SEQUENTIAL BUCKLING: A VARIATIONAL ANALYSIS

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Abstract. We examine a variational problem from elastic stability theory: a thin elastic strut on an elastic foundation. The strut has infinite length, and its lateral deflection is represented by $u : \mathbb{R} \to \mathbb{R}$. Deformation takes place under conditions of prescribed total shortening, leading to the variational problem

$$\inf \left\{ \frac{1}{2} \int u''^2 + \int F(u) : \frac{1}{2} \int u'^2 = \lambda \right\}. \tag{0.1}$$

Solutions of this minimization problem solve the Euler-Lagrange equation

$$u'''' + pu'' + F'(u) = 0, \qquad -\infty < x < \infty. \tag{0.2}$$

The foundation has a nonlinear stress-strain relationship F', combining a *destiffening* character for small deformation with subsequent *stiffening* for large deformation. We prove that for every value of the shortening $\lambda > 0$ the minimization problem has at least one solution. In the limit $\lambda \to \infty$ these solutions converge on bounded intervals to a periodic profile, that is characterized by a related variational problem.

We also examine the relationship with a bifurcation branch of solutions of (0.2), and show numerically that all minimizers of (0.1) lie on this branch This information provides an interesting insight into the structure of the solution set of (0.1).

Key words. fourth-order, Swift-Hohenberg equation, Extended Fisher-Kolmogorov equation, localization, localized buckling, concentration-compactness, destiffening, restiffening, destabilization, restabilization.

AMS subject classifications. 34C11, 34C25, 34C37, 49N99, 49R99, 73C50, 73H05, 73H10, 73K05, 73K20, 73N20, 73Q05, 73V25, 86A60.

1. Introduction.

1.1. Localized buckling.

Long elastic structures that are loaded in the longitudinal direction can buckle in a localized manner. By this we mean that the lateral deflection is concentrated on a small section of the total length of the structure. A well-known example of this localization phenomenon is the axially loaded cylinder, which buckles in a localized diamond-like pattern [28, 14, 9]. Another example, one which will be the subject of this paper, is the strut on a foundation: a thin elastic layer confined laterally by a different elastic material.

One area of application in which the model of a strut on a foundation has received extensive attention is that of Structural Geology. In this context the strut represents a thin layer of rock that is embedded in a different type of rock, and the longitudinal compression is the result, directly or indirectly, of tectonic plate movement. In the geological context the most common constitutive assumptions are those of viscous, or visco-elastic, materials; however, there is a case to be made for the importance of elastic effects in the deformation process [21, p. 302], and this is the situation we consider here. An introduction to this field can be found in [21, Chapters 10–15].

Observed geological folds commonly display a certain degree of periodicity. Much of the initial work in this area, initiated by Biot in the late 1950's [1], centered on using the observed period to determine—by doing a parameter fit on the strut model—some of the material properties involved. In the 1970's, with the coming of powerful computational techniques, a consensus arose that folds can be formed in a sequential manner, as depicted by Figure 1.1 [5, 6]. The fold initiates around an

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imperfection, and as the applied shortening increases, the initial folds lock up and cease to grow, while new folds spawn at neighbouring locations. At a given time the resulting profile shows a periodic section flanked by decaying tails; as the shortening increases the periodic section widens. Similar examples of localization followed by spreading are found in axially loaded cylinders [14, 9], in sandwich structures [7], and in kink banding in layered materials [10]. The survey paper [7] discusses these examples from a common perspective.

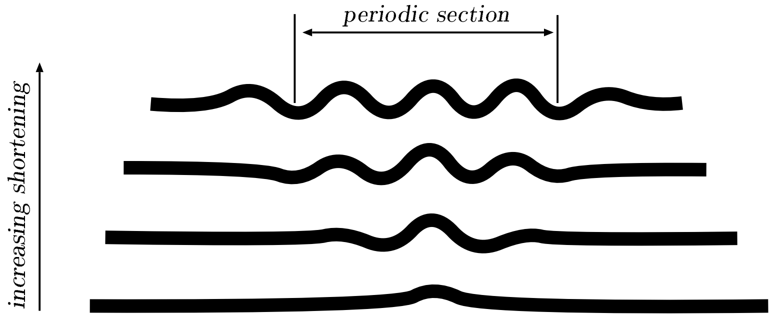

FIG. 1.1. *Folds can form in a sequential manner, driven by increasing shortening (schematic)*

1.2. The Modelling. In this paper we investigate the issues of localization and subsequent spreading of deformation for a model of an elastic strut confined by an elastic foundation. We will make a number of important simplifications, and therefore we now discuss the derivation of the equations in some detail.

Our starting point is a thin Euler strut (a strut whose cross-sections remain planar and orthogonal to the center line) of infinite length. Throughout the paper we assume a two-dimensional setting. The independent variable x measures arc length, and we characterize the configuration of the strut by the center-line angle $\theta = \theta(x)$. The strain energy associated with the bending of the strut is equal to $(EI/2)\int {\theta'}^2(x)dx$. E is Young's modulus and I is the moment of inertia of the cross-section.

The strut is assumed to rest on a foundation of Winkler type, as shown in Figure 1.2. The force response q of this foundation is a function of the local vertical displacement $u(x)$ only, i.e. $q(x) = f(u(x))$. Because of the local character of this response, the strain energy associated with the foundation is equal to $\int F(u(x))dx$, where $F' = f$, $F(0) = 0$. The vertical displacement u and the angle θ are related by $u'(x) = \sin\theta(x)$.

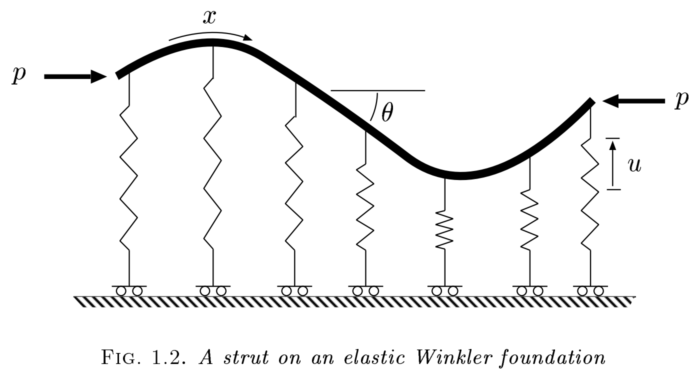

FIG. 1.2. *A strut on an elastic Winkler foundation*

After non-dimensionalization the total strain energy for the strut and its foundation is therefore given by

$$\mathcal{W}(\theta) = \frac{1}{2}\int_{-\infty}^{\infty} \theta'^2(x)\,dx + \int_{-\infty}^{\infty} F(u(x))\,dx.$$

where it is understood that $u' = \sin\theta$, $u(-\infty) = 0$. We also define the shortening of the strut, the amount the end points approach each other because of the deformation $\theta(\cdot)$:

$$\mathcal{J}(\theta) := \int_{-\infty}^{\infty} (1 - \cos\theta(x))\,dx.$$

In engineering it is common to differentiate between dead and rigid loading. In dead loading the external force acting on the structure (in Figure 1.2 the in-plane load p) is prescribed ('controlled' is the usual word, reflecting the possibility of a varying load). In rigid loading a load is applied, but the controlled parameter is the displacement (or some other measure of the deformation). Here the load plays the role of an implied quantity. The two forms of loading share the same equilibria, but the stability properties of these equilibria depend on the form of loading: as a general rule, localized buckles are unstable under dead loading, and stable under rigid loading. (An example of dead loading from daily life is a human being standing on a beer can. As soon as the buckle appears the can collapses completely, showing the instability of the localized buckle under dead loading. However, under rigid loading conditions a variety of localized buckles are witnessed [28]).

With this in mind, we minimize the strain energy $\mathcal{W}$ under a prescribed value λ of the total shortening, i.e. under the condition $\mathcal{J} = \lambda$. While this is a well-posed problem, and one that we intend to return to in subsequent publications, the nonlinearities present render the analysis difficult. We therefore consider a partial linearization of this problem instead, by assuming that u' is small and replacing

$$\theta' = \frac{u''}{\sqrt{1-u'^2}} \qquad \text{by} \qquad u'', \tag{1.1}$$

and

$$1 - \cos\theta = 1 - \sqrt{1-u'^2} \qquad \text{by} \qquad \frac{1}{2}u'^2. \tag{1.2}$$

(Note that in doing so we eliminate nonlinearities of a geometrical nature, but retain the nonlinearity in the function F, which is more of a material kind. We discuss this issue further in Section 7.) The resulting problem, the central problem in this paper, is:

Find a function $u \in H^2(\mathbb{R})$ that solves the minimization problem

$$\inf\{W(u) : J(u) = \lambda\} \tag{1.3a}$$

where the strain energy W and total shortening J are given by

$$W(u) = \frac{1}{2}\int u''^2 + \int F(u), \qquad \text{and} \qquad J(u) = \frac{1}{2}\int u'^2. \tag{1.3b}$$

A solution u satisfies the Euler-Lagrange equation

$$W'(u) - pJ'(u) = 0, \tag{1.4}$$

for some $p \in \mathbb{R}$, where primes denote Fréchet derivatives, which is equivalent to

$$u'''' + pu'' + f(u) = 0, \qquad \text{on } \mathbb{R}. \tag{1.5}$$

The Lagrange multiplier p is physically interpreted as the in-plane load that is required to enforce the prescribed amount of shortening. Without this load, i.e. when minimizing W without constraint, the sole minimizer would be the trivial state $u \equiv 0$.

Equation (1.5), for various forms of the nonlinearity f, has a history too lengthy to discuss in detail here. Suffice it to mention that it is known, among other names, as the stationary Swift-Hohenberg equation or the stationary Extended Fisher-Kolmogorov equation, and that it appears in a host of different applications. We refer the interested reader to the survey articles [2, 3, 18].

1.3. The nonlinearity F. The results of this paper depend in a very sensitive manner on the properties of F. In order to describe this we introduce some terminology. Recall that F itself is the potential energy associated with the foundation springs, $F'(u) = f(u)$ is the force associated with a deflection u, and F'' is the *marginal stiffness*.

In the engineering literature *destiffening* refers to a decrease in marginal stiffness, or in everyday language, a weakening of the material. For this model, destiffening refers to a decrease of $F''(u)$ as u moves away from zero (in either positive or negative direction).

The opposite of destiffening is stiffening, which applies to an increase in marginal stiffness as $|u|$ moves away from zero. Although we briefly dwell on such functions in the next section, a more interesting property is what we call *de/restiffening*, or *restiffening* for short: $F''(u)$ decreases for small $|u|$ and becomes increasing for large $|u|$. Throughout this paper we assume a fixed function of restiffening type:

$$F(u) = \frac{1}{2}u^2 - \frac{1}{4}u^4 + \frac{\alpha}{6}u^6, \qquad \alpha \geq \frac{1}{4}. \tag{1.6}$$

Besides the restiffening property this function also has some other desirable qualities, such as

- F is even;
- $F(u) > 0$ if $u \neq 0$;
- $uF'(u) \geq 0$.

We will return to these issues in Section 7, where we discuss in some detail the relationship between the results and the function F.

1.4. Results. In this paper we bring together a number of results concerning the minimization problem (1.3), (1.6).

The existence of solutions of the minimization problem (1.3) is not immediate, since the domain is unbounded and therefore minimizing sequences need not be compact. The nonlinearity F is crucial to this issue. To illustrate this, we mention that in the next section we show that a stiffening function F leads to non-existence:

> *if $2F(u)/u^2 > F''(0)$ for all $u \neq 0$, then minimizing sequences are never compact, and the infimum is not achieved.*

In the parlance of the beginning of this paper, minimizing sequences *de-localize* and spread out. In Section 2 we show how the restiffening property of (1.6) guarantees the existence of a minimizer.

The role of λ in Problem (1.3) is that of a pure parameter: properties of Problem (1.3) for one value of λ are completely decoupled from those for a different value. In addition, minimizers need not be unique. If we choose a minimizer for each value of λ, and denote it by u_λ, then these observations imply that the map $\lambda \mapsto u_\lambda$ may have no continuity properties whatsoever.

In fact, however, the situation is different. The numerical results in Section 5 indicate that there is a strong evolutionary aspect, in that the map $\lambda \mapsto u_\lambda$ is 'mostly' continuous. In addition, we prove in Section 3 that the evolution suggested by Figure 1.1 is essentially correct:

THEOREM. *For any sequence $\lambda_n \to \infty$, a subsequence $u_{\lambda_{n'}}$ converges, after an appropriate translation, to a periodic function $u_\#$. This convergence is uniform on bounded sets.*

The periodic function $u_\#$ solves a related variational problem (see Section 3). In Section 4 we discuss some symmetry properties of this function.

In Section 5 we introduce a numerical method to search for minimizers of (1.3), based on a constrained gradient flow. Figure 1.3 shows some of the results of this calculation. While the form of this curve is unusual at first sight, in Section 6 we present an interpretation of this curve in terms of a bifurcation diagram of a related problem (equation (1.5) for prescribed p). This interpretation, while non-rigorous, gives a satisfactory explanation and raises a few interesting questions as well. We conclude, in Section 7, with some comments on the choice of the nonlinearity F.

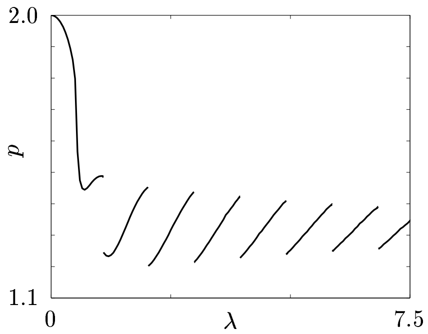

FIG. 1.3. *Plot of the load p_λ associated with a minimizer against λ*

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2. Existence of minimizers. The existence of minimizers of problem (1.3) is a non-trivial problem because of the potential lack of compactness on the unbounded domain $\mathbb{R}$. To illustrate this we consider the case of a completely stiffening function F, one for which $F''(u) > F''(0)$ if $u \neq 0$, or slightly more generally, one for which

$2F(u)/u^2 > F''(0)$ if $u \neq 0$. We then have for any $u \in H^2(\mathbb{R})$,

$$W(u) = \frac{1}{2}\int u''^2 + \int F(u) > \frac{1}{2}\int u''^2 + \frac{1}{2}\int u^2 \geq 2J(u) \tag{2.1}$$

where the final inequality follows from the observation that

$$\int u'^2 = -\int u''u \leq \frac{1}{2}\int u''^2 + \frac{1}{2}\int u^2.$$

We infer that for any $\lambda > 0$, we have $\inf\{W(u) : J(u) = \lambda\} \geq 2\lambda$, and for any given u this inequality is strict: $W(u) > 2J(u)$.

We now construct an explicit minimizing sequence u_n of problem (1.3) for this potential F. Set

$$u_n(x) = a_n e^{-x^2/n} \sin x, \qquad x \in \mathbb{R}, \tag{2.2}$$

where $a_n \in \mathbb{R}$ is chosen so that $J(u_n) = \lambda$. Note that $a_n \to 0$ as $n \to \infty$. An explicit calculation shows that $W(u_n) \to 2\lambda$; therefore, with the remarks made above in mind, we conclude that $\inf\{W(u) : J(u) = \lambda\} = 2\lambda$, and that the infimum is not attained.

Contained in the argument above is a snippet of information that we will use several times in the proofs that follow. For easy reference we make it a Lemma:

Lemma 2.1. *Let $I \subset \mathbb{R}$ be an interval, bounded or otherwise, and let $u \in H^2(I)$ be such that $uu' = 0$ on ∂I. Then*

$$2\int_I u'^2 \leq \int_I u''^2 + \int_I u^2.$$

As above, the proof follows by partial integration.

The Theorem below shows that, in contrast to the example above, the infimum is attained if F is not of completely stiffening type, but has a destiffening character for small u (i.e. $F''(u) < F''(0)$ for small $u \neq 0$). We can interpret the situation in the following way. A destiffening quality ($F'' < F''(0)$) favours localized deformation, therefore causing minimizing sequences to be compact, resulting in the existence of minimizers on unbounded domains. A stiffening potential favours de-localization, spreading, of the deformation, as illustrated by the minimizing sequence (2.2). If the two characters are combined, as in the potential (1.6), then the destiffening character for small u is sufficient to guarantee the existence of minimizers, regardless of the behaviour for large u. On the other hand, the restiffening character in F becomes noticeable for larger values of λ, in which an equilibrium between localizing and spreading effects creates a periodic structure. We will return to this issue in the next section.

Note that on a bounded interval, given appropriate boundary conditions, a minimizer always exists. One might wonder whether the problem would not be simplified by working on a bounded interval instead of on $\mathbb{R}$. In fact, we expect a strong correspondence between the (non-)existence of minimizers on $\mathbb{R}$ and the form of the minimizers on large but bounded intervals: if existence holds on $\mathbb{R}$, then minimizers on intervals will be localized and largely independent of the size of the interval; but if there is non-existence on $\mathbb{R}$, then minimizers on the interval will be spread out, with a small amplitude, similar to the sequence u_n above. (See [8] for a discussion of the purely stiffening nonlinearity on a bounded interval). From the point of view of the developments later in this paper, the current problem, with a restiffening foundation, is fundamentally different from the purely stiffening case. In addition, we will use the unbounded domain in the convergence result of Theorem 3.1 and in the comparison with a bifurcation diagram on $\mathbb{R}$ in Section 6. With this in mind we choose to consider the problem on the unbounded domain $\mathbb{R}$.

Throughout this paper we define

$$W_\lambda = \inf_{u\in C_\lambda} W(u), \qquad C_\lambda = \{u \in H^2(\mathbb{R}) : J(u) = \lambda\}.$$

THEOREM 2.2. *Let F be as given in (1.6). Then for each $\lambda > 0$ there exists $u \in C_\lambda$ that minimizes (1.3).*

Before we prove this theorem we derive some auxiliary properties.

LEMMA 2.3. *For all $\lambda > 0$,*

1. $W_\lambda < 2\lambda$.
2. *If $u_n \in \mathcal{C}_\lambda$ is a minimizing sequence of W, then*

$$\limsup_{n\to\infty} \|u_n\|_{L^\infty(\mathbb{R})} \le M\lambda$$

for some constant M.

Proof. Define the explicit sequence

$$u_\varepsilon(x) = a_\varepsilon \varepsilon^{1/2} \operatorname{sech}(\varepsilon x)\cos x$$

where a_ε is chosen such that $J(u_\varepsilon) = \lambda$ (note that $a_\varepsilon = O(1)$ as $\varepsilon \to 0$). This sequence satisfies $W(u_\varepsilon) \to 2\lambda$, implying $W_\lambda \le 2\lambda$. For the strict inequality we compute

$$\begin{aligned}\lambda &= \frac{1}{2}\int u_\varepsilon'(x)^2\,dx \\ &= \frac{1}{2}a_\varepsilon^2\left\{\varepsilon\int \operatorname{sech}^2(\varepsilon x)\sin^2 x\,dx - 2\varepsilon^2\int \operatorname{sech}(\varepsilon x)\operatorname{sech}'(\varepsilon x)\cos x\sin x\,dx\right.\\ &\qquad\left. + \,\varepsilon^3\int(\operatorname{sech}'(\varepsilon x))^2\cos^2 x\,dx\right\}. \end{aligned} \tag{2.3}$$

Note that

$$\begin{aligned}&\int \operatorname{sech}(\varepsilon x)\operatorname{sech}'(\varepsilon x)\cos x\sin x\,dx\\ &\qquad = \frac{1}{4\varepsilon}\int(\operatorname{sech}^2(\varepsilon x))'\sin 2x\,dx = \frac{1}{2\varepsilon}\int \operatorname{sech}^2(\varepsilon x)\cos 2x\,dx\\ &\qquad = \frac{1}{\varepsilon^2}\sqrt{\frac{\pi}{2}}\,\widehat{(\operatorname{sech}^2)}\left(\frac{2}{\varepsilon}\right),\end{aligned}$$

where $\hat{}$ denotes the Fourier transform

$$\hat v(\omega) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} v(x)e^{-i\omega x}\,dx.$$

Since $\operatorname{sech}^2 \in \mathcal{S}$, the set of smooth rapidly decreasing functions, we have $\widehat{(\operatorname{sech}^2)} \in \mathcal{S}$, and therefore

$$\int \operatorname{sech}(\varepsilon x)\operatorname{sech}'(\varepsilon x)\cos x\sin x\,dx = o(\varepsilon^k) \qquad \text{for all } k\in\mathbb{N}.$$

Using the same ideas to estimate the first and third terms in (2.3) we find

$$\int \operatorname{sech}^2(\varepsilon x)\sin^2 x\,dx = \frac{1}{2}\int \operatorname{sech}^2(\varepsilon x)\,dx + o(\varepsilon^k)$$
$$\int (\operatorname{sech}'(\varepsilon x))^2\cos^2 x\,dx = \frac{1}{2}\int (\operatorname{sech}'(\varepsilon x))^2\,dx + o(\varepsilon^k),$$

for all $k \in \mathbb{N}$. Consequently (2.3) implies

$$a_\varepsilon^2 = 4\lambda(1+O(\varepsilon^2))\left(\int_{\mathbb{R}} \operatorname{sech}^2(y)\,dy\right)^{-1}.$$

Using this we compute

$$\frac{1}{2}\int u_\varepsilon^2 = \lambda(1+O(\varepsilon^2)), \tag{2.4}$$
$$\int u_\varepsilon^4 = c_1\varepsilon(1+O(\varepsilon^2)), \tag{2.5}$$
$$\int u_\varepsilon^6 = c_2\varepsilon^2(1+O(\varepsilon^2)), \tag{2.6}$$
$$\frac{1}{2}\int {u_\varepsilon''}^2 = \lambda(1+O(\varepsilon^2)), \tag{2.7}$$

for some constants $c_1, c_2 > 0$. For the last equality above we apply the same argument as for $\int {u_\varepsilon'}^2$ to eliminate the cross-product terms. Uniting these estimates we conclude that

$$W(u_\varepsilon) = \lambda(2+O(\varepsilon^2)) - c_1\varepsilon,$$

and hence

$$\inf_{\mathcal{C}_\lambda} W(u) < 2\lambda.$$

For part 2 we first note that since $\alpha > 3/16$, there exists $\beta > 0$ such that

$$F(u) \geq \frac{\beta}{2}u^2 \qquad \text{for} \qquad u \in \mathbb{R}. \tag{2.8}$$

By part 1 we can restrict our attention to minimizing sequences that satisfy $W(u_n) \leq 2\lambda$; we have

$$\|u_n\|^2_{L^\infty(\mathbb{R})} \leq C\,\|u_n\|^2_{H^1(\mathbb{R})} \leq 2C\max\{1, 1/\beta\}\,W(u_n) \leq 4C\lambda\max\{1,1/\beta\}. \qquad \square$$

Remark 2.1. The proof of part 1 of the Lemma above uses the relative importance of the destiffening quartic term: the destiffening is of order ϵ, while the 'noise' associated with the non-constant amplitude in u_n is of order ϵ^2 (as shown by the estimates (2.4) and (2.7)). It follows that for a destiffening character of higher order, e.g. a function F of the type $u^2/2 - u^8/8 + \alpha u^{10}/10$, this method of proof does not apply, since the destiffening will be dwarfed by the noise. However, numerical tests have shown that for such functions F the minimization problem still admits solutions, and that the assertion of the Lemma still holds.

Corollary 2.4. *Let u_n be a minimizing sequence for problem (1.3). Then*

$$\liminf_{n\to\infty} \|u_n\|_{L^\infty(\mathbb{R})} = m(\lambda) > 0.$$

Proof. If $\|u_n\|_{L^\infty(\mathbb{R})} \to 0$, then

$$\frac{\frac{1}{2}\int u_n''^2 + \frac{1}{2}\int u_n^2}{W(u_n)} \to 1 \qquad \text{as} \quad n\to\infty. \tag{2.9}$$

Since

$$2\lambda = \int {u_n'}^2 = -\int u_n'' u_n \le \frac{1}{2}\int u_n''^2 + \frac{1}{2}\int u_n^2,$$

we infer from (2.9) that $\liminf W(u_n) \ge 2\lambda$, which contradicts part 1 of Lemma 2.3. □

In addition we need an a priori result on minimizers, which is proved in the appendix:

Lemma 2.5. *Let $u \in H^2(\mathbb{R})$ be a solution of (1.3). Then $p < 2$.*

We now continue with the proof of the main theorem of this section.

Proof of Theorem 2.2. The proof follows quite closely the outline of the examples given in [12, 13]. Let u_n be a minimizing sequence, and consider $\rho_n = {u_n'}^2/2$, so that $\rho_n \ge 0$ and $\int \rho_n = 1$. Of the three possibilities for this sequence, vanishing, dichotomy, and compactness, we show that neither vanishing nor dichotomy can occur, leaving compactness as the only possibility.

Vanishing can not occur. Suppose that

$$\sup_x \int_{x-R}^{x+R} {u_n'}^2 \to 0 \qquad \text{for all} \qquad R > 0.$$

We can choose $x = 0$ as the location of a maximum of each $|u_n|$, and by Corollary 2.4 we then have $u_n(0) \ge m(\lambda) > 0$ (changing u_n into $-u_n$ if necessary). Consequently

$$\liminf_{n\to\infty} \int_{\mathbb{R}} F(u_n) \ge \liminf_{n\to\infty} \frac{\beta}{2}\int_{-R}^{R} u_n^2 \ge \beta m(\lambda)^2 R$$

which is unbounded as $R \to \infty$. This contradicts $\limsup W(u_n) < 2\lambda$.

Dichotomy can not occur. For any given $\lambda > 0$, dichotomy is contradicted, proving compactness of the minimizing sequence and therefore existence of a minimizer, if

$$W_\lambda < W_{\theta\lambda} + W_{(1-\theta)\lambda} \tag{2.10}$$

for all $\theta \in (0,1)$ (see [12, 13]). We shall show that (2.10) holds for all $\lambda > 0$ and $\theta \in (0,1)$.

Define

$$A = \{\mu > 0 : (1.3) \text{ has a solution for all } 0 < \lambda \le \mu\}.$$

First we show that A is non-empty. There exist $\bar{u}, \delta > 0$ such that $2F(u) - uf(u) \ge \delta u^4$ for all $|u| \le \bar{u}$. Choose λ_0 small enough to ensure that $2M\lambda \le \bar{u}$ for all $0 < \lambda < \lambda_0$, and pick $0 < \lambda < \lambda_0$. Let v_n be a minimizing sequence such that $J(v_n) = \lambda$; without loss of generality we suppose that $\|v_n\|_{L^\infty(\mathbb{R})} \leq \bar{u}$. Then

$$\begin{aligned}\frac{d}{d\mu}\frac{W(\mu v_n)}{J(\mu v_n)}\bigg|_{\mu=1} &= \frac{1}{J(v_n)^2}\left(J(v_n)W'(v_n)v_n - W(v_n)J'(v_n)v_n\right)\\ &= \frac{1}{J(v_n)}\left(\int v_n f(v_n) - 2\int F(v_n)\right)\\ &\leq -\frac{\delta}{\lambda}\int v_n^4. \end{aligned} \tag{2.11}$$

Since $\|v_n'\|^2_{L^2(\mathbb{R})} = 2\lambda$ is bounded and $\|v_n\|_{L^\infty(\mathbb{R})} > m(\lambda)$, the last term above is bounded away from zero as $n \to \infty$. Therefore W_λ/λ is a strictly decreasing function of λ for $0 < \lambda < \lambda_0$; this shows that $(0, \lambda_0) \subset A$, since we have for any $\theta \in (0,1)$

$$\begin{aligned}W_\lambda = \lambda\frac{W_\lambda}{\lambda} &< \lambda\left(\theta\frac{W_{\theta\lambda}}{\theta\lambda} + (1-\theta)\frac{W_{(1-\theta)\lambda}}{(1-\theta)\lambda}\right)\\ &= W_{\theta\lambda} + W_{(1-\theta)\lambda}.\end{aligned}$$

To show that A is open, suppose that there exists a sequence $\lambda_n + \varepsilon_n \notin A$, $\lambda_n + \varepsilon_n \downarrow \lambda$, $\lambda \in A$, and $\varepsilon_n \to 0$, such that

$$W_{\lambda_n+\varepsilon_n} = W_{\lambda_n} + W_{\varepsilon_n}.$$

Since $\liminf W_{\varepsilon_n}/\varepsilon_n = 2$—by an argument similar to that of Corollary (2.4)—this implies

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}(W_{\lambda+\varepsilon} - W_\lambda) \geq 2. \tag{2.12}$$

But, since $\lambda \in A$, there exists $u \in \mathcal{C}_\lambda$ with $W(u) = W_\lambda$, and by Lemma 2.5 the associated load satisfies $p < 2$. Then

$$\frac{d}{d_\mu}W(\mu u)\bigg|_{\mu=1} = W'(u)u = pJ'(u)u = p\,\frac{d}{d_\mu}J(\mu u)\bigg|_{\mu=1}.$$

Consequently

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}(W_{\lambda+\varepsilon} - W_\lambda) \leq p < 2,$$

contradicting (2.12).

Finally, we show that A is closed by the following claim: If u and v are minimizers of W at the respective values of λ, then

$$\inf\{W(z) : J(z) = J(u) + J(v)\} < W(u) + W(v). \tag{2.13}$$

This proves that A is closed by the following argument. Suppose $A \supset (0, \lambda_0)$; for all $\theta \in (0,1)$ we have functions u and v that minimize W under the constraints $J(u) = \theta\lambda_0$ and $J(v) = (1-\theta)\lambda_0$. Inequality (2.13) then reduces to (2.10), implying that problem (1.3) also has a solution for $\lambda = \lambda_0$.

To prove (2.13) we choose two sequences $x_n \geq 0$, $y_n \leq 0$, with $x_n \to \infty$ and $y_n \to -\infty$ with certain properties detailed below. We introduce a notation for integrals over a part of $\mathbb{R}$:

$$W_{[a,b]}(u) = \frac{1}{2}\int_a^b u''^2 + \int_a^b F(u) \qquad \text{and} \qquad J_{[a,b]}(u) = \frac{1}{2}\int_a^b u'^2.$$

Setting $p = \max\{p_u, p_v\}$ (the maximum of the two values of the load associated with u and v) we require of the sequences x_n, y_n that there exists an $\varepsilon > 0$ such that

$$p \leq \min\left\{\frac{W_{[x_n,\infty)}(u)}{J_{[x_n,\infty)}(u)}, \frac{W_{(-\infty,y_n]}(v)}{J_{(-\infty,y_n]}(v)}\right\} - \varepsilon \qquad \text{for all } n.$$

This is possible since u and v are small at infinity, and therefore

$$\limsup W_{[x_n,\infty)}(u)/J_{[x_n,\infty)}(u) \geq 2.$$

In addition we assume that $(u,u')(x_n) = (v,v')(y_n)$ for all n. This is also possible since for large K the set $\{(u,u')(x) : x > K\} \subset \mathbb{R}^2$ is a spiral around the origin. The same is true for $\{(v,v')(x) : x < -K\}$; but for v the spiral rotates in the opposite direction. It follows that there is a countably infinite set of intersections of the two spirals, corresponding to pairs (x_n, y_n) with $x_n \to \infty$, $y_n \to -\infty$.

Now pick $\hat{u}, \hat{v} \in H^2(\mathbb{R})$ such that $\operatorname{supp}\hat{u} \subset (-\infty, 0)$ and $\operatorname{supp}\hat{v} \subset (0,\infty)$, and that in addition $J'(u)\hat{u} = J'(v)\hat{v} = 1$. Define

$$z_n(x) = \begin{cases} u(x_n + x) + \gamma_n \hat{u}(x_n + x) & \text{for } x < 0 \\ v(y_n + x) + \gamma_n \hat{v}(y_n + x) & \text{for } x > 0. \end{cases}$$

Here γ_n is fixed by the requirement $J(z_n) = J(u) + J(v)$:

$$\begin{aligned} J(z_n) = \frac{1}{2}\int_{-\infty}^{x_n} u'^2 + \gamma_n \int_{-\infty}^{x_n} u'\hat{u}' + \frac{\gamma_n^2}{2}\int_{-\infty}^{x_n} \hat{u}'^2 \\ + \frac{1}{2}\int_{y_n}^{\infty} v'^2 + \gamma_n \int_{y_n}^{\infty} v'\hat{v}' + \frac{\gamma_n^2}{2}\int_{y_n}^{\infty} \hat{v}'^2. \end{aligned}$$

Since $\operatorname{supp}\hat{u} \cap [x_n, \infty) = \emptyset$,

$$\gamma_n \int_{-\infty}^{x_n} u'\hat{u}' = \gamma_n \int_{-\infty}^{\infty} u'\hat{u}' = \gamma_n J'(u)\hat{u} = \gamma_n,$$

so that

$$\begin{aligned} J(z_n) &= \frac{1}{2}\int_{-\infty}^{x_n} u'^2 + \frac{1}{2}\int_{y_n}^{\infty} v'^2 + 2\gamma_n + C\gamma_n^2 \\ &= J(u) + J(v) - J_{[x_n,\infty)}(u) - J_{(-\infty,y_n]}(v) + 2\gamma_n + C\gamma_n^2, \end{aligned}$$

where $C = (1/2)\int(\hat{u}'^2 + \hat{v}'^2)$. It follows that γ_n satisfies

$$\gamma_n = \frac{1}{2}J_{[x_n,\infty)}(u) + \frac{1}{2}J_{(-\infty,y_n]}(v) - \frac{C}{2}\gamma_n^2$$

as $n \to \infty$. Note that $\gamma_n \to 0$.

Putting all together,

$$\begin{aligned}
W(z_n) &= W_{(-\infty,x_n]}(u+\gamma_n\hat{u}) + W_{[y_n,\infty)}(v+\gamma_n\hat{v}) \\
&= W(u+\gamma_n\hat{u}) + W(v+\gamma_n\hat{v}) - W_{[x_n,\infty)}(u+\gamma_n\hat{u}) - W_{(-\infty,y_n]}(v+\gamma_n\hat{v}) \\
&= W(u) + W(v) + \gamma_n(W'(u)\hat{u} + W'(v)\hat{v}) + O(\gamma_n^2) - W_{[x_n,\infty)}(u) \\
&\qquad - W_{(-\infty,y_n]}(v) \\
&= W(u) + W(v) + \gamma_n(p_u+p_v) + O(\gamma_n^2) - W_{[x_n,\infty)}(u) - W_{(-\infty,y_n]}(v) \\
&\le W(u) + W(v) + 2\gamma_n p - (p+\varepsilon)(J_{[x_n,\infty)}(u) + J_{(-\infty,y_n]}(v)) + O(\gamma_n^2) \\
&\le W(u) + W(v) - 2\varepsilon\gamma_n + O(\gamma_n^2).
\end{aligned}$$

This last inequality proves the claim (2.13) and therefore Theorem 2.2. ∎

The definition of W_λ provides no explicit continuity properties with respect to variation of λ. However, the variational character can be exploited to derive an interesting semi-convexity property.

LEMMA 2.6. *There exists $C > 0$ such that*

$$\frac{d^2}{d\lambda^2} W_\lambda \le \frac{C}{\lambda} \qquad \text{for all} \qquad \lambda > 0,$$

in the sense of distributions.

Proof. Note that $u^2 f'(u) - uf(u) \le 24F(u)$ for all $u \in \mathbb{R}$. Choose $\lambda > 0$, and let u achieve W_λ. Setting $v_h = u\sqrt{1+h/\lambda}$, we have $J(v_h) = \lambda + h$ and

$$\begin{aligned}
\frac{d^2}{dh^2} W(v_h)\Big|_{h=0} &= \frac{1}{4\lambda^2}\left\{W''(u)\cdot u\cdot u - W'(u)\cdot u\right\} \\
&= \frac{1}{4\lambda^2}\int (u^2 f'(u) - uf(u)) \\
&\le \frac{6W(u)}{\lambda^2} \le \frac{12}{\lambda}.
\end{aligned}$$

This implies the result. ∎

Lemma 2.6 implies that the left and right derivatives of W_λ with respect to λ are well defined. Note that the Euler-Lagrange equation (1.4) implies that if W_λ is achieved at $\lambda = \lambda_0$ by u_{λ_0}, with load p_{λ_0}, then $\partial W_\lambda/\partial\lambda(\lambda_0-) \ge p_{\lambda_0} \ge \partial W_\lambda/\partial\lambda(\lambda_0+)$. It follows that any jumps in p_λ must be downward (for increasing λ).

3. Appearance of a periodic section. In the introduction we mentioned the locking-up and spreading of the deformation as the shortening increases. If this process is continued, we expect a periodic section to build up, flanked by spreading tails. The following theorem makes this precise for the model considered in this paper.

THEOREM 3.1. *For any sequence $\lambda_n \to \infty$, a subsequence $u_{\lambda_{n'}}$ converges, after an appropriate translation, to a periodic function $u_\#$. This convergence is in $C^k(K)$ for all $k \ge 0$ and for all compact sets $K \subset \mathbb{R}$. The periodic function $u_\#$ solves the minimization problem*

$$M_\# = \inf\left\{\frac{W(u)}{J(u)} : u \in H^2_{\mathrm{loc}}(\mathbb{R}) \text{ periodic}\right\} \tag{3.1}$$

In addition, as $\lambda_{n'} \to \infty$, $p_{\lambda_{n'}} \to M_\#$.

In the formulation of this theorem, as in the rest of this paper, the functionals W and J will be defined on periodic functions $u \in H^2_{\rm loc}(\mathbb{R})$ by restricting the integrals to a period and normalizing, i.e. if u has period T, then

$$W(u) = \frac{1}{2T}\int_0^T u''^2 + \frac{1}{T}\int_0^T F(u).$$

Before entering the details of the proof, we should briefly comment on the appearance of the new minimization problem (3.1). If u minimizes W/J among all periodic functions, then by choosing periodic test functions $\phi \in H^2_{\rm loc}(\mathbb{R})$ with the same period and considering the perturbations $u + \varepsilon\phi$ we derive the Euler-Lagrange equation

$$0 = \frac{1}{J(u)}\left\{W'(u)\cdot\phi - \frac{W(u)}{J(u)}\,J'(u)\cdot\phi\right\}.$$

Comparison with equation (1.4) shows that u solves the same ODE as u_λ, and the load is numerically equal to the optimal quotient $W(u)/J(u) = M_\#$.

We conjecture that the solution of (3.1) is unique for the function F that we consider in this paper. However, it is not difficult to construct a different function F for which uniqueness does not hold. (One could construct of a function F which is identical to (1.6) over the range of $u_\#$, but is different for (much) larger values of $|u|$. Then $u_\#$ remains a local minimum for the minimization problem (3.1), but an additional minimum may exist with a much larger amplitude. By adjusting F this function can be given the same value of the ratio W/J as $u_\#$).

Proof. The proof falls apart in five steps.

Step 1. $\limsup_{\lambda\to\infty} W_\lambda/\lambda \le M_\#$. Indeed, if v is a periodic function, then $v_\lambda(x) = \eta(|x| - \mu)v(x)$ belongs to C_λ for some $\mu = \mu(\lambda)$. Here η is a smooth cut-off function satisfying

$$\eta(x) = \begin{cases} 1 & x \le 0 \\ 0 & x \ge 1. \end{cases}$$

Then $W(v_\lambda)/\lambda = W(v_\lambda)/J(v_\lambda) \to W(v)/J(v)$ as $\lambda \to \infty$; therefore

$$\limsup_{\lambda\to\infty} \frac{W_\lambda}{\lambda} \le \limsup_{\lambda\to\infty} \frac{W(v_\lambda)}{\lambda} = \frac{W(v)}{J(v)},$$

from which it follows that $\limsup_{\lambda\to\infty} W_\lambda/\lambda \le M_\#$.

Step 2. *Translation of u_λ and construction of a periodic function w_λ.*

We first note that by the assumption $\alpha \ge 1/4$ the nonlinearity F is increasing in $|u|$. This implies that $p \ge 0$ by the equation (obtained by multiplying (1.5) by u and integrating):

$$p\int u'^2 = \int u''^2 + \int uf(u). \tag{3.2}$$

As a result the origin is a saddle-focus for equation (1.5) (when viewed as a dynamical system in x), and orbits in the stable and unstable manifold oscillate around zero.

For a given λ we divide $\mathbb{R}$ into intervals $[x_i, x_{i+1})$ delimited by the stationary points x_i of u_λ. Note that the oscillation mentioned above implies that none of the intervals $[x_i, x_{i+1})$ is unbounded. We calculate the ratio r_i of the local values of W and J for each of these intervals,

$$r_i = \frac{\frac{1}{2}\int_{x_i}^{x_{i+1}} u_\lambda''^2 + \int_{x_i}^{x_{i+1}} F(u_\lambda)}{\frac{1}{2}\int_{x_i}^{x_{i+1}} u_\lambda'^{\,2}}.$$

For large $|x|$, $F(u_\lambda) \sim u_\lambda^2/2$, and therefore $\liminf_{i\to\pm\infty} r_i \geq 2$. Since $W(u_\lambda)/\lambda$ is a convex combination of $\{r_i\}$,

$$\frac{W(u_\lambda)}{\lambda} = \sum_{i\in\mathbb{Z}} \frac{r_i}{2\lambda}\int_{x_i}^{x_{i+1}} u_\lambda'^{\,2}, \qquad \sum_{i\in\mathbb{Z}} \frac{1}{2\lambda}\int_{x_i}^{x_{i+1}} u_\lambda'^{\,2} = \frac{J(u_\lambda)}{\lambda} = 1,$$

and since $W(u_\lambda)/\lambda < 2$, there exists $i \in \mathbb{Z}$ such that r_i is minimal among all r_i, and for this i we have $r_i < 2$. Fixing i we translate u_λ such that the interval $[x_i, x_{i+1})$ becomes $[0, T/2)$. The periodic function w_λ, with period T, is now defined to be equal to u_λ on $[0, T/2)$, and to be even around 0 and around $T/2$, as shown in Figure 3.1. Note that by the choice of i we have

$$\frac{W(w_\lambda)}{J(w_\lambda)} = r_i < \frac{W(u_\lambda)}{\lambda}. \tag{3.3}$$

Remark also that this inequality implies that any localized function has a ratio W/J that is strictly larger than $M_\#$. To indicate the dependence of T on λ we write T_λ.

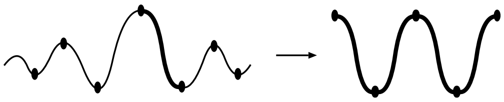

FIG. 3.1. *A section between two stationary points is replicated*

Step 3. $\lim_{\lambda\to\infty} W(w_\lambda)/J(w_\lambda) = M_\#$. This follows from the sequence of inequalities

$$\begin{aligned} M_\# &\leq \liminf_{\lambda\to\infty} \frac{W(w_\lambda)}{J(w_\lambda)} \leq \limsup_{\lambda\to\infty} \frac{W(w_\lambda)}{J(w_\lambda)} \\ &\leq \limsup_{\lambda\to\infty} \frac{W(u_\lambda)}{\lambda} \leq M_\#. \end{aligned}$$

Step 4. *The sequence $\{u_\lambda\}$ is bounded in $H^4_{\rm loc}(\mathbb{R})$, and the sequence $\{w_\lambda\}$ is bounded in $H^2_{\rm loc}(\mathbb{R})$.* This result depends crucially on the destiffening-restiffening character of F via the Lemma

LEMMA 3.2. *Fix $K \in \mathbb{R}$. There exists $M > 0$ such that if $p \leq K$ and $u \in L^\infty(\mathbb{R})$ solves equation (1.5), then $\|u\|_{L^\infty(\mathbb{R})} \leq M$.*

Note that the order of the quantifiers is important: the lemma states that if u is bounded, then it is bounded by a constant independent of u and p (subject to $p \leq K$). We defer the proof of this lemma to the end of this section.

Since the functions u_λ satisfy equation (1.5) and $p_\lambda < 2$, the sequence $\{u_\lambda\}$ is bounded in $L^\infty(\mathbb{R})$. Standard elliptic estimates (e.g. [23, Theorem 11.1]) then give the boundedness of $\{u_\lambda\}$ in H^4 on compact sets. Since the cut-and-paste operation by which w_λ is constructed does not conserve H^4-regularity, the functions w_λ only enjoy the same regularity properties up to H^2-regularity.

As a consequence of the H^4-boundedness, u'_λ, u''_λ, and u'''_λ are all bounded in $L^\infty(\mathbb{R})$ independently of λ; additionally T_λ is bounded from below, since if $T_\lambda \to 0$, then by the bound on u'_λ, $\|u_\lambda\|_{L^\infty(0,T_\lambda)} = \|w_\lambda\|_{L^\infty(\mathbb{R})} \to 0$, so that we have $\liminf W(w_\lambda)/J(w_\lambda) \geq 2$. This contradicts (3.3).

Step 5. *Convergence.* Since w_λ is bounded in H^2_{loc} uniformly in λ, we can choose a sequence that converges weakly in $H^2_{\mathrm{loc}}(\mathbb{R})$ to a limit function w_∞.

1. If T_λ is bounded along this sequence, then—possibly after extracting a subsequence—T_λ and $J(w_\lambda)$ converge, and w_∞ is periodic with a finite period. The weak convergence implies that $W(w_\infty) \leq \liminf W(w_\lambda)$, so that w_∞ is a solution of the minimization problem (3.1).

2. If T_λ is unbounded, then note that w_λ and u_λ have the same weak limit w_∞. We choose a subsequence such that p_λ, which is bounded between 0 and 2, converges. The weak convergence of u_λ in H^4_{loc} implies that w_λ satisfies equation (1.5) with limit load p_∞. This load lies necessarily between 0 and 2; this implies, as above, that solutions tending to zero oscillate around zero, contradicting the monotonicity of w_λ. We conclude that case 2 does not occur.

If we pick $\delta > 0$ such that $(0,\delta)$ is included in $(0, T_\lambda/2)$ for all λ, and $\phi \in C_c^\infty((0,\delta))$, then

$$p_\lambda = \frac{W'(u_\lambda)\cdot\phi}{J'(u_\lambda)\cdot\phi} = \frac{W'(w_\lambda)\cdot\phi}{J'(w_\lambda)\cdot\phi} \to \frac{W'(w_\infty)\cdot\phi}{J'(w_\infty)\cdot\phi}$$

by the weak convergence of w_λ. By the remark made before the beginning of the proof, the fact that w_∞ minimizes the ratio W/J among all periodic functions implies that w_∞ also satisfies equation (1.5) with $p = M_\#$. Therefore $p_\lambda \to M_\#$.

Step 6. *Conclusion.* The functions u_λ and w_∞ solve the same differential equation (1.5), for loads p_λ and $M_\#$ that satisfy $p_\lambda \to M_\#$. We have $u_\lambda \rightharpoonup w_\infty$ in $H^2(0,\delta)$; using standard elliptic theory it follows that u_λ converges to w_∞ in $C^k(0,\delta)$ for all $k \in \mathbb{N}$. The classical result of continuous dependence on initial data then extends this to any compact set K. This concludes the proof of the Theorem. ◻

We end this section with the proof of Lemma 3.2.

Proof. We first prove the lemma under the condition $|p| \leq K$. Suppose that $p_n \in [-K, K]$ and u_n satisfy equation (1.5), with $\|u_n\|_{L^\infty(\mathbb{R})} \to \infty$. Set $\gamma_n = \|u_n\|^{-1}_{L^\infty(\mathbb{R})}$, so that $\gamma_n \to 0$, and define

$$v_n(x) = \gamma_n u_n(\gamma_n x).$$

Then

$$v_n'''' + p_n\gamma_n^2 v_n'' + \gamma_n^4 v_n - \gamma_n^2 v_n^3 + \alpha v_n^5 = 0.$$

Since v_n is uniformly bounded, classical elliptic estimates (e.g., [23]) imply that $v_n \rightharpoonup v_\infty$ in $H^4_{\mathrm{loc}}(\mathbb{R})$, after extraction of a subsequence. The limit v_∞ therefore satisfies the equation

$$v_\infty'''' + \alpha v_\infty^5 = 0 \qquad \text{on } \mathbb{R} \tag{3.4}$$

which has no non-zero bounded solution (see e.g. [19]). This contradicts the fact that $\|v_n\|_{L^\infty(\mathbb{R})} = 1$.

If we release the lower bound on p, and assume that $p_n \to -\infty$, then we define in addition

$$\delta_n = \max\{\gamma_n, |p_n|^{1/2} \gamma_n^2\}$$

and

$$v_n(x) = \gamma_n u_n(\delta_n x).$$

Since v'''' and pv'' are both positive operators if $p < 0$, the unboundedness of p is irrelevant for the elliptic estimates. The limit equation is

$$\overline{\gamma} v_\infty'''' - \overline{\delta} v_\infty'' + v_\infty^5 = 0 \qquad \text{on } \mathbb{R},$$

where $\overline{\gamma}, \overline{\delta} \in [0,1]$ and $\overline{\gamma} + \overline{\delta} \neq 0$. For none of the possible combinations of $\overline{\gamma}$ and $\overline{\delta}$ does this equation have a bounded non-zero solution. $\square$

4. The periodic function $u_\#$. In the previous section we showed that there exists a solution $u_\#$ to the variational problem

$$M_\# = \inf \left\{ \frac{W(u)}{J(u)} : u \in H^2_{\text{loc}}(\mathbb{R}) \text{ periodic} \right\}$$

and that it is the limit, on compact sets, of solutions u_λ of Problem (1.3). In this section we discuss a number of issues concerning this periodic function $u_\#$.

4.1. Critical buckling load. Going back to the model of an axially loaded strut, let us briefly examine the behaviour under dead loading, rather than rigid loading; i.e. we fix the load p and seek an associated response. The appropriate energy for this loading situation is [24, p. 50]

$$\mathcal{L}(u) = W(u) - pJ(u), \tag{4.1}$$

which is often called the total potential or the Lagrangian. Note that equilibria of $\mathcal{L}$ again satisfy equation (1.4); both dead and rigid loading lead to the same equilibria, but the stability properties differ.

For small values of p, $\mathcal{L}$ is a positive definite function of u, and the trivial response, $u \equiv 0$, is the unique global minimizer. When p passes a threshold value there will be profiles with a negative Lagrangian, so that the zero response is no longer optimal, and can be improved upon by a non-zero deflection. Thus we can define a critical load p_c, such that

$$\inf_{u \in H^2(\mathbb{R})} \mathcal{L}(u) = 0 \qquad \text{if } p < p_c$$
$$\inf_{u \in H^2(\mathbb{R})} \mathcal{L}(u) < 0 \qquad \text{if } p > p_c.$$

Note that if $\inf \mathcal{L}(u) < 0$, then in fact $\inf \mathcal{L}(u) = -\infty$, by replication of an appropriate function u.

An alternative, but equivalent, way of representing the statements above is

$$p_c = \inf_{u \in H^2(\mathbb{R})} \frac{W(u)}{J(u)}.$$

Here the connection with the previous section becomes clear.

4.2. Symmetry of the minimizer. Variational problems very similar to that of $\inf \mathcal{L}$ arise in the study of polymeric materials under tension [11, 16]. It is interesting to note that the concept of a critical load (p_c), that has its origin in a mechanical viewpoint, is mirrored very closely by the ideas presented in [11], notably Theorem 6.1.

While the settings of [11, 16] are slightly different from the current one, some of the proofs carry over immediately. By adapting Lemmas 3.3 and 3.6 of [16] we find

LEMMA 4.1 ([16]).

1. $u_\#$ is even about any critical point;

2. if $u_\#$ has a zero, then it is odd about this zero.

As for the condition that $u_\#$ have zeros, this is easily proved:

LEMMA 4.2. *$u_\#$ has a zero.*

Proof. Suppose that $u_\# > 0$ on $\mathbb{R}$. For $\mu > 1$, define $v_\mu = \max u_\# - \mu((\max u_\#) - u_\#)$. Then $\int v_\mu''^2 = \mu^2 \int u_\#''^2$, $\int v_\mu'^2 = \mu^2 \int u_\#'^2$, and $\int F(v_\mu) \le \int F(u_\#)$ provided $v_\mu \ge 0$ (recall that $F'(u) \ge 0$ if $u \ge 0$). Therefore

$$\frac{W(v_\mu)}{J(v_\mu)} \le \frac{\frac{\mu^2}{2}\int u_\#''^2 + \int F(u_\#)}{\frac{\mu^2}{2}\int u_\#'^2} < \frac{W(u_\#)}{J(u_\#)},$$

which contradicts the minimality of $u_\#$. □

In summary, $u_\#$ is both odd and even.

5. Numerical computation of minimizers.

5.1. Procedure. The computation of global minimizers in a non-convex setting suffers from the potential existence of a large number of local minimizers. The problem at hand—that of minimizing W for prescribed values of J—appears to be particularly demanding from this point of view, since the associated Euler-Lagrange equation (1.5) is expected to have a large number of homoclinic solutions. Champneys and Toland [4] showed the existence of a multitude of homoclinic orbits bifurcating from $p = -2$ for a related problem ($\alpha = 0$), which they numerically tracked into the $p > 0$ domain. These orbits are 'multi-modal', 'repeated' versions of a primary orbit. In addition the existence of many 'multi-bump' homoclinics has been shown, which consist of N copies of a given homoclinic, separated by large distances.

However, there is evidence that many of these homoclinic orbits are not constrained minimizers. There is a folk theorem, that received some backup in [22], that local stability under rigid loading is related to the change of J along an equilibrium path: if J decreases, then the solution is stable, and it is unstable otherwise. This would disqualify many equilibria off-hand. For the multi-bump homoclinics an additional argument suggests that they can never be stable (see again [22]). Based on this circumstantial evidence, we conjecture that the number of constrained local minimizers is in fact very limited. The numerical evidence of this section supports this conjecture, and we shall return to a further discussion of the issue in Section 6.

We therefore adopt the following procedure to seek a global minimizer of problem (1.3) for given λ. Starting from quasi-random initial data (satisfying $J = \lambda$) we solve the constrained gradient flow problem

$$u_t = -u_{xxxx} - p u_{xx} - f(u), \qquad x \in \mathbb{R},\ t > 0 \tag{5.1}$$

$$J(u(\cdot, t)) = \lambda, \qquad t > 0. \tag{5.2}$$

Here $p = p(t)$ is a priori unknown, and is determined as part of the solution. This problem has a strictly decreasing Lyapunov function (the functional W), and converges rapidly to a stationary solution, that we assume to be a local minimizer. By repeating this process for a 'large' number of different random initial data we collect a number of local minimizers. We select the solution with the lowest value of W as the global minimizer of W under the condition $J = \lambda$.

For the computation of solutions of the constrained dynamical system (5.1-5.2) we restrict the problem to a finite domain $(-L, L)$, with L suitably large, and impose the boundary conditions of a simply supported beam ($u(\pm L) = u_{xx}(\pm L) = 0$). An equivalent variational formulation follows by multiplying the equation by a test function v with $v(\pm L) = 0$ and integrating:

$$\int_{-L}^{L} u_t v \, dx + \int_{-L}^{L} u_{xx} v_{xx} \, dx - p \int_{-L}^{L} u_x v_x \, dx + \int_{-L}^{L} f(u) v \, dx = 0. \tag{5.3}$$

We now determine an approximation to $u(x,t)$ by using the finite-element method to give a semi-discretization of (5.3) [25]. To do this we approximate $u(x,t)$ by the function $U_h(x,t) = \sum U_i(t)\phi_i(x) + \sum U_{xi}(t)\psi_i(x)$. Here ϕ_i and ψ_i are piecewise cubic functions defined on a uniform mesh of spacing $h := 2L/N$ so that

$$\phi_i(-L + jh) = \psi_i'(-L + jh) = \delta_{ij} \qquad \text{and} \qquad \psi_i(-L + jh) = \phi'(-L + jh) = 0 \quad \text{for } i, j = 0 \ldots N.$$

The space S_h is the span of the functions ϕ_i ($i = 1, \ldots, N-1$) and ψ_i ($i = 0, \ldots, N$) (such that the imposed boundary condition $u = 0$ is incorporated into the solution space). We set $U \in \mathbb{R}^{2N}$ equal to $U = U(t) = (U_1, \ldots, U_{N-1}, U_{x0}, \ldots, U_{xN})$. Now we require that U_h should satisfy (5.3) for all functions $V \in S_h$. Setting $V = \phi_i$ or $V = \psi_i$ this leads to the following system of Ode's for U and P:

$$AU_t + BU - PCU + D = 0, \tag{5.4}$$

where the $2N \times 2N$ matrices A, B, and C are given by

$$\begin{aligned} A_{ij} &= \int \phi_i \phi_j \qquad 1 \le i, j \le N-1, \\ B_{ij} &= \int \phi_i'' \phi_j'' \qquad 1 \le i, j \le N-1, \\ C_{ij} &= \int \phi_i' \phi_j' \qquad 1 \le i, j \le N-1, \end{aligned}$$

with similar entries for other ranges of i and j. The components D_i of the zero-order term D in (5.4) are numerical approximations, using Simpson's rule, of the integral

$$\begin{aligned} &\int f(U_h)\phi_i \qquad 1 \le i \le N-1 \\ &\int f(U_h)\psi_{i-N} \qquad N \le i \le 2N. \end{aligned}$$

The in-plane load $p(t)$ is determined as part of the solution and the necessary and sufficient condition comes from the integral constraint (5.2), which reads in discretized form

$$\frac{1}{2} U^T C U = \lambda. \tag{5.5}$$

The system (5.4-5.5) is then an index-2 differential-algebraic equation. Differentiating (5.5) with respect to time we find

$$U^T C U_t = 0. \tag{5.6}$$

We solved (5.4) and (5.6) using DDASSL, a backward-difference form differential-algebraic equation solver [20]. We choose to replace the constraint (5.5) by (5.6) since the latter provides a DAE system of index one, which DDASSL is designed to handle. It is verified after calculation that the deviation from (5.5) due to accumulation of numerical error is acceptably small (relative error less than 0.01).

5.2. Results. Figure 5.1 shows a plot of the load p_λ as a function of λ. The initial data sample size is 25.

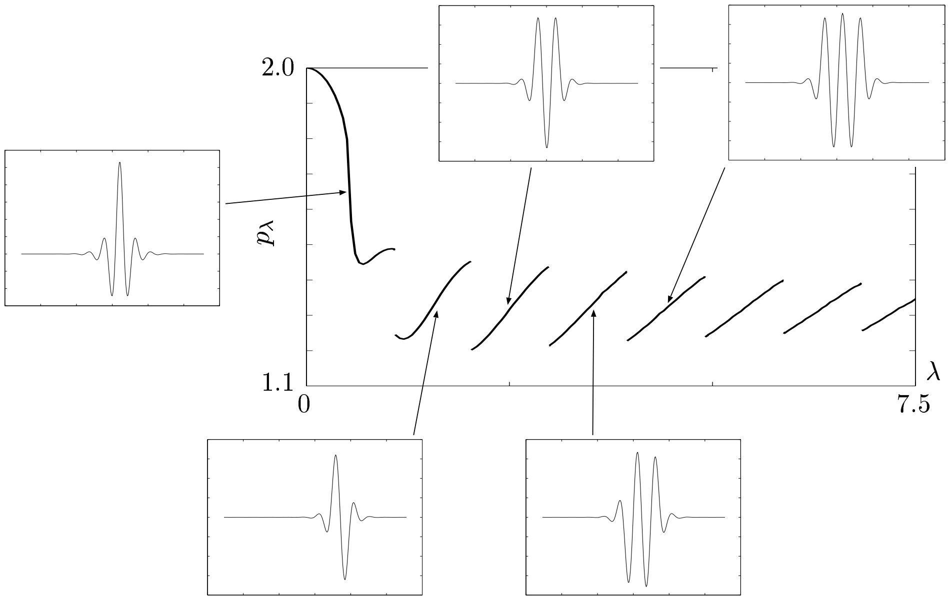

FIG. 5.1. *Results of the numerical minimization* ($\alpha = 0.3$)

A number of features of this graph merit special mention.

1. The graph decomposes into a collection of continuous curves. The apparent discontinuities in this figure are actual discontinuities; the change in λ causes local minima to move relative to each other, and at these discontinuities the global minimum jumps from one local minimum to another. Also, it appears that the continuous curves are projections of continua of solutions in state space (note that comparison is not trivial because of the interference of the discretization; also, we do not want to impose any symmetry).
2. Theorem 3.1 states that for any sequence $\lambda_n \to \infty$, $p_{\lambda_n} \to M_\#$. In Figure 5.1 we recognize this convergence in the decrease of the vertical extent of the graph as λ increases.
3. On the continuous parts of the curve, the solution has either odd or even symmetry. At the jumps the solution switches from one to the other.
4. The load is not a continuous function of λ; but all jumps are downward. Compare this to Lemma 2.6.

In the next section we give an interpretation of the form of Figure 5.1.

6. Correspondence with the bifurcation diagram. In this section we briefly change our perspective: instead of problem (1.3) we consider the ODE (1.5),

$$u'''' + pu'' + f(u) = 0, \qquad \text{on } \mathbb{R}, \tag{1.5}$$

where p is a prescribed parameter. A solution of (1.3) also solves (1.5), but the opposite is not true. As we mentioned in the previous section, there are many solutions of (1.5) that are strongly suspected of not even being local constrained minimizers.

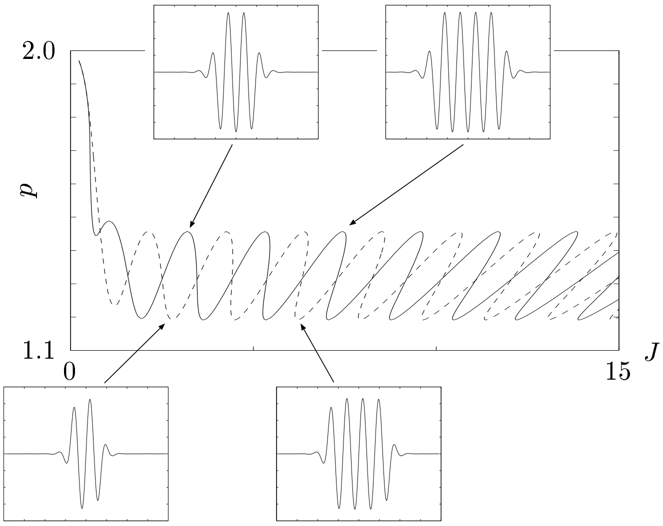

FIG. 6.1. *Bifurcation diagram for equation (1.5) showing curves of even (continuous line) and odd (dashed line) solutions bifurcating from $p = 2$. Here $\alpha = 0.3$.*

Figure 6.1 shows a bifurcation plot of equation (1.5). At $p = 2$, at zero J, a Hamiltonian-Hopf bifurcation creates four homoclinic orbits. Two of these are even, and each the opposite of the other ($u_2 = -u_1$); the other two are odd, and again each other's opposite. In Figure 6.1 we identify the two even and the two odd solutions and thus draw two curves in total.

The initial part of the figure, near $p = 2$, is typical for a destiffening nonlinearity. The oscillating behaviour for larger values of J, however, is related to the competing de-stiffening and re-stiffening qualities. It is shown in [19] how the restiffening nature (more specifically, the fact that $F(u) > F(0)$ for $u \neq 0$) implies that along the curve p must be bounded from below. Woods and Champneys [27, 26] show that the snaking behaviour can be explained as the result of a collision of the unstable manifold of zero with the stable manifold of a family of periodic orbits parametrized by p. When $p = M_\#$, this periodic orbit is exactly the function $u_\#$ of Theorem 3.1.

When we combine this figure and Figure 5.1 into one diagram (Figure 6.2) there is a strong suggestion that all minimizers lie on the bifurcation curve. If we elevate this numerical suggestion to the status of hypothesis, that is, if we suppose that all minimizers of Problem (1.3) lie on this bifurcation diagram, then the jumps from one curve to the other result from a simple energy argument. In a graph of load against deflection, strain energy is represented by area under the graph. More precisely, if we have a continuum of solutions v_s of equation (1.4), parametrized by s, with associated

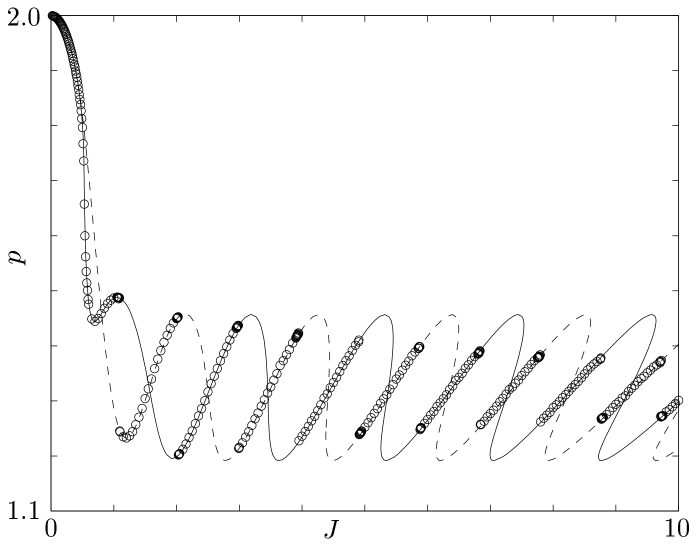

FIG. 6.2. *Combination of Figures 5.1 and 6.1*

load p_s, then

$$\begin{aligned} W(v_{s_2}) - W(v_{s_1}) &= \int_{s_1}^{s_2} W'(v_s) \cdot \frac{dv_s}{ds}\, ds \\ &= \int_{s_1}^{s_2} p_s J'(v_s) \cdot \frac{dv_s}{ds}\, ds \\ &= \int_{s_1}^{s_2} p_s\, dJ(v_s), \end{aligned}$$

with a slight abuse of notation in the last integral.

To explain the jumps, let us assume, to start with, that for some interval of values of λ all minimizers lie on a given continuum of solutions v_s. This is shown schematically in Figure 6.3. At the critical value λ_c the two areas E_1 and E_2 are equal, implying that the strain energies at **A** and **B** are equal. As λ passes through the critical value, the minimum in the strain energy jumps from the top to the bottom curve.

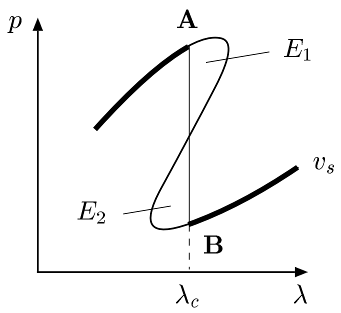

FIG. 6.3. *The thick line indicates the minimizer under constrained* λ

In the case of the problem as stated in (1.3), the numerical results clearly indicate that both the branches of solutions in Figure 6.1 contain minimizers. We therefore need to take both curves into account when searching for jumps. As an example, Figure 6.4 shows a blow-up of the first jump in Figure 5.1, where the minimum passes from the even to the odd branch. Again the jump corresponds to an equal-area condition. The other jumps arise in the same manner.

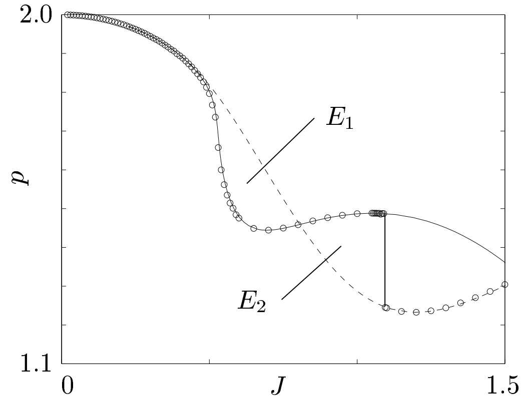

FIG. 6.4.

In summary, if we make the assumption that all global minimizers of (1.3) lie on the bifurcation diagram of Figure 6.1, then the form of Figure 5.1 follows readily from energy comparison.

The assumption that all global minimizers lie on the bifurcation diagram is a strong one. As of yet there is no conclusive argument why this might be the case. For some specific classes of solutions of (1.5) it has been shown that they are or are not locally minimal (see above) but these results depend in a critical manner on the structure of the solutions involved. A complete classification of all solutions of (1.5) is still a distant goal, and therefore doing an exhaustive search is not an option.

To complicate matters, the numerical results suggest that local optimality does not guarantee membership of the bifurcation diagram in Figure 6.1. As mentioned before, the algorithm used for finding the global minimizer runs a constrained gradient flow algorithm starting from random initial data; the function that the algorithm stabilizes at for large time is assumed to represent a local minimum. This procedure is repeated a number of times, and the local minimum with least strain energy is tagged as the global minimum. This is the solution that appears in Figures 5.1 and 6.2.

In Figure 6.5 we show an excerpt of Figure 6.2, but this time we plot not only the global minimum but all of the local minima that were found along the way. In addition to the solutions that we would expect, those that lie on the two bifurcation curves, other solutions appear with a different structure. Of course, 'local optimality' has been established in a crude manner, so this could well be a numerical artefact. The question of the relationship between the minimization problem (1.3) and Figure 6.1 remains an interesting one, however, that merits to be studied in more detail.

7. The nonlinearity F. In this paper we concentrate entirely on functions F of the form (1.6). Of course the class of functions for which one can derive the same results is much larger, and in this section we give some indication as to which properties of F enter into play. In addition, the existence result (Theorem 2.2) and

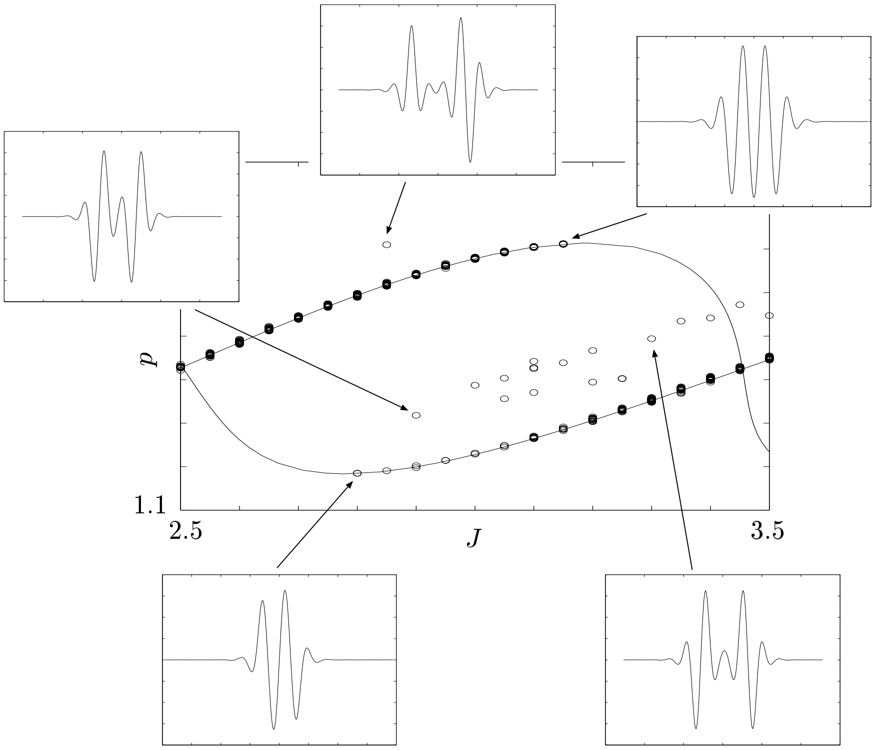

FIG. 6.5. *Every circle represents a 'local minimizer' that was found numerically (see text).*

the convergence result (Theorem 3.1) differ in their requirements, and we shall also comment on this issue.

The term destiffening was defined in the introduction as a decrease in the marginal stiffness $F''(u)$ as u moves away from zero. The actual property used in the proofs, however, is the combination '$F'''(0) = 0$ and $F''''(0) < 0$' (in Lemma 2.3, part 1). The function (1.6) satisfies both of these formulations of the destiffening character, but the function $F(u) = u^2/2 - u^6/6 + \alpha u^8/8$, for instance, satisfies only the first of the two. As remarked on page 8, the proof of Lemma 2.3 does not apply to this latter function, but numerical results suggest that the assertion of the Lemma ($W_\lambda < 2\lambda$) holds nonetheless. At this stage we must conclude that there is a grey area between these two formulations of 'destiffening'.

If we tolerate this lack of accuracy for the moment, we can assert that the destiffening nature is crucial for the existence proof, via the same property $W_\lambda < 2\lambda$ and the estimate (2.11). However, destiffening alone is not sufficient to guarantee existence for all $\lambda > 0$. If F takes negative values (assuming $F(0) = 0$), say $F(\bar{u}) < 0$, then for sufficiently large values of λ we can create admissible profiles with large *negative* strain energy. As an example, consider

$$u_k(x) = \bar{u}\,\eta(|x| - k),$$

where η is a smooth cut-off function such that $\eta \equiv 1$ on $(-\infty, -1]$ and $\eta \equiv 0$ on $[1, \infty)$. If $k > 1$, then $J(u_k)$ is independent of k, but $W(u_k)$ takes arbitrarily large negative values as $k \to \infty$. Since we therefore have $\inf_{\mathcal{C}_\lambda} W = -\infty$, the existence question is absurd. In order to avoid this degeneracy, we need to assume $F(u) \geq 0$ (the possibility $F(\bar{u}) = 0$, $\bar{u} \neq 0$ leads to non-compactness of minimizing sequences; however, such sequences can be adapted to regain compactness, so that the existence of a minimizer is not compromised).

To summarize, the main characteristics of F that lead to existence are the destiffening nature and this positivity property. The function (1.6) meets these constraints if and only if $\alpha \geq 3/16$.

Turning to the convergence of minimizers as the end-shortening λ tends to infinity (Theorem 3.1), simple positivity of F is not sufficient. One can construct counter-examples where $F(u)$ is small, but positive, for large $|u|$; minimizers for such nonlinearities are unbounded in the L^∞-norm, and therefore do not converge. Some form of stiffening for larger u is necessary to prevent this runaway. As before, no sharp condition is known, but Lemma 3.2, which provides the all-important L^∞ bound, can proved for all F with

$$F'(s) \sim s^{q-1} \qquad \text{as } |s| \to \infty, \qquad \text{with } q > 2,$$

without any change in the proof. For such functions F the statement of Theorem 3.1 should hold unchanged.

In addition, for the convergence result we assume that $\alpha \geq 1/4$, so that $p \geq 0$ (see (3.2)), and solutions necessarily oscillate at infinity. This property is used twice in the proof of Theorem 3.1. We conjecture that $\alpha \geq 1/4$ is unnecessarily restrictive, however, and that $\alpha \geq 3/16$ should suffice for both existence of minimizers and the convergence for large λ.

While dwelling on the subject of the nonlinearity F, we might also comment on the requirement of restiffening itself, i.e. the fact that we assume a relatively complex structure in the response of the elastic foundation. It is true that the combination of initial destiffening and subsequent stiffening appears artificial. However, there is good reason to assume that both the destiffening and the subsequent stiffening characters are present in actual examples of elastic struts on foundations—not in the foundation response, but in other elements in the model. For instance, in linearizing the higher-order terms in the equation—that is, by replacing $\mathcal{W}$ and $\mathcal{J}$ by W and J—a destiffening property that is present in the original formulation has been discarded. Mühlhaus [17] gives a heuristic argument for this fact, and it can be verified by doing a small-amplitude development of the appropriate nonlinear terms.

Similarly, a foundation that does not have the local response of the Winkler foundation that we consider here, but 'feels' the proximity of the layer at adjoining sites, has a strongly stiffening character for large deformations. This is illustrated by Figure 7.1, where the material indicated by the hashing, being squashed by the bends in the strut, will exert a large force on the strut in the opposite direction. This is an inherently non-local effect that can not be captured with a Winkler foundation. In summary, the various simplifying assumptions that we have made during the modelling process have removed the destiffening and subsequent stiffening characteristics from the formulation, forcing us to re-introduce them via the foundation response.

With these arguments in mind we chose to consider a mathematical model that has the nature, if not the exact form, of the mechanical problem. We hope that the ideas of this paper will be amenable to future extension.

Appendix. Proof of Lemma 2.5. It is relatively simple to prove that for any minimizer u the load necessarily satisfies $p \leq 2$. If u minimizes W at constant J, with

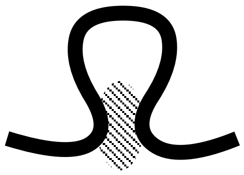

FIG. 7.1. *Squashed material exerts a non-local force on the strut*

associated load p, then u is a stationary point of the functional

$$\mathcal{L}(u) = W(u) - pJ(u).$$

Since the constraint is one-dimensional, the second derivative of $\mathcal{L}$ at u can not have more than one unstable eigenmode.

On the other hand, suppose that $p > 2$, and let $\phi \in C_c^\infty(\mathbb{R})$ satisfy

$$\int \left(\phi''^2 - p\phi'^2 + \phi^2\right) < 0.$$

If $|x|$ is sufficiently large, then $F''(u(x)) \leq 1$ and therefore, setting $\psi_K(x) = \phi(x-K)$,

$$\mathcal{L}''(u) \cdot \phi_K \cdot \phi_K = \int \left({\phi_K''}^2 - p{\phi_K'}^2 + F''(u){\phi_K}^2\right) < 0,$$

both for large and for small K. It follows that $\mathcal{L}$ has at least two unstable directions, and this contradicts the assumption that u is a minimum.

When we write equation (1.5), for $p = 2$, as a four-dimensional dynamical (Hamiltonian) system, then the linear part of this system is given by a matrix which is not diagonizable. Using normal form theory, for every $k \in \mathbb{N}$ we can transform the system to a system given by the Hamiltonian

$$H = \frac{1}{2}|p|^2 + \langle p, Jq\rangle + P(|q|^2, \langle p, Jq\rangle) + O(|p|^{2k}, |q|^{2k}) \qquad \text{(A.1)}$$

(see e.g. [15, Chapter VII]). Here P is a polynomial in its two arguments, whose lowest-order terms are quadratic. For our purposes the only important term in $P(u, v)$ is au^2, or equivalently $a|q|^4$. The calculations done by Woods [26] show that $a > 0$.

By hypothesis, the orbit represented by $(p(t), q(t))$ converges to the origin as $t \to \infty$. Since the system is linear in the limit of small amplitude, it follows that (p, q) must converge to solutions of the linear problem. More accurately, if we choose $t_n \to \infty$, and rescale by setting

$$(p_n, q_n)(t) = \frac{1}{|(p,q)(t_n)|}(p,q)(t - t_n),$$

then the functions (p_n, q_n) converge on compact subsets to bounded solutions of the linear problem. Since all such solutions satisfy $p \equiv 0$, it follows that $p = o(|q|)$ as $t \to \infty$.

We next transform (p, q) to polar coordinates (r, R, θ, Θ), given by

$$q_1 = r\cos\theta \qquad q_2 = r\sin\theta$$

$$p_1 = R\cos\theta - \left(\frac{\Theta}{r}\right)\sin\theta \qquad p_2 = R\sin\theta + \left(\frac{\Theta}{r}\right)\cos\theta.$$

The Hamiltonian then takes the form

$$H(r,R,\theta,\Theta) = \frac{1}{2}\left(R^2 + \left(\frac{\Theta}{r}\right)^2\right) - \Theta + P(r^2,\Theta) + O\left(\left|R^2 + \left(\frac{\Theta}{r}\right)^2\right|^k + r^{2k}\right).$$

The result $p = o(|q|)$ translates to $R/r, \Theta/r^2 \to 0$, which implies, together with $H = 0$, that

$$\Theta \sim \frac{1}{2}R^2 + ar^4.$$

We can then calculate an estimate of the rate of decay of Θ:

$$\dot{\Theta} = -\frac{\partial H}{\partial \theta} = O(R^{2k} + r^{2k}) = O(\Theta^{k/2}).$$

It follows that for $k \geq 4$ the rate of decay of Θ is too small to be compatible with the condition that $u \in H^2(\mathbb{R})$, or

$$\int^{\infty} \left(|p|^2 + |q|^2\right) < \infty,$$

since

$$|p|^2 + |q|^2 \geq R^2 + r^2 \geq c\Theta$$

for some $c > 0$, in the limit $t \to \infty$.

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