcal{N}_0(q,v,\xi{+}\mathrm{D}\mathcal{E}_0(q)). \tag{2.8}$$

4. $\mathcal{M}_0(q,v,\xi)\ge\langle v,\xi\rangle$ *for all* $(q,v,\xi)\in T\mathbf{Q}\otimes T^*\mathbf{Q}$.
5. $\mathcal{M}_0(q,v,\xi)\ge\mathcal{M}_0(q,0,\xi)$ *for all* $(q,v,\xi)\in T\mathbf{Q}\otimes T_q^*\mathbf{Q}$.

We briefly comment on these. Assumptions 1–3 make the prior discussion precise. Note that $\mathcal{N}_0$ is assumed to be independent of the time horizon T. This is a common feature of convergence results of this type; see e.g. [Bra02, Ch. 3], or the examples later in this paper, and note that this independence also is implicitly present in condition (2.6) for simple EDP-convergence.

Assumption 4 implies that $\mathcal{E}_0$ and $\mathfrak{D}_0^T$ satisfy a chain rule similar to Lemma 2.4:

$$\mathcal{E}_0(q(T)) + \mathfrak{D}_0^T(q) \ge \mathcal{E}_0(q(0)) \qquad \text{for any } q\in AC([0,T],\mathbf{Q}).$$

Assumption 5 is satisfied at positive ε, since by the conditions on dissipation potentials we have $\mathcal{R}_\varepsilon(q,v)\ge\mathcal{R}_\varepsilon(q,0)$ for all q and v, so that

$$\mathcal{M}_\varepsilon(q,v,\xi) = \mathcal{R}_\varepsilon(q,v)+\mathcal{R}_\varepsilon^*(q,\xi{-}\mathrm{D}\mathcal{E}_\varepsilon(q)) \ge \mathcal{R}_\varepsilon(q,0)+\mathcal{R}_\varepsilon^*(q,\xi{-}\mathrm{D}\mathcal{E}_\varepsilon(q)) = \mathcal{M}_\varepsilon(q,0,\xi).$$

Since the property $\mathcal{R}_\varepsilon(q,v)\ge\mathcal{R}_\varepsilon(q,0)$ can be interpreted as characterizing gradient flows (see Remark 2.6), Assumption 5 formulates that the limiting structure $\mathcal{M}_0$ preserves this aspect of the gradient-flow nature. If we impose a continuity requirement on $\mathcal{N}_0$, then Assumption 5 can also be derived through the Γ-convergence limit—we show this in the next Lemma. In the next section both Assumptions 4 and 5 will be essential in recovering a dissipation-potential formulation of $\mathcal{M}_0$.

Lemma 2.10. *Assume all of Assumption 2.9 except part 5; instead, assume that $\mathcal{N}_0$ is continuous. Then*

$$\mathcal{N}_0(q,v,\xi) \ge \mathcal{N}_0(q,0,\xi) \quad \text{and} \quad \mathcal{M}_0(q,v,\xi) \ge \mathcal{M}_0(q,0,\xi) \qquad \text{for all } (q,v,\xi) \in T\mathbf{Q}\otimes T^*\mathbf{Q}. \tag{2.9}$$

Proof. Fix $q^0 \in \mathbf{Q}$. By working in local coordinates and taking sufficiently small T, we can choose a curve $q_0 : [0,T] \to \mathbf{Q}$ to satisfy $q_0(t) = q^0 + tv$, for any $v \in T_{q^0}\mathbf{Q}$. Similarly, for sufficiently small T we can choose $\mathcal{F}$ such that $-\mathrm{D}\mathcal{F}$ is a constant $\xi \in T^*_{q^0}\mathbf{Q}$ on the affine curve q_0.

By the continuity of $\mathcal{N}_0$, $\mathfrak{D}_0^T(q_0,\mathcal{F})$ is finite; therefore we can find a recovery sequence $q_\varepsilon \to q_0$ for $\mathfrak{D}_\varepsilon^T(\cdot,\mathcal{F})$. Define the time-rescaled curve $r_\varepsilon(s) := q_\varepsilon(s/\lambda)$ for $s \in [0,\lambda T]$; the sequence r_ε converges to a limit r_0 in $AC([0,\lambda T],\mathbf{Q})$ with $r_0(s) = q_0(s/\lambda)$. We then calculate

$$\begin{aligned}
\int_0^T \mathcal{N}_0\big(q_0(t),\dot q_0(t),-\mathrm{D}\mathcal{F}(q_0(t))\big)\,\mathrm{d}t &= \lim_{\varepsilon\to0}\int_0^T \mathcal{N}_\varepsilon\big(q_\varepsilon(t),\dot q_\varepsilon(t),-\mathrm{D}\mathcal{F}(q_\varepsilon(t))\big)\,\mathrm{d}t\\
&= \lim_{\varepsilon\to0}\frac{1}{\lambda}\int_0^{\lambda T} \mathcal{N}_\varepsilon\big(r_\varepsilon(s),\lambda\dot r_\varepsilon(s),-\mathrm{D}\mathcal{F}(r_\varepsilon(s))\big)\,\mathrm{d}s\\
&\ge \liminf_{\varepsilon\to0}\frac{1}{\lambda}\int_0^{\lambda T} \mathcal{N}_\varepsilon\big(r_\varepsilon(s),\dot r_\varepsilon(s),-\mathrm{D}\mathcal{F}(r_\varepsilon(s))\big)\,\mathrm{d}s\\
&\ge \frac{1}{\lambda}\int_0^{\lambda T} \mathcal{N}_0\big(r_0(s),\dot r_0(s),-\mathrm{D}\mathcal{F}(r_0(s))\big)\,\mathrm{d}s\\
&= \int_0^T \mathcal{N}_0\big(q_0(t),\dot q_0(t)/\lambda,-\mathrm{D}\mathcal{F}(q_0(t))\big)\,\mathrm{d}t.
\end{aligned}$$

Using the continuity of $\mathcal{N}_0$ we find that

$$\frac{1}{T}\int_0^T \mathcal{N}_0\big(q^0+tv,v,\xi\big)\,\mathrm{d}t \ge \frac{1}{T}\int_0^T \mathcal{N}_0\big(q^0+tv,0,\xi\big)\,\mathrm{d}t,$$

and taking the limit $T \to 0$ we find the first inequality in (2.9). The second follows by the definition (2.8). □

Remark 2.11. For the results of this paper it would also be sufficient to require the Γ-convergence of $\mathfrak{D}_\varepsilon^T$ only on sequences of curves with uniformly bounded energy $\mathcal{E}_\varepsilon$. □

2.5 Primal-dual maps

For fixed $q \in \mathbf{Q}$, the map $(v,\xi) \mapsto \mathcal{M}_0(q,v,\xi)$ constructed in the previous section may have various different properties, and we study them next.

Let X be a real reflexive Banach space; we will apply the results below to the case $X = T_q\mathbf{Q}$ and $X^* = T_q^*\mathbf{Q}$, for a fixed $q \in \mathbf{Q}$, but the development below holds more generally. Recall that any functional $\mathcal{R} : X \to [0,\infty]$ is a *dissipation potential* if it is convex, lower semicontinuous, non-negative, and zero at zero.

Definition 2.12. Let $G : X \times X^* \to \mathbb{R}$ satisfy $G(v,\xi) \ge \langle v,\xi\rangle$.

(a) We say that G is a *dual dissipation sum* if there exists a dissipation potential $\mathcal{R}$ such that

$$G(v,\xi) = \mathcal{R}(v) + \mathcal{R}^*(\xi).$$

(b) We say that G has a *contact-equivalent dissipation potential* if there exists a dissipation potential $\widetilde{\mathcal{R}}$ such that

$$\mathcal{C}_G = \{(v,\xi) : G(v,\xi) = \langle v,\xi\rangle\} = \operatorname{graph}(\partial\widetilde{\mathcal{R}}). \tag{2.10}$$

(c) We say that G has a *force-dependent dissipation potential* if for all $\xi \in X^*$ there exists a dissipation potential $\overline{\mathcal{R}}_\xi$ such that

$$G(v,\xi) = \overline{\mathcal{R}}_\xi(v) + (\overline{\mathcal{R}}_\xi)^*(\xi).$$

Lemma 2.13. *Let $G : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ satisfy $G(v,\xi) \geq \langle v,\xi\rangle$.*

1. *In each of the three cases above the dissipation potentials are uniquely characterized by G.*

2. *If G is a dual dissipation sum $\mathcal{R}\oplus\mathcal{R}^*$, then $\mathcal{R}$ also is a contact-equivalent dissipation potential for G (i.e. (a) $\Longrightarrow$ (b)). The potential $\mathcal{R}$ also satisfies the conditions of being a force-dependent dissipation potential ((a) $\Longrightarrow$ (c)), even though $\mathcal{R}$ does not actually depend on ξ.*

3. *Assume that G satisfies*

$$\forall\xi \in X^* : \quad G(\cdot,\xi) \text{ is lower semi-continuous and convex,} \tag{2.11a}$$
$$G(v,\xi) \geq G(0,\xi) \text{ for all } v \in X, \tag{2.11b}$$

and has a contact-equivalent dissipation potential $\widetilde{\mathcal{R}}$. If $\widetilde{\mathcal{R}}$ is superlinear, then G also has a force-dependent dissipation potential $\overline{\mathcal{R}}_\xi$ (this is a qualified (b) $\Longrightarrow$ (c)). It is possible that $\overline{\mathcal{R}}_\xi \neq \widetilde{\mathcal{R}}$.

Proof. To prove the uniqueness of the potentials, first consider case (a). If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two dissipation potentials, then

$$\mathcal{R}_1(v) - \mathcal{R}_2(v) = \mathcal{R}_2^*(\xi) - \mathcal{R}_1^*(\xi) \qquad \text{for all } (v,\xi) \in X \times X^*.$$

It follows that both sides are constant, and by the normalization condition $\mathcal{R}_i(0) = 0$ the potentials coincide. The proof of case (c) is identical. Finally, in case (b), if two dissipation potentials represent G, then they have the same subdifferential; again they are equal up to a constant, and this constant vanishes for the same reason.

Part 2 of the Lemma follows from the definition. To prove part 3, first note that by the superlinearity and Lemma 2.2, for each $\xi \in X^*$ there exists $v_\xi \in \partial\widetilde{\mathcal{R}}(\xi)$; since $\mathcal{C}_G = \operatorname{graph}(\partial\widetilde{\mathcal{R}})$, this implies that $G(v_\xi,\xi) = \langle v_\xi,\xi\rangle$. Define for each $\xi \in X^*$ the function $\overline{\mathcal{R}}_\xi : X \to [0,\infty]$ by

$$\overline{\mathcal{R}}_\xi(v) := G(v,\xi) - G(0,\xi).$$

Since $G(0,\xi) \overset{(2.11b)}{\leq} G(v_\xi,\xi) = \langle v_\xi,\xi\rangle < \infty$, the difference above is well-defined. By (2.11a) and (2.11b), the function $\overline{\mathcal{R}}_\xi$ is convex and lower semicontinuous, and satisfies $\overline{\mathcal{R}}_\xi(0) = 0 = \min_v \overline{\mathcal{R}}_\xi(v)$. To calculate the dual $\overline{\mathcal{R}}_\xi^*(\xi)$, note that v_ξ minimizes the convex function $v \mapsto G(v,\xi) - \langle v,\xi\rangle$, with value zero, so that

$$\overline{\mathcal{R}}_\xi^*(\xi) = \sup_{v\in X} \langle v,\xi\rangle - \overline{\mathcal{R}}_\xi(v) = \sup_{v\in X} [\langle v,\xi\rangle - G(v,\xi)] + G(0,\xi) = G(0,\xi).$$

It follows that $G(v,\xi) = \overline{\mathcal{R}}_\xi(v) + \overline{\mathcal{R}}_\xi^*(\xi)$. The fact that $\widetilde{\mathcal{R}}$ and $\overline{\mathcal{R}}_\xi$ may be different is illustrated by the example of Section 3. □

2.6 Tilt- and contact tilt-EDP-convergence

We now define two new convergence concepts, *EDP-convergence with tilting* and *contact EDP-convergence with tilting.*

Definition 2.14. Let the sequence $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ of gradient systems satisfy Assumption 2.9, and recall that the limiting function $\mathcal{M}_0$ is given by (2.8). The sequence $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ converges

1. in the sense of **EDP-convergence with tilting**, or **tilt-EDP-convergence**, to a limit $(\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$ if for all $q \in \mathbf{Q}$, $\mathcal{M}_0(q, \cdot, \cdot)$ is a dual dissipation sum with potential $\mathcal{R}_0(q, \cdot)$.
2. in the sense of **contact EDP-convergence with tilting**, or **contact tilt-EDP-convergence**, to a limit $(\mathbf{Q}, \mathcal{E}_0, \widetilde{\mathcal{R}}_0)$ if for all $q \in \mathbf{Q}$, $\mathcal{M}_0(q, \cdot, \cdot)$ has a contact-equivalent dissipation potential $\widetilde{\mathcal{R}}_0(q, \cdot)$.

The two convergences are also written as $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{tiEDP}} (\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$ and $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{coEDP}} (\mathbf{Q}, \mathcal{E}_0, \mathcal{R}_0)$.

We add the simple EDP convergence for completeness and comparison:

Lemma 2.15. *Let the sequence $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ of gradient systems satisfy Assumption 2.9. If for all $q \in \mathbf{Q}$ the function $\mathcal{M}_0(q, \cdot, \cdot)$ has a force-dependent dissipation potential, then the sequence $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ converges in the **simple EDP sense** of Definition 2.7.*

Remark 2.16. The opposite implication does not hold: if the sequence converges in the simple EDP sense, then it follows that there exists a dissipation potential $\mathcal{R}_0$ such that $\mathcal{M}_0(q, v, -\mathrm{D}\mathcal{E}_0(q)) = \mathcal{R}_0(q, v) + \mathcal{R}_0^*(q, -\mathrm{D}\mathcal{E}_0(q))$. In order to have a force-dependent dissipation potential, however, we need information about $\mathcal{M}_0(q, v, \xi)$ for all values of ξ, not just $\xi = -\mathrm{D}\mathcal{E}_0(q)$. □

Proof of Lemma 2.15. Assume that $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ satisfies Assumption 2.9, and that the limit function $\mathcal{M}_0$ has a force-dependent dissipation potential $\overline{\mathcal{R}}_\xi$. Under Assumption 2.9, part 1 of Definition 2.7 is automatically satisfied. By taking $\mathcal{F} = 0$ in the Γ-convergence statement of $\mathfrak{D}_\varepsilon^T$ in Assumption 2.9, we recover the Γ-convergence in part 2 of Definition 2.7. The fact that $\overline{\mathcal{R}}_\xi$ is a force-dependent dissipation potential implies that

$$\mathcal{N}_0(q, v, 0) = \mathcal{M}_0(q, v, -\mathrm{D}\mathcal{E}_0(q)) = \overline{\mathcal{R}}_{-\mathrm{D}\mathcal{E}_0(q)}(q, v) + \overline{\mathcal{R}}^*_{-\mathrm{D}\mathcal{E}_0(q)}(q, -\mathrm{D}\mathcal{E}_0(q)).$$

Therefore the limit $\mathfrak{D}_0^T$ is given as a sum $\overline{\mathcal{R}}_{-\mathrm{D}\mathcal{E}_0} \oplus \overline{\mathcal{R}}^*_{-\mathrm{D}\mathcal{E}_0}$, thus fulfilling (2.6). □

In each of the three cases, the convergence uniquely fixes a limiting dissipation potential $\mathcal{R}_0(q, \cdot)$, $\widetilde{\mathcal{R}}_0(q, \cdot)$, or $\overline{\mathcal{R}}_{0,\xi}(q, \cdot)$.

2.7 Properties of tilt- and contact tilt-EDP-convergence

In Section 1.3 we described how we want the new convergence concepts to be such that tilting the energies does not change the limiting dissipation potentials. The definitions above have been constructed with this aim in mind, and we now check that indeed the two tilted convergence concepts have this property.

Lemma 2.17 (In tilt and contact tilt limits, the dissipation potential is independent of the tilts)**.** *Let $\Box$ signify either tilt-EDP-convergence or contact tilt-EDP-convergence.*

If

$$(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\Box} (\mathbf{Q}, \mathcal{E}, \mathcal{R}),$$

then for all $\widetilde{\mathcal{F}} \in C^1(\mathbf{Q})$ we have

$$(\mathbf{Q}, \mathcal{E}_\varepsilon + \widetilde{\mathcal{F}}, \mathcal{R}_\varepsilon) \xrightarrow{\Box} (\mathbf{Q}, \mathcal{E} + \widetilde{\mathcal{F}}, \mathcal{R}).$$

Note that the limiting dissipation potential $\mathcal{R}$ is the same for all $\widetilde{\mathcal{F}}$.

Proof. Since $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\Box} (\mathbf{Q}, \mathcal{E}, \mathcal{R})$, Assumption 2.9 is satisfied for the sequence $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$. For both tilt- and contact tilt-convergence, we first check that the perturbed sequence $(\mathbf{Q}, \mathcal{E}_\varepsilon + \widetilde{\mathcal{F}}, \mathcal{R}_\varepsilon)$ also satisfies Assumption 2.9.

The Γ-convergence requirement $\mathcal{E}_\varepsilon + \widetilde{\mathcal{F}} \xrightarrow{\Gamma} \mathcal{E} + \widetilde{\mathcal{F}}$, part 1 of Assumption 2.9, follows directly from the properties of Γ-convergence and the continuity of $\widetilde{\mathcal{F}}$. For parts 2 and 3 we take any $\mathcal{F} \in C^1(\mathbf{Q})$ as in the Assumption and observe that

$$\widetilde{\mathfrak{D}}^T_\varepsilon(q, \mathcal{F}) := \int_0^T \Big[\mathcal{R}_\varepsilon(q, \dot q) + \mathcal{R}^*_\varepsilon(q, -\mathrm{D}\mathcal{E}_\varepsilon(q) - \mathrm{D}\widetilde{\mathcal{F}}(q))\Big]\,\mathrm{d}t = \mathfrak{D}^T_\varepsilon(q, \mathcal{F} + \widetilde{\mathcal{F}}).$$

Therefore $\widetilde{\mathfrak{D}}^T_\varepsilon(\cdot, \mathcal{F})$ Γ-converges to $\mathfrak{D}^T_0(\cdot, \mathcal{F} + \widetilde{\mathcal{F}})$, and we have

$$\mathfrak{D}^T_0(q, \mathcal{F}+\widetilde{\mathcal{F}}) = \widetilde{\mathfrak{D}}^T_0(q, \mathcal{F}) := \int_0^T \widetilde{\mathcal{N}}_0(q, \dot q, -\mathrm{D}\mathcal{F}(q))\,\mathrm{d}t, \quad \text{with} \quad \widetilde{\mathcal{N}}_0(q, v, \eta) := \mathcal{N}_0(q, v, \eta - \mathrm{D}\widetilde{\mathcal{F}}(q)).$$

Thererfore $\widetilde{\mathfrak{D}}^T_\varepsilon$, $\widetilde{\mathfrak{D}}^T_0$, and $\widetilde{\mathcal{N}}_0$ satisfy parts 2 and 3.

Defining

$$\widetilde{\mathcal{M}}_0(q, v, \xi) := \widetilde{\mathcal{N}}_0(q, v, \xi + \mathrm{D}\mathcal{E}(q) + \mathrm{D}\mathcal{F}(q)),$$

we find

$$\widetilde{\mathcal{M}}_0(q, v, \xi) = \mathcal{N}_0(q, v, \xi + \mathrm{D}\mathcal{E}(q)) = \mathcal{M}_0(q, v, \xi). \tag{2.12}$$

This identity establishes parts 4 and 5, and therefore the sequence $(\mathbf{Q}, \mathcal{E}_\varepsilon + \mathcal{F}, \mathcal{R}_\varepsilon)$ satisfies Assumption 2.9.

The identity $\widetilde{\mathcal{M}}_0 = \mathcal{M}_0$ in (2.12) also implies that the sequence $(\mathbf{Q}, \mathcal{E}_\varepsilon + \mathcal{F}, \mathcal{R}_\varepsilon)$ satisfies the same convergence as the unperturbed sequence sequence $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$. $\Box$

Next we consider relations between the three convergence concepts. Up to a technical requirement the three concepts are ordered:

Lemma 2.18. *We have*

$$\textit{tilt-EDP-cvg} \implies \textit{contact tilt-EDP-cvg}$$

and

$$\left.\begin{array}{r} \textit{contact tilt-EDP-cvg} \\ \widetilde{\mathcal{R}}_0(q, \cdot)\ \textit{superlinear for all } q \end{array}\right\} \implies \textit{simple EDP-cvg.}$$

In addition, if tilt-EDP-convergence holds, then all three convergences hold and the dissipation potentials coincide: $\mathcal{R}_0 = \widetilde{\mathcal{R}}_0 = \overline{\mathcal{R}}_{0,\xi}$.

Proof of Lemma 2.18. Both arrows follow directly from Lemma 2.13. Part 2 of Lemma 2.13 implies that in the case of tilt-EDP-convergence all three convergences hold, and the potentials coincide. □

Lemma 2.19 (Alternative characterization of tilt-EDP-convergence)**.** *Consider a sequence* $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ *of gradient systems, and a fixed gradient system* $(\mathbf{Q}, \mathcal{E}, \mathcal{R})$*. The following are equivalent:*

1. $(\mathbf{Q}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{tiEDP}} (\mathbf{Q}, \mathcal{E}, \mathcal{R})$*;*

2. *For each* $\widetilde{\mathcal{F}} \in C^1(\mathbf{Q})$*,* $(\mathbf{Q}, \mathcal{E}_\varepsilon{+}\mathcal{F}, \mathcal{R}_\varepsilon) \xrightarrow{\text{EDP}} (\mathbf{Q}, \mathcal{E}{+}\mathcal{F}, \mathcal{R})$*.*

The proof is just a reshuffling of definitions.

Remark 2.20. The important thing to note here is that the problems with simple EDP-convergence, in having force-dependent dissipation potentials, can not be solved simply by requiring simple EDP-convergence for all tilted versions of the systems with a single dissipation potential. By Lemma 2.19 this requirement is equivalent to tilt-EDP-convergence, and therefore is too strong: in the two examples of Sections 3 and 4 tilt-EDP-convergence does not hold.

The benefit of the intermediate concept of contact tilt-EDP-convergence lies in the combination of tilting, which allows the convergence to roam over all of (v, ξ)-space, with restriction to the contact set, which allows the connection between $\mathcal{M}_0$ and $\mathcal{R}_0$ to focus on the case of contact. We comment more on this in Section 5. □

3 Contact tilt-EDP-convergence for a model with a wiggly dissipation

3.1 Model and convergence results

We study a family $(\mathbb{R}, \mathcal{E}, \mathcal{R}_\varepsilon)$, $\varepsilon > 0$, of gradient systems, where the energy is independent of ε while the dissipation strongly oscillates in the state variable q, namely

$$\mathcal{R}_\varepsilon(q, v) = \frac{\mu(q, q/\varepsilon)}{2} v^2,$$

where $\mu \in \mathrm{C}^0(\mathbb{R}^2)$ is 1-periodic in the second variable, i.e. $\mu(q, y{+}1) = \mu(q, y)$, and satisfies the bound $0 < \underline{m} \leq \mu(q, y) \leq \overline{m} < \infty$.

Setting

$$\mathcal{R}_{\text{eff}}(q, v) = \frac{\overline{\mu}(q)}{2} v^2 \quad \text{with } \overline{\mu}(q) := \int_0^1 \mu(q, y)\,\mathrm{d}y$$

we obtain the following convergence result. The solutions q^ε of the gradient flow

$$0 = \mu(q^\varepsilon, q^\varepsilon/\varepsilon)\dot{q}^\varepsilon + \mathrm{D}\mathcal{E}(q^\varepsilon)$$

converge to the solution q of the gradient flow

$$0 = \overline{\mu}(q)\dot{q} + \mathrm{D}\mathcal{E}(q). \tag{3.1}$$

Theorem 3.1 (Contact tilt-EDP-convergence). *We have* $(\mathbb{R},\mathcal{E},\mathcal{R}_\varepsilon) \xrightarrow{\text{coEDP}} (\mathbb{R},\mathcal{E},\mathcal{R}_{\text{eff}})$, *where* $\mathcal{R}_{\text{eff}}(q,\cdot)$ *is quadratic and is independent of* $\mathcal{E}$.

If $\mu(q,\cdot)$ *is not constant, we have simple EDP-convergence for a non-quadratic* $\overline{\mathcal{R}}_0(q,\cdot)$ *that depends on* $\mathcal{E}$*, and there is no tilt-EDP-convergence.*

Proof. The tilted dissipation functional has the form

$$\mathfrak{D}^\eta_\varepsilon(q) = \int_0^T \mathcal{N}_\varepsilon(q,\dot q, -\mathrm{D}\mathcal{F}(q))\,\mathrm{d}t \text{ with } \mathcal{N}_\varepsilon(q,v,\eta) = \mathcal{R}_\varepsilon(q,v) + \mathcal{R}^*_\varepsilon(q,\eta{-}\mathrm{D}\mathcal{E}(q)).$$

Hence, we obtain the special form

$$\mathcal{N}_\varepsilon(q,v,\eta) = \widehat{\mathcal{N}}(q,q/\varepsilon,v,\eta{-}\mathrm{D}\mathcal{E}(q)) \text{ with } \widehat{\mathcal{N}}(q,y,v,\xi) = \frac{\mu(q,y)}{2}v^2 + \frac{\xi^2}{2\mu(q,y)}.$$

The Γ-limit $\mathfrak{D}^\eta_0$ of $\mathfrak{D}^\eta_\varepsilon$ was calculated in [DFM17, Thm. 2.4] by slightly generalizing the results in [Bra02]. Indeed, our integrand $\widehat{\mathcal{N}}$ satisfies exactly the same assumptions as N in [DFM17, Eqn. (3,3)]; thus the approach there (see Prop. 3.6 and 3.7) can be used on our situation again. We arrive at

$$\mathfrak{D}^\eta_0(q) = \int_0^T \mathcal{N}_0(q,\dot q,\eta)\,\mathrm{d}t \text{ with } \mathcal{N}_0(q,v,\eta) = \mathcal{M}_0(q,\dot q,\eta{-}\mathrm{D}\mathcal{E}(q)),$$

where the effective dissipation structure $\mathcal{M}_0$ is given by homogenization, namely

$$\mathcal{M}_0(q,v,\xi) = \inf\Big\{ \int_{s=0}^1 \widehat{\mathcal{N}}(q,z(s),vz'(s),\xi)\,\mathrm{d}s \;\Big|\; z\in \mathbf{H}^1_v \Big\} \tag{3.2a}$$

$$= \inf\Big\{ \int_{s=0}^1 \Big(\frac{\mu(q,z(s)\big(vz'(s)\big)^2}{2} + \frac{\xi^2}{2\mu(q,z(s))}\Big)\,\mathrm{d}s \;\Big|\; z\in \mathbf{H}^1_v \Big\} \tag{3.2b}$$

$$= \inf\Big\{ \int_{y=0}^1 \Big(\frac{\mu(q,y)v^2}{2b(y)} + \frac{b(y)\xi^2}{2\mu(q,y)}\Big)\,\mathrm{d}y \;\Big|\; b(y)>0,\ \int_0^1 b(y)\,\mathrm{d}y = 1 \Big\}, \tag{3.2c}$$

where $\mathbf{H}^1_v := \{\, z\in \mathrm{H}^1(]0,1[) \mid z(1) = z(0)+\mathrm{sign}(v)\,\}$. As in [DFM17] this result strongly depends on the 1-periodicity of $\mu(q,\cdot)$ and on the fact that $y=q/\varepsilon$ is a scalar variable.

The first observation is that $\mathcal{M}_0$ is not given by a dual pair $\mathcal{R}_{\text{eff}} \oplus \mathcal{R}^*_{\text{eff}}$. For this, we use that $\mathcal{M}_0(q,\cdot,\cdot)$ can be evaluated explicitly on the two axes, namely

$$\mathcal{M}_0(q,0,\xi) = \frac{1}{\mu_{\max}(q)}\xi^2 \text{ with } \mu_{\max}(q) = \max\{\,\mu(q,y) \mid y\in[0,1]\,\}, \tag{3.3a}$$

$$\mathcal{M}_0(q,v,0) = \frac{\mu_{1/2}(q)}{2}v^2 \text{ with } \mu_{1/2}(q) := \Big(\int_0^1 \sqrt{\mu(q,y)}\,\mathrm{d}y\Big)^2. \tag{3.3b}$$

The first result is seen via (3.2c) by concentrating b near maximizers of $\mu(q,\cdot)$. The second follows from (3.2b) by minimizing $\int_0^1 m(z)z'^2\,\mathrm{d}y$ subject to $z(1)=z(0)+1$, which leads to $\mu_{1/2}(q)$ as given above.

If $\mu(q,\cdot)$ is not constant we have $\mu_{1/2}(q) < \mu_{\max}(q)$, such that there is no tilt-EDP-convergence.

Clearly, we have the lower bound $\mathcal{M}_0(q,v,\xi)\geq \xi v$, which follows from the lower bound

$$\frac{\mu(q,z(s))\big(vz'(s)\big)^2}{2} + \frac{\xi^2}{2\mu(q,z(s))} \geq |v|z'(s)\xi \tag{3.4}$$

for the integrand in (3.2b) (where equality holds if and only if $m(q,z(s))|v|z'(s)=\xi$) and integration over $s\in[0,1]$ using the boundary condition for z.

The contact set $\mathsf{C}_{\mathcal{M}_0}(q)$, defined similarly to (2.10),

$$\mathcal{C}_{\mathcal{M}_0}(q) := \big\{(v,\xi) : \mathcal{M}_0(q,v,\xi) = \langle v,\xi\rangle\big\},$$

can be constructed as follows. For $v=0$ we have to solve $\mathcal{M}_0(q,0,\xi)=\xi\,0=0$, which gives $\xi=0$. For $v\neq 0$ we can use (3.2b), where now by coercivity a minimizer $Z\in\mathbf{H}^1_v$ exists. From

$$\mathcal{M}_0(q,v,\xi) = \int_0^1 \Big(\frac{\mu(q,Z(s))\big(Z'(s)\big)^2}{2} + \frac{\xi^2}{2\mu(q,Z(s))}\Big)\,\mathrm{d}s = \xi v = \int_0^1 |v|Z'(s)\xi\,\mathrm{d}s.$$

By the lower estimate (3.4) we conclude that Z must satisfy $m(q,Z(s))|v|Z'(s)=\xi$ for a.a. $s\in[0,1]$. Integrating over s we find $vM^*(q)=\xi$, and the contact set reads

$$\mathsf{C}_{\mathcal{M}_0}(q) = \{\,(v,\xi)\in\mathbb{R}^2 \mid \mathcal{M}_0(q,v,\xi)=\xi v\,\} = \{\,(v,\overline{\mu}(q)v) \mid v\in\mathbb{R}\,\},$$

This, gives the desired linear kinetic relation and the quadratic effective dissipation potential $\mathcal{R}_{\text{eff}}(q,v)=\frac{\overline{\mu}(q)}{2}v^2$.

By the abstract result in Lemma 2.18 we have also simple EDP-convergence with the dissipation potential

$$\overline{\mathcal{R}}(q,v) := \mathcal{M}_0(q,v,-\mathrm{D}\mathcal{E}(q)) - \mathcal{M}_0(q,0,-\mathrm{D}\mathcal{E}(q)).$$

Because we have shown that $\mathcal{M}_0$ is not of the form $\Phi(q,v)+\Psi(q,\xi)$, we conclude that $\overline{\mathcal{R}}(q,\cdot)$ depends on $\mathcal{E}$. Moreover, $v\mapsto\overline{\mathcal{R}}(q,v)$ is not quadratic. □

We emphasize that the gradient-flow equation obtained from simple EDP-convergence is indeed the same as the equation obtained from contact tilt-EDP-convergence:

$$0=\partial_v\overline{\mathcal{R}}(q,\dot q)+\mathrm{D}\mathcal{E}(q) = \partial_v\mathcal{M}_0(q,\dot q,-\mathrm{D}\mathcal{E}(q))+\mathrm{D}\mathcal{E}(q). \tag{3.5}$$

This form can be more explicit by using the fact that $\mathcal{M}_0(q,\cdot,\cdot)$ only depends on v^2 and ξ^2 and is homogeneous of degree one in these variables, viz.

$$\mathcal{M}_0(q,v,\xi) = \big(\xi^2{+}\overline{\mu}(q)^2v^2\big)\Phi\Big(q,\frac{\xi^2}{\xi^2{+}\overline{\mu}(q)^2v^2}\Big).$$

This follows from (3.2c). The function $\Phi:\mathbb{R}\times[0,1]\to\mathbb{R}$ is continuous and satisfies

$$\Phi(q,0)=\frac{\mu_{1/2}(q)}{2\overline{\mu}(q)^2},\quad \Phi(q,1/2)=\frac{1}{2\overline{\mu}(q)},\quad \Phi(q,1)=\frac{1}{2\mu_{\max}(q)},\quad \Phi(q,s)\geq\frac{\sqrt{s(1{-}s)}}{\overline{\mu}(q)},$$

where the last relation follows from $\mathcal{M}_0(q,v,\xi)\geq\xi v$. With this, (3.5) takes the form

$$0=2\overline{\mu}(q)^2\dot q\,\Psi\Big(q,\frac{\mathrm{D}\mathcal{E}(q)^2}{\mathrm{D}\mathcal{E}(q)^2{+}\overline{\mu}(q)^2\dot q^2}\Big)+\mathrm{D}\mathcal{E}(q),\ \text{ where }\ \Psi(q,s)=\Phi(q,s)-s\partial_s\Phi(q,s).$$

Using $\Phi'(q,1/2)=0$ we have $\Psi(q,1/2)=\Phi(q,1/2)=1/(2\overline{\mu}(q))$, and conclude that (3.5) is indeed equivalent to (3.1).

Certainly this form of the equation involving the nonlinear kinetic relation

$$v\mapsto\xi=\partial_v\mathcal{M}_0(q,v,-\mathrm{D}\mathcal{E}(q)) = 2\overline{\mu}(q)^2v\,\Psi\Big(\frac{\mathrm{D}\mathcal{E}(q)^2}{\mathrm{D}\mathcal{E}(q)^2{+}\overline{\mu}(q)^2v^2}\Big)$$

is "less natural" than the effective equation (3.1) featuring the simple linear kinetic relation $v\mapsto\xi=\overline{\mu}(q)v$.

3.2 Comments

Remark 3.2 (Validity of conjecture $\mathcal{M}_0 \le \mathcal{R}_{\text{eff}} \oplus \mathcal{R}^*_{\text{eff}}$, see [DFM17, Sec. 5.4]). In this example, we can easily show that the sum of the dual pair $\mathcal{R}_{\text{eff}} \oplus \mathcal{R}^*_{\text{eff}}$ is always bigger than $\mathcal{M}_0$. To see this, we insert a special competitor into the characterization (3.2c). The choice $\widehat{b} : y \mapsto \mu(q,y)/\overline{\mu}(q)$ is an admissible competitor, and we find

$$\mathcal{M}_0(q,v,\xi) \le \int_0^1 \Big(\frac{\mu(q,y)v^2}{2\widehat{b}(y)} + \frac{\widehat{b}(y)\xi^2}{2\mu(q,y)} \Big) \,\mathrm{d}y = \frac{\overline{\mu}(q)v^2}{2} + \frac{\xi^2}{2\overline{\mu}(q)} = \mathcal{R}_{\text{eff}}(q,v) + \mathcal{R}^*_{\text{eff}}(q,\xi).$$

□

Remark 3.3 (Bipotential and non-convexity). Clearly, $\mathcal{M}_0(q,\cdot,\xi)$ is convex. Following the ideas in [DFM17] it is possible to show that $\mathcal{M}_0(q,v,\cdot)$ is convex as well. Indeed, neglecting the dependence on q, assuming $v>0$, we define $\mathcal{W}(\xi,h) = \int_0^1 \sqrt{\xi^2{+}2h\mu(y)}\,\mathrm{d}y$ and find

$$\mathcal{M}_0(v,\xi) = v\mathcal{W}(\xi, H(v,\xi)) - H(v,\xi), \ \text{ where } 1 = v\mathcal{W}(\xi,H(v,\xi)),$$

i.e. $h = H(v,\xi)$ is implicitly defined by the last relation. Using the implicit function theorem one finds $\mathrm{D}^2_\xi \mathcal{M}_0(v,\xi) = v\big(\mathrm{D}^2_\xi\mathcal{W} - (\mathrm{D}_\xi\mathrm{D}_h\mathcal{W})^2/\mathrm{D}^2_h\mathcal{W}\big)|_{h=H(v,\xi}$, which is non-negative because $\mathcal{W}$ is convex in ξ and concave in μ.

However, in general $\mathcal{M}_0$ is not jointly convex in v and ξ. This can be seen by evaluating $\mathcal{M}_0$ at three points:

$$\mathcal{M}_0(v_0,0) = \frac{\mu_{1/2}\,v_0^2}{2}, \quad \mathcal{M}_0(0,\overline{\mu}v_0) = \frac{(\overline{\mu}v_0)^2}{2\mu_{\max}}, \quad \mathcal{M}_0(\tfrac12 v_0, \tfrac12\overline{\mu}v_0) = \frac{\overline{\mu}v_0^2}{4},$$

where the last relation uses that the point lies on the contact set. As this point also lies in the middle of the first two, convexity can only hold if we have

$$\frac{\overline{\mu}v_0^2}{4} \le \frac12 \Big(\frac{\mu_{1/2}\,v_0^2}{2} + \frac{(\overline{\mu}v_0)^2}{2\mu_{\max}} \Big) \quad \Longleftrightarrow \quad \overline{\mu} \le \mu_{1/2} + (\overline{\mu})^2/\mu_{\max}.$$

Choosing $\mu(y) = \alpha + |2y{-}1|^\gamma$ for $y \in [0,1]$, where α is sufficiently small and γ sufficiently big (e.g. $\gamma \ge 4$), we find a contraction to convexity. □

Remark 3.4 (Convergence of Riemannian distance). It is interesting to note that we may look at the gradient system $(\mathbb{R},\mathcal{E},\mathcal{R}_\varepsilon)$ also as a metric gradient system $(\mathbb{R},\mathcal{E},\mathcal{D}_\varepsilon)$, where the associated distances $\mathcal{D}_\varepsilon : \mathbb{R}\times\mathbb{R} \to [0,\infty[$ are defined via

$$\begin{aligned}\mathcal{D}_\varepsilon(q_0,q_1)^2 &:= \inf\Big\{ \int_0^1 2\mathcal{R}_\varepsilon(q,\dot q)\,\mathrm{d}s \,\Big|\, q(0)=q_0,\ q(1)=q_1,\ q \in \mathrm{H}^1(]0,1[) \Big\} \\ &= \Big| \int_{q_0}^{q_1} \sqrt{\mu(q,q/\varepsilon)}\,\mathrm{d}q \Big|^2.\end{aligned}$$

Obviously, the distances $\mathcal{D}_\varepsilon$ converge to the limit distance $\mathcal{D}_0$ given by

$$\mathcal{D}_0(q_0,q_1)^2 = \Big| \int_{q_0}^{q_1} \int_0^1 \sqrt{\mu(q,y)}\,\mathrm{d}y\,\mathrm{d}q \Big|^2 = \Big| \int_{q_0}^{q_1} \sqrt{\mu_{1/2}(q)}\,\mathrm{d}q \Big|^2$$

with $\mu_{1/2}(q)$ from (3.3b). ($(\mathbb{R},\mathcal{D}_\varepsilon)$ converges to $(\mathbb{R},\mathcal{D}_0)$ in the Gromov-Hausdorff sense.)

For non-constant $\mu(q,\cdot)$ we have $\mu_{1/2}(q) < \overline{\mu}(q)$, and conclude that the limit $\mathcal{D}_0$ of the distances $\mathcal{D}_\varepsilon$ is different from the effective distance $\mathcal{D}_{\text{eff}}$ obtained from $\mathcal{R}_{\text{eff}}$, namely

$$\mathcal{D}_{\text{eff}}(q_0,q_1)^2 = \Big| \int_{q_0}^{q_1} \sqrt{\overline{\mu}(q)}\,\mathrm{d}q \Big|^2 = \Big| \int_{q_0}^{q_1} \Big(\int_0^1 \mu(q,y)\,\mathrm{d}y \Big)^{1/2} \mathrm{d}q \Big|^2.$$

Hence, predictions using $\mathcal{D}_0$ instead of $\mathcal{D}_{\text{eff}}$ would give too little dissipation. In particular, the general theory from [Sav11] does not apply, because $\mathcal{E}$ is not uniformly geodesically λ-convex for all $\mathcal{D}_\varepsilon$. □

4 The wiggly-energy example from [DFM17]

In [DFM17] a wiggly-energy model was considered, where the gradient system $(\mathbb{R},\mathcal{E}_\varepsilon,\mathcal{R})$ has the form

$$\mathcal{E}_\varepsilon(t,q) = \mathcal{U}(q) + \varepsilon\mathcal{W}(q,q/\varepsilon) - \ell(t)q, \quad \mathcal{R}(q,\dot q) = \frac{\varrho(q)}{2}\dot q^2. \tag{4.1}$$

It was shown that the systems converge in the sense of contact tilt-EDP convergence with a limit system $(\mathbb{R},\mathcal{E}_0,\mathcal{R}_{\text{eff}})$, where $\mathcal{E}_0(t,q) = \mathcal{U}(q) - \ell(t)q$ and the effective dissipation potential strongly depends on the wiggly part $\mathcal{W}$.

The following theorem summarizes the results in [DFM17] that show that $(\mathbb{R},\mathcal{E}_\varepsilon,\mathcal{R})$ converges in the sense of *contact* tilt-EDP-convergence, but not in the stronger sense of tilt-EDP-convergence. Here the loading ℓ acts in a natural way as a time-dependent tilt. Indeed, the notion of tilt-EDP-convergence was developed while studying this model.

To obtain an explicit result we restrict ourself to a special case of the much more general result in [DFM17] and assume the following explicit expressions:

$$\mathcal{U}(q) = \frac{k}{2}q^2, \quad \mathcal{W}(q,y) = A\cos y, \quad \mathcal{R}(q,v) = \frac{\mu}{2}v^2 \quad \text{with } k,A,\mu>0. \tag{4.2}$$

Theorem 4.1. *Consider the $(\mathbb{R},\mathcal{E}_\varepsilon,\mathcal{R})$ of gradient systems given through (4.1) and (4.2). Then, the following holds:*

(A) The dissipation functionals $\mathfrak{D}^T_\varepsilon$ defined via (2.3) weakly Γ-converge in $\mathrm{H}^1([0,T])$ to $\mathfrak{D}^T_0 : q \mapsto \int_0^T \mathcal{M}_0(q,\ell(t){-}\mathrm{D}\mathcal{U}(q))\,\mathrm{d}t$ with

$$\mathcal{M}_0(v,\xi) = \inf\Big\{ \int_0^1 \Big(\frac{\varrho}{2}\big(vz'(s)\big)^2 + \frac{\big(\xi{+}A\sin z(s)\big)^2}{2\varrho} \Big)\mathrm{d}s \;\Big|\; z \in H^1_v \Big\}, \tag{4.3}$$

where $H^1_v = \{\, z \in \mathrm{H}^1([0,1]) \mid z(1) = z(0) + \mathrm{sign}(v) \,\}$.

(B) $\mathcal{M}_0$ satisfies $\mathcal{M}_0(v,\xi) \geq v\xi$ for all $v,\xi \in \mathbb{R}$, and

$$\mathcal{M}_0(v,\xi) = v\xi \quad \Longleftrightarrow \quad \varrho v = \mathrm{sign}(\xi)\sqrt{\max\{x^2{-}A^2,0\}}.$$

(C) We have the contact tilt-EDP-convergence $(\mathbb{R},\mathcal{E}_\varepsilon,\mathcal{R}) \xrightarrow{\text{coEDP}} (\mathbb{R},\mathcal{E}_0,\mathcal{R}_{\text{eff}})$ with

$$\mathcal{E}_0(t,q) = \mathcal{U}(q) - \ell(t)q \quad \textit{and} \quad \mathcal{R}_{\text{eff}}(v) = \int_0^{|v|} \sqrt{A^2{+}(\varrho w)^2}\,\mathrm{d}w,$$

(D) tilt-EDP-convergence does not hold.

The above theorem can be derived as for the wiggly-dissipation model $(\mathbb{R}, \mathcal{E}^{(3)}, \mathcal{R}_\varepsilon^{(3)})$ discussed before, where "$^{(3)}$" indicates the previous section. However, there is a major difference in the two results.

In both cases we start with a quadratic dissipation potential $\mathcal{R}_\varepsilon^{(3)}(q,v) = \mu(q)v^2/2$ and $\mathcal{R}(v) = \varrho v^2/2$. In the previous section the effective dissipation potential $\mathcal{R}_{\text{eff}}^{(3)}$ reads $v \mapsto \overline{\mu}(q)v^2/2$ and, hence, is still quadratic and solely depends on the family $\mathcal{R}_\varepsilon^{(3)}$. In contrast, in the present case $\mathcal{R}_{\text{eff}}$ is no longer quadratic, and explicitly depends on the amplitude A, which is a microscopic information stemming from the family $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$. Thus, we see EDP-convergence in a convergence for the pair $(\mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ and cannot be described by doing a convergence of for the family $(\mathcal{R}_\varepsilon)_{\varepsilon>0}$ alone.

5 Understanding the two new convergence concepts

The new convergence concepts of tilt- and contact tilt-EDP-convergence are based upon simultaneous convergence of all tilted versions of the gradient system. In this section we explain why this choice is successful in deriving effective kinetic relations, without falling prey to the same problem as simple EDP-convergence. This will also allow us to explain in a different manner why tilt-convergence is not sufficient, and why the contact version can be considered 'more natural'. The discussion in this section is necessarily formal.

Two observations are central:

Observation 1: Gradient-flow solutions solve a Hamiltonian system. Solutions of the gradient-flow system $(\mathbf{Q}, \mathcal{E}, \mathcal{R})$ can be obtained as solutions of the global minimization problem

$$\inf \big\{ \mathcal{E}(q(T)) - \mathcal{E}(q(0) + \mathfrak{D}(q) : q(0) = q^0 \big\}, \qquad q^0 \text{ given},$$

and the minimal value is zero.

At the same time, stationary points of the functional above are solutions of a Hamiltonian system. In the simple case $\mathbf{Q} = \mathbb{R}$ and $\mathcal{R}(q,v) = \mathcal{R}(v) = \frac{1}{2}v^2$, for instance, stationary points satisfy

$$\ddot{q} = \mathcal{E}'(q)\mathcal{E}''(q). \tag{5.1}$$

It may seem paradoxical that gradient-flow solutions are also solutions of a Hamiltonian system. In this example it is easy to recognize that solutions of the gradient flow $\dot{q} = -\mathcal{E}'(q)$ also solve (5.1), by calculating

$$\ddot{q} = -\frac{d}{dt}\mathcal{E}'(q) = -\mathcal{E}''(q)\dot{q} = \mathcal{E}''(q)\mathcal{E}'(q).$$

In general, the gradient-flow solutions form a strict subset of all solutions of the Hamiltonian system; this subset is automatically reached when the functional is minimized without constraint on the end point $q(T)$. For minimization with different conditions on the end point, however, minimizers will still be solutions of the Hamiltonian system, but no longer gradient-flow solutions.

Observation 2: The limit $\mathcal{M}_0$ is obtained by relaxation. In the limit $\varepsilon \to 0$ in the example in the previous section, the limiting functional $\mathcal{M}_0(q,v,\xi)$ is obtained through *relaxation.* This is best recognized in the formulas (3.2), specifically (3.2b): $\mathcal{M}_0$ is defined through a minimization of rescaled versions of $\mathcal{R}_\varepsilon$ and $\mathcal{R}_\varepsilon^*$, for a given value of ξ, and under a constraint on the curves z. Because of this constraint, the final value $z(1)$ is not free, and consequently the minimization need not result in a gradient-flow solution z. The *non*-gradient-flow nature of z therefore is a consequence of the multi-scale construction of $\mathcal{M}_0$, in which we impose a fixed macroscopoic velocity v, and minimize over microscopic degrees of freedom under that constraint.

However, when v and ξ are such that $\mathcal{M}_0(q, v, \xi) = \langle v, \xi \rangle$, solutions of the minimization problem *are* gradient-flow solutions (see the discussion following (3.4)). We therefore have the following situation:

1. For general v and ξ the value of $\mathcal{M}_0$ and the corresponding optimizer z may not be relevant as representations of the limit $\varepsilon \to 0$ of gradient-flow solutions q_ε;

2. For those v and ξ satisfying contact, i.e. $\mathcal{M}_0(q, v, \xi) = \langle v, \xi \rangle$, optimizers z are of gradient-flow type, and may represent the behaviour of solutions q_ε.

This explains why contact tilt-EDP-convergence is a natural choice: it connects the relaxation $\mathcal{M}_0$ with a dissipation potential $\mathcal{R}$ exactly at those values of v and ξ where optimizers are of gradient-flow type. In fact, Lemma 2.18 implies that if simple EDP-convergence yields a limiting dissipation potential that does depend on the force—this is exactly the case of a problematic kinetic relation—then tilt-EDP-convergence *can not* hold.

6 Tilting in Markov processes

Many gradient flows arise from the large deviations of Markov processes, and the tilting of the previous sections has a natural counterpart in this context. In this section we explore this connection.

6.1 Gradient flows and large deviations of Markov processes

In [MPR14] we showed the following general result: Suppose that Q^n is a sequence of Markov processes in $\mathbf{Q}$, that is reversible with respect to their stationary measure $\mu^n \in \mathcal{P}(\mathbf{Q})$. Assume that the following two large-deviation principles hold:

1. The invariant measures μ^n satisfy a large-deviation principle with rate function $S : \mathbf{Q} \to [0, \infty]$, i.e.
$$\mu^n \sim \exp\bigl(-nS\bigr), \qquad \text{as } n \to \infty;$$

2. The time courses of Q^n satisfy a large-deviation principle in $C([0,T]; \mathbf{Q})$ with rate function $I : C([0,T]; \mathbf{Q}) \to [0, \infty]$, i.e.
$$\operatorname{Prob}\Bigl(Q^n \approx q \,\Big|\, Q^n_0 \approx q(0)\Bigr) \sim \exp\bigl(-nI(q)\bigr), \qquad \text{as } n \to \infty. \tag{6.1}$$

Then I can be written as

$$I(q) = \tfrac{1}{2}S(q(T)) - \tfrac{1}{2}S(q(0)) + \int_0^T \bigl[\mathcal{R}(q, \dot q) + \mathcal{R}^*\bigl(q, -\tfrac{1}{2}\mathrm{D}S(q)\bigr)\bigr]\, dt, \tag{6.2}$$

for some symmetric dissipation potential $\mathcal{R}$. This result can be interpreted as follows.

- The functional I is non-negative, and with probability one a sequence of realizations Q^n of the stochastic process converges (along subsequences) to a curve q satisfying $I(q) = 0$. The property $I(q) = 0$ therefore identifies the limiting behaviour of the stochastic process Q^n.

- Curves q satisfying $I(q) = 0$ are solutions of the gradient-flow equation $\dot q = \mathrm{D}_\xi \mathcal{R}^*(q, -\frac12 \mathrm{D}S(q))$; therefore there is a one-to-one mapping between the functional I and the gradient system $(\mathbf{Q}, \frac12 S, \mathcal{R})$.

Over the last few years, a number of well-known gradient systems has been recognized as arising in this way. For instance, the 'diffusion' or 'heat' equation $\partial_t \rho = \Delta\rho$ arises as the limit of independent ('diffusing') Brownian particles [AD*11, AD*13], with the well-known Wasserstein-Entropy gradient structure; as the limit of the simple symmetric exclusion process describing particles hopping on a lattice [AD*13], with a gradient structure of a mixing entropy and a modified Wasserstein distance; and as the limit of oscillators that exchange energy ('heat') [PRV14], with a gradient structure consisting of an alternative logarithmic entropy and again a modified Wasserstein distance. Rate-independent systems arise from taking further limits [BoP16], and extensions to GENERIC have also been recognized [DPZ13].

In the next two sections we study how *tilting* enters this structure.

6.2 The static case

We first consider a non-dynamic case: X^n is a random variable in $\mathbf{Q}$, with law $\mu^n \in \mathcal{P}(\mathbf{Q})$. One example of this arises in the stochastic-process example above: if the initial state Q_0^n is drawn from the invariant measure μ^n of the process, then Q_t^n also has law μ^n for all time $t \geq 0$, and $X^n := Q_t^n$ for fixed t therefore is an example of the situation we are considering.

In previous sections we have implicitly used a property that is well known in the context of energetic modelling: *Energies are additive.* More precisely, when combining energies that arise from different phenomena, the energy of the total system is simply the sum of the individual energies. In this way, given an energy $\mathcal{E}$, the perturbed energy $\mathcal{E} + \mathcal{F}$ arises naturally as the sum of the original energy $\mathcal{E}$ and the external potential $\mathcal{F}$.

We now connect this additivity property with tilting of random variables. In the stochastic context, *tilting* a sequence of random variables X^n means considering a new sequence $X^{\mathcal{F},n}$ with law

$$\mu^{\mathcal{F},n}(A) := \frac{\displaystyle\int_A \mathrm{e}^{-n\mathcal{F}(q)}\,\mu^n(dq)}{\displaystyle\int_{\mathbf{Q}} \mathrm{e}^{-n\mathcal{F}(q)}\,\mu^n(dq)}. \tag{6.3}$$

This has the effect of giving higher probability to $q \in \mathbf{Q}$ for which $\mathcal{F}(q)$ is smaller: it 'tilts' the distribution in the direction of lower values of $\mathcal{F}$.

If μ^n satisfies a large-deviation principle with rate function S, as in the case of the stochastic process above, and satisfies a tail condition, then Varadhan's and Bryc's Lemmas (see e.g. [Ell85, Th. II.7.2]) imply that $\mu^{\mathcal{F},n}$ also satisfies a large-deviation principle, with 'tilted' rate function $S^{\mathcal{F}}$:

$$\mu^{\mathcal{F},n} \sim \exp\bigl(-nS^{\mathcal{F}}\bigr), \qquad S^\eta(q) := S(q) + \mathcal{F}(q) + \text{constant},$$

where the constant is chosen such that $\inf S^{\mathcal{F}} = 0$. This result can be understood by remarking that from $\mu^n \sim \mathrm{e}^{-nS}$ we find

$$\mathrm{e}^{-n\mathcal{F}}\mu^n \sim \mathrm{e}^{-n\mathcal{F}-nS},$$

which leads to the first two terms in $S^{\mathcal{F}}$; the constant in $S^{\mathcal{F}}$ arises from the normalization constant in (6.3).

The additivity property for energies thus has a counterpart for random variables in the form of the tilting of (6.3); the two concepts, addition of energies and tilting of random variables, coincide in the large-deviation limit $n \to \infty$.

6.3 The dynamic case

In the setup in the previous sections, not only are energies assumed to be additive, but also the dissipation function $\mathcal{R}$ is assumed to be independent of the tilting: addition of $\mathcal{F}$ changes the energy but not the dissipation. This assumption has its origin in the modelling background of mechanical gradient flows, in which the dissipation functional $\mathcal{R}$ defines the force-to-velocity relationship $\mathrm{D}_\xi \mathcal{R}^*(q, \cdot)$, which is assumed to be independent of the driving energy.

We now show that the same independence arises naturally for gradient systems that arise in the context of Markov processes. As in Section 6.1 we consider a Markov process Q^n in $\mathbf{Q}$; let L^n be its generator. (For instance, if Q^n solves the stochastic differential equation in $\mathbb{R}^d$,

$$dQ^n_t = b^n(Q^n_t)\,dt + \sigma^n(Q^n_t)\,dW_t,$$

then

$$(L^n f)(q) = b^n(q)\nabla f(q) + \frac{1}{2}\sigma^n(q)\sigma^n(q)^T \Delta f(q). \qquad)$$

In the dynamic context, tilting can be written in terms of the generator through the Fleming-Sheu logarithmic transform [Fle82, She85],

$$(L^{\mathcal{F},n} f)(q) := \mathrm{e}^{n\mathcal{F}(q)} L^n(\mathrm{e}^{-n\mathcal{F}} f)(q) - \mathrm{e}^{n\mathcal{F}(q)} f(q)(L^n \mathrm{e}^{-n\mathcal{F}})(q).$$

Let $Q^{\mathcal{F},n}$ be generated by $L^{\mathcal{F},n}$; if Q^n has invariant measure μ^n, then $Q^{\mathcal{F},n}$ has invariant measure $\mathrm{e}^{-n\mathcal{F}}\mu^n$.

In the derivation of the characterization (6.2), $\mathcal{R}^*$ is found by taking the limit in a scaled version of L^n, as follows. Define the *nonlinear generator*

$$(H^n f)(q) := \frac{1}{n}\mathrm{e}^{-nf(q)}(L^n \mathrm{e}^{nf})(q),$$

and its limit, in a sense to be defined precisely (see [FeK06, Ch. 6, 7]),

$$(Hf)(q) := \lim_{n\to\infty} H_n f(q).$$

In a successful large-deviation result, the operator H operates on f only through its derivative $\mathrm{D}f$, which allows us to identify

$$Hf(q) = \mathcal{H}(q, \mathrm{D}f(q)).$$

The dual dissipation function $\mathcal{R}^*$ is then defined by

$$\mathcal{R}^*(q,\xi) := \mathcal{H}\big(q, \xi + \tfrac{1}{2}\mathrm{D}S(q)\big) - \mathcal{H}\big(q, \tfrac{1}{2}\mathrm{D}S(q)\big).$$

Given this structure, we can now show how tilting does not affect $\mathcal{R}^*$. If we replace L^n by $L^{\mathcal{F},n}$ in this procedure, then

$$\begin{aligned}(H^{\mathcal{F},n}f)(q) &:= \frac{1}{n}\mathrm{e}^{-nf(q)}(L^{\mathcal{F},n}\mathrm{e}^{nf})(q)\\ &= \frac{1}{n}\mathrm{e}^{-nf(q)}\mathrm{e}^{n\mathcal{F}(q)}L^n(\mathrm{e}^{-n\mathcal{F}}\mathrm{e}^{nf})(q) - \frac{1}{n}\mathrm{e}^{-nf(q)}\mathrm{e}^{n\mathcal{F}(q)}\mathrm{e}^{nf(q)}L^n\mathrm{e}^{-n\mathcal{F}}(q)\\ &= H^n(f-\mathcal{F})(q) - H^n(-\mathcal{F})(q)\\ &\to H(f-\mathcal{F})(q) - H(-\mathcal{F})(q) \qquad \text{as } n\to\infty\\ &= \mathcal{H}(q,\mathrm{D}f(q)-\mathrm{D}\mathcal{F}(q)) - \mathcal{H}(q,-\mathrm{D}\mathcal{F}(q)).\end{aligned}$$

The dissipation potential $\mathcal{R}^{\mathcal{F},*}$ associated with the large deviations of the tilted process $Q^{\mathcal{F},n}$, with tilted invariant-measure rate functional $S^{\mathcal{F}} = S + \mathcal{F} + \text{constant}$, then satisfies

$$\begin{aligned}\mathcal{R}^{\mathcal{F},*}(q,\xi) &= \Big[\mathcal{H}\big(q,\xi+\tfrac12\mathrm{D}S^{\mathcal{F}}(q)-\mathrm{D}\mathcal{F}(q)\big) - \mathcal{H}(q,-\mathrm{D}\mathcal{F}(q))\Big]\\ &\qquad\qquad - \Big[\mathcal{H}\big(q,+\tfrac12\mathrm{D}S^{\mathcal{F}}(q)-\mathrm{D}\mathcal{F}(q)\big) - \mathcal{H}(q,-\mathrm{D}\mathcal{F}(q))\Big]\\ &= \mathcal{H}\big(q,\xi+\tfrac12\mathrm{D}S(q)\big) - \mathcal{H}\big(q,+\tfrac12\mathrm{D}S(q)\big)\\ &= \mathcal{R}^*(q,\xi).\end{aligned}$$

In other words, tilting replaces the invariant-measure large-deviation functional S by $S^{\mathcal{F}} = S + \mathcal{F} + \text{constant}$, and leaves $\mathcal{R}$ untouched.

Summarizing, there is a strong analogy between the modification of energies by addition, and the modification of stochastic processes by tilting. In both cases the dissipation function is expected to be unaffected; in the mechanical context this is a modelling postulate, and in the stochastic context it is a consequence of the structure of the tilting.

Regardless of whether the gradient-flow structure arises directly from a modelling argument or indirectly through a large-deviation principle, the behaviour under modification of the energy is therefore the same.

7 Discussion

One can interpret the introduction of η into a given gradient system $(\mathbf{Q},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ as the addition of a component in the system that generates an additional energy without changing the kinetic relation. This is a first step towards a further goal: generalize the convergence concepts of this paper to the case in which two independent gradient systems $(\mathbf{Q}^{1,2},\mathcal{E}_\varepsilon^{1,2},\mathcal{R}_\varepsilon^{1,2})$ are connected by adding a shared energy component $\mathcal{F}_\varepsilon : \mathbf{Q}^1\times\mathbf{Q}^2 \to \mathbb{R}\cup\{+\infty\}$. The aim is to define a convergence concept for the individual systems that implies convergence of the joint system under reasonable conditions on the joint energy $\mathcal{F}_\varepsilon$. We leave this for future work.

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