Friction Factor Estimation for Turbulent Flow in Corrugated Pipes with Rough Walls

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Outline

1. Problem Setting

2. Two-Equation Turbulence Models and BCs
   - Reynolds Averaging, $k - \epsilon$ and $k - \omega$ Models.
   - Law of the Wall

3. Turbulent Flow in Conventional Pipes
   - Smooth Wall Case
   - Rough Wall Case

4. Friction Factor Computations
Where We Are Now

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Flexible Pipes

- Respond well to bending
- Easy to install
- Excellent strength/length ratio
- Corrugated
- Rough walls
Simulation of Turbulent Flows

3 basic approaches:
- DNS - Direct Numerical Simulation
- LES - Large-Eddy Simulations
- RANS - Reynolds-Averaged Navier-Stokes

← DNS solution

← RANS solution
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Equations Describing the Dynamics of Flow

Incompressible flow equations:

\[
\frac{\partial \tilde{u}_j}{\partial x_j} = 0 \quad \leftarrow \text{continuity equation}
\]

\[
\rho \left[ \frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} \right] = -\frac{\partial p}{\partial x_i} + \frac{\partial \tilde{T}_{ij}^{(v)}}{\partial x_j} \quad \leftarrow \text{NS equation}
\]

\(\tilde{T}_{ij}^{(v)}\) - stress due to viscous forces

Newtonian fluid hypothesis:

\[
\tilde{T}_{ij}^{(v)} = \mu \left[ \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right].
\]
Reynolds Averaging

Reynolds decomposition:

\[
\tilde{u}_i = U_i + u_i, \\
\tilde{p} = P + p, \\
\tilde{T}^{(v)}_{ij} = T^{(v)}_{ij} + \tau^{(v)}_{ij},
\]

\(U_i, P, T^{(v)}_{ij}\) - mean components; \(u_i, p, \tau^{(v)}_{ij}\) - fluctuating components.

\[
\frac{\partial U_j}{\partial x_j} = 0.
\]

\[
\rho \left[ \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right] = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ T^{(v)}_{ij} - \rho \langle u_iu_j \rangle \right].
\]

\(\rho \langle u_iu_j \rangle\) is unknown ← closure problem
Eddy Viscosity Approximation

Newtonian type closure, proposed by Boussinesq:

\[
\sigma_{ij} \equiv -\rho \langle u_i u_j \rangle = \mu_T \left[ \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right]
\]

\(\mu_T\) - "turbulence viscosity" (eddy viscosity), \([\text{N/m}^2\cdot\text{s}]\) (not constant).

\(k\) - specific turbulence kinetic energy, \([\text{N}\cdot\text{m/kg=\text{m}^2/\text{s}^2}]\).
\(\epsilon\) - turbulence dissipation, \([\text{m}^2/\text{s}^3]\).
\(\omega\) - turbulence dissipation per unit turbulence kinetic energy, \([1/\text{s}]\).

\[\mu_T = \rho C_\mu \frac{k^2}{\epsilon}\]

\[\mu_T = \rho \frac{k}{\omega}\]

Both modeled on dimensional grounds.
Define the NS operator as

\[ N(\tilde{u}_i) = \rho \frac{\partial \tilde{u}_i}{\partial t} + \rho \tilde{u}_k \frac{\partial \tilde{u}_i}{\partial x_k} + \frac{\partial \tilde{p}}{\partial x_i} - \mu \frac{\partial^2 \tilde{u}_i}{\partial x_k^2}, \]

Take the following moment of NS operator

\[ \langle u_i N(\tilde{u}_j) + u_j N(\tilde{u}_i) \rangle = 0 \Rightarrow \text{equation for } \rho \langle u_i u_j \rangle. \]

Turbulence kinetic energy (per unit mass)

\[ k \equiv \frac{1}{2} \langle u_i u_i \rangle = \frac{1}{2} [\langle u_1^2 \rangle + \langle u_2^2 \rangle + \langle u_3^2 \rangle]. \]

Take the trace of the Reynolds stress equation

\[ \rho \frac{\partial k}{\partial t} + \rho U_j \frac{\partial k}{\partial x_j} = \sigma_{ij} \frac{\partial U_i}{\partial x_j} - \rho \epsilon + \frac{\partial}{\partial x_j} \left[ (\mu + \frac{\mu_T}{\sigma_k}) \frac{\partial k}{\partial x_i} \right]. \]
Turbulence Dissipation Equation - Outline of Derivation

Take the following moment of NS operator

$$2\frac{\mu}{\rho} \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial N(u_i)}{\partial x_j} \right\rangle = 0 \Rightarrow \text{equation for } \epsilon = \frac{\mu}{\rho} \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\rangle.$$

$$\rho \frac{\partial \epsilon}{\partial t} + \rho U_j \frac{\partial \epsilon}{\partial x_j} = C_{\epsilon 1} \frac{\epsilon}{k} \sigma_{ij} \frac{\partial U_i}{\partial x_j} - C_{\epsilon 2} \rho \frac{\epsilon^2}{k} + \frac{\partial}{\partial x_j} \left[ \left( \mu + \frac{\mu_T}{\sigma_\epsilon} \right) \frac{\partial \epsilon}{\partial x_j} \right].$$

$C_{\epsilon 1}, C_{\epsilon 2}, \sigma_\epsilon$ - modeling constants.

$$\rho \frac{\partial \omega}{\partial t} + \rho U_j \frac{\partial \omega}{\partial x_j} = \alpha \frac{\omega}{k} \sigma_{ij} \frac{\partial U_i}{\partial x_j} - \beta \rho \omega^2 + \frac{\partial}{\partial x_j} \left[ (\mu + \sigma_\omega \mu_T) \frac{\partial \omega}{\partial x_j} \right].$$

$\alpha, , \beta, \sigma_\omega$ - modeling constants.
Boundary Conditions

Law of the Wall

\[ U_p^+ = \frac{1}{\kappa} \ln y_p^+ + B. \]
Law of the Wall

\[
U_p^+ = \frac{1}{\kappa} \ln y_p^+ + B,
\]

where \( U_p^+ = \frac{U_p}{u_*} \), \( y_p^+ = \frac{u_* y_p}{\nu} \).

\[ u_* = \sqrt{\frac{\tau_w}{\rho}}, \quad \tau_w \text{ is unknown!} \]

From Prandtl’s assumption (\( \nu_T = \kappa u_* y \)) and the balance of production and dissipation of turbulent kinetic energy:

\[ u_* = C_{\mu}^{1/4} k_p^{1/2}. \]
Law of the Wall as Boundary Condition

\[ \frac{U_p}{u_*} = \frac{1}{\kappa} \ln y_p^+ + B. \]

Multiply by \( u_*^2 \)

\[ U_p u_* = u_*^2 \left( \frac{1}{\kappa} \ln y_p^+ + B \right). \]

Replace \( u_*^2 \) by \( \tau_w/\rho \), and \( u_* \) by \( C_\mu^{1/4} k_p^{1/2} \)

\[ U_p C_\mu^{1/4} k_p^{1/2} = \frac{\tau_w}{\rho} \left( \frac{1}{\kappa} \ln y_p^+ + B \right). \]

The skin friction force at the wall (or wall stress), \( \tau_w \):

\[ \tau_w = \frac{\rho C_\mu^{1/4} k_p^{1/2}}{\frac{1}{\kappa} \ln y_p^+ + B} U_p. \]
The stress at the wall (or anywhere else) can be computed as a function of the velocity gradient (Newtonian Fluid approximation).

\[ \tau_w = (\mu + \mu_T) \frac{\partial U_p}{\partial n}. \]

\[ (\mu + \mu_T) \frac{\partial U_p}{\partial n} - \frac{\rho C_\mu^{1/4} k_p^{1/2}}{1/\kappa \ln y_p^+ + B} U_p = 0, \quad y_p^+ = \frac{\rho C_\mu^{1/4} k_p^{1/2} y_p}{\mu}. \]

\[ a(k_p, \epsilon_p) \frac{\partial U_p}{\partial n} + b(k_p) U_p = 0. \quad \leftarrow \text{Robin BC with variable coefficients} \]

Zero-flux BC (no turb. energy transfer through the boundary)

\[ \mathbf{n} \cdot \nabla k_p = 0, \quad \epsilon_p = \frac{C_\mu^{3/4} k_p^{3/2}}{\kappa y_p}, \quad \omega_p = \frac{C_\mu^{-1/4} k_p^{1/2}}{\kappa y_p} \]

\( y_p \) - distance from the wall, free parameter
Sensitivity of Solution to the Choice of $y_p^+$

Solution is not changing for $50 < y_p^+ < 300$
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The Moody Diagram
Fully Developed Flow

\[ \frac{L_e}{D} \approx 4.4Re^{1/6}, \text{ for turbulent flow} \]
Computational Domain and BCs

Inflow/Outflow

\[ \mathbf{U}(r, 0) = \mathbf{U}(r, L) \]
\[ k(r, 0) = k(r, L) \]
\[ \epsilon(r, 0) = \epsilon(r, L) \]
\[ \omega(r, 0) = \omega(r, L) \]
\[ P(r, 0) = P_{in} \]
\[ P(r, L) = P_{out} \]

Symmetry axis

\[ \mathbf{n} \cdot \nabla \mathbf{U} = 0 \]
\[ \mathbf{n} \cdot \nabla k = 0 \]
\[ \mathbf{n} \cdot \nabla \epsilon = 0 \]
\[ \mathbf{n} \cdot \nabla \omega = 0 \]

Wall

\[ (\mu + \mu_T) \frac{\partial U_p}{\partial n} = \frac{\rho C_1^{1/4} k_p^{1/2}}{\frac{1}{\kappa} \ln y_p^+ + B} U_p \]
\[ \mathbf{n} \cdot \nabla k_p = 0 \]
\[ \epsilon_p = \frac{C_3^{3/4} k_p^{3/2}}{\kappa y_p^+}, \quad \omega_p = \frac{C_1^{-1/4} k_p^{1/2}}{\kappa y_p^+} \]
Meshes and Solution Procedure
Computed vs. Measured Friction Factor
Computed vs. Measured Friction Factor

Friction factor computed with different turbulence models:

- $k-\varepsilon$, $B=5.5$
- $k-\varepsilon$, $B=5.0$
- $k-\omega$, $B=5.5$
- $k-\omega$, $B=5.0$
- Measurement
Law of the Wall for Rough Walls

\[ \frac{U}{u_\ast} = \frac{1}{\kappa} \ln \left( \frac{y_p}{e} \right) + 8.5 \]

where:
- \( U \) is the mean flow velocity
- \( u_\ast \) is the friction velocity
- \( y_p \) is the distance from the wall to the point of interest
- \( e \) is the roughness height
- \( \kappa \) is the von Karman constant

The graph shows the variation of friction factor \( f \) with Reynolds number \( Re \) for different roughness heights. The equation is valid for turbulent flow in conventional pipes.
Law of the Wall for Rough Walls

\[ \frac{U}{u_*} = \frac{1}{\kappa} \ln \left( \frac{y_p}{e} \right) + 8.5 = \frac{1}{\kappa} \ln y_p^+ + \left[ 8.5 - \frac{1}{\kappa} \ln e^+ \right] \]

- \( e \) - roughness height
- \( e^+ \) - non-dimensional roughness
"Combined" Law of the Wall

Introduce a "combined" law of the wall:

\[ \frac{U}{u_*} = \frac{1}{\kappa} \ln y_p^+ + B^*, \]

where

\[ B^* = B + \theta(8.5 - B - \frac{1}{\kappa} \ln e^+), \]

Hydrodynamic smoothness - \( \theta = 0 \) (\( e^+ < e_1^+ \))

Full roughness - \( \theta = 1 \) (\( e^+ > e_2^+ \))

Transition - \( \theta = \theta(e^+) \), \( e_1^+ < e^+ < e_2^+ \)

\[ \theta = \sin \left( \frac{\pi \ln(e^+/e_1^+)}{2 \ln(e_2^+/e_1^+)} \right), \quad e_1^+ = 2.25, \quad e_2^+ = 90. \]
Computed vs. Measured Friction Factor

Friction factor for different values of wall roughness

- $e/D=0.0005$
- $e/D=0.001$
- $e/D=0.005$
- $e/D=0.01$
- $e/D=0.05$

$Re$ vs. $f$
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Pressure across the pipe is not constant in corrugated pipes

\[ P(r, 0) = P(r, L) + \Delta P \]
Initial Mesh
Adapted Mesh
Convergence vs. Iteration Number (Adaptive Solver)
A Typical Solution
Friction Factor vs. Wall Roughness
Conclusions

- $k - \epsilon$ model is \textit{slightly better} than $k - \omega$ model
- To get a realistic estimation of the friction factor the laws for smooth and rough wall have to be \textit{combined}
- The roughness of the fabric has a secondary influence on the friction factor
- To obtain a considerable decrease in friction factor shape optimisation for the steel spiral should be considered