1. (5 pts) A graph is $k$-regular if each vertex has degree exactly $k$. Show that the edges of every $k$-regular bipartite graph can be partitioned into $k$ disjoint perfect matchings.

Solution: Let $X$ and $Y$ denote the left and right side of the graph. Note that $|X| = |Y|$ as the number of edges adjacent to $X$ is $k|X|$ and the number of edges adjacent to $Y$ is $k|Y|$.

Consider a subset $S$ of $X$. The number of edges adjacent to $S$ is $k|S|$. These edges are adjacent to vertices in $N(S)$ in $Y$ (and to nothing else). But if $|N(S)| < |S|$, then some vertex in $N(S)$ must have degree strictly more than $k$, contradicting that the graph is $k$-regular.

So, $|N(S)| \geq |S|$ for every subset $S$. By Hall’s theorem there is left-saturated matching. Thus there is a perfect matching.

If we remove a perfect matching, we end up with a $k-1$ regular graph. We now repeat the argument above to remove another perfect matching and so on until the graph is empty. This gives us the partition of the edges into disjoint perfect matchings.

2. (10 pts) Show that for any bipartite graph $G$, the maximum size of a matching is exactly equal to the size of the minimum vertex cover. Recall that a subset of vertices $S$ is a vertex cover if every edge has at least one endpoint in $S$.

[Hint: Construct a directed network where any integral flow corresponds to a matching. You might want to put infinite capacity on some edges. Apply max-flow min-cut theorem to this graph. Stare at this min-cut.]

Solution: Let $L$ and $R$ denote the left and right sides of $G$. Connect every vertex in $L$ with an edge from source $s$ with a capacity 1 edge. Every vertex in $R$ is connected to sink $t$ with a capacity 1 edge. For edges between $L$ and $R$, we direct them from $L$ to $R$ and give them capacity $\infty$.

As seen in class, any integral flow corresponds to a matching. Now consider the max-flow. This is integral and its value is equal to size of the maximum matching. Call this value $m$. Let $v$ denote the value of the minimum vertex cover. We want to show that $v = m$.

First, recall that $v \geq m$ because for every edge in the maximum matching we need at least unique one vertex to cover it. So it suffices to show that $v \leq m$.

By the max-flow min-cut theorem, there is some min-cut of value $m$. Consider some such cut $(s \cup A \cup B, t \cup (L - A) \cup (R - B))$ where $A \subseteq L$ and $B \subseteq R$ denote the set of vertices that lie on the source-side of the min-cut.

As $m$ is finite (the cut $\{s\}, V - \{s\}$ has capacity $|L|$), there cannot be any edges from $A$ to $R - B$ as these edges contribute $\infty$ to the capacity of the cut. So the capacity of this cut is precisely $(|L - A| + |B|)$.

Now we claim that $(L - A) \cup B$ is a valid vertex cover. This is because the only edges that will not be covered by this set of vertices are the ones going between $A$ and $R - B$. But we know that there are no such edges (as the min-cut has finite capacity). As the smallest vertex cover can be no worse, so $v \leq m$.

3. (10 pts) Let $H$ be a 3-uniform hypergraph on $n$ vertices, with $m$ hyperedges. Assume that $m \geq n/3$. An independent set in $H$ is a subset of the vertices that does not contain any hyperedge.
Using the probabilistic method with alterations, show that $H$ contains an independent set of size at least $\frac{2n^{3/2}}{3\sqrt{3m}}$.

**Solution:** Consider a random subset of the vertices of $H$, by including each vertex with probability $p$, independently of all other vertices. Call this set $X$. Let $Y$ be the set of edges spanned by $X$ (note that an edge is spanned by $X$ if all three of its vertices are in $X$). For each such edge, remove one of its vertices from $X$. The resulting set is an independent set, with at least $|X| - |Y|$ vertices. Hence, by the averaging principle, there exists an independent set with at least

$$E[|X| - |Y|] = np - mp^3$$

vertices. The right-hand side is maximised for $p = \sqrt{\frac{n}{3m}}$ (which is actually a probability by the lower bound on $m$), which yields the desired result.

4. (10 pts) Let $A$ be a matrix. A submatrix of $A$ is obtained by deleting rows and columns from $A$. A submatrix is called constant, if all its entries are equal.

Show that for all $j \geq 2$, there exists an $n \times n \{0, 1\}$-matrix (with $n = \lfloor 2^{j/2} \rfloor$) that has no constant $j \times j$ submatrix.

**Solution:** We will show that with positive probability, a suitable chosen random $n \times n$-matrix does not have a constant $j \times j$ submatrix. Let $A$ be a random $n \times n \{0, 1\}$-matrix obtained by letting each entry be either 0 or 1, each with probability 1/2, independently of all other entries. There are $\binom{n}{j}^2 j \times j$-submatrices, and each of these submatrices is constant with probability $2^{1-j^2}$. Hence, by the union bound, the probability that there is a constant $j \times j$ submatrix is at most $\binom{n}{j}^2 2^{1-j^2}$. Using that $\binom{n}{j}^2 \leq \frac{n^j}{j!}$ and $n \leq 2^{j/2}$, we find that

$$\left(\frac{n}{j}\right)^2 2^{1-j^2} \leq \frac{(2^{j/2})^2}{j!} 2^{1-j^2} = \frac{2}{j!} \leq \frac{1}{2},$$

where the final inequality follows from the assumption $j \geq 2$. 