Hierarchies
Today

1. Some more familiarity with Hierarchies
2. Examples of some basic upper and lower bounds
3. Survey of recent results (possible material for future talks)
Hierarchies

1. Lovasz-Schrijver (LS), LS+
2. Sherali Adams
3. Lasserre
4. Mixed Hierarchy (recently used)

Idea: \( P = \text{conv} \text{(subset } S \text{ of } \{0,1\}^n) \)
\( x = (x_1, ..., x_n) \in P = \text{prob. distribution over } S \)

Want to approximate P by putting linear constraints on \( x_i \).
Hierarchies try to capture some kind of local correlations
Variable $x_S$ intended to model $\pi_{i \in S} x_i$

$x_i^2 = x_i$, so $x_S = x_{S \cup \{i\}}$ if $i \in S$

Take LP, multiply each constraint by
$\Pi_{i \in I} x_i \Pi_{j \in J} (1 - x_j)$ for all $I \cup J = S$ $|S| \leq t$

$x_S$ should satisfy additional constraints
Applying this to constraint $1 \geq 0$

$x_{12} \geq 0$
$x_1 - x_{12} \geq 0$
$x_2 - x_{12} \geq 0$
$1 - x_1 - x_2 + x_{12} \geq 0$

Same as: $M_S(x)$ PSD and $M_S(g_l \ast x)$ PSD
Alternate Description

\[ x_{\{S,\alpha\}} \quad \alpha \in \{0,1\}^S \]

Non-negativity \[ x_{\{S,\alpha\}} \geq 0 \]
Consistency: \[ x_{\{12,(00)\}} + x_{\{12,(01)\}} = x_{\{1,(0)\}} = x_{\{13,(00)\}} + x_{\{13,(01)\}} \]
Sum up to 1.

Pf: \[ x_{\{S,\alpha\}} = \sum_{T \in S \cap \alpha^{-1}(0)} -1^{\lvert T \rvert} x_{\{S \cap \alpha^{-1}(1) \cup T\}} \] (inclusion-exclusion)

Alternate view: Define a probability distribution \( D(S) \) on each subset \( S \).
\( D_S \) and \( D_T \) consistent on \( S \cap T \)

Sherali Adams solution \( \Leftrightarrow \) local distributions \( D(S) \) that satisfy LP constraints. (will make more sense later)
Knapsack

Single constraint:
Max \( \sum_i p_i x_i \) \quad s. t. \quad \sum_i a_i x_i \leq B

Sherali Adams: Multiply by
\[ \Pi_{i \in I} x_i \Pi_{j \in J} (1 - x_j) \quad \text{for all } I \cup J = S \quad |S| \leq t \]

\[ a_i = 1 \quad B = 1.99 \]
\[ x_1 + \ldots + x_n \leq 1.99 \]
Can determine \( x_{\{i,j\}} = 0 \). How?

Yet LP has integrality gap about 2, even for linear rounds.
Knapsack Gap

\[ x_1 + \ldots + x_n \leq 2 - \epsilon \]

Consider solution \( x_i = p = \frac{2-\epsilon}{n(1+\alpha(1-\epsilon))} \)

\( x_S = 0 \) for \( 2 \leq |S| \leq \alpha n \)

Claim: Solution valid for \( \alpha n \) rounds of SA.

Proof: 1) Valid distribution on S.
2) Constraints: Suffices to check multiplication with \( \pi_{i\in S} (1 - x_i) \) only

\[
\sum_{i \notin S} x_i \leq (2 - \epsilon)(1 - \sum_{i \in S} x_i )
\]
Max independent set

\[ x_i + x_j \leq 1 \quad \text{for } (i, j) \in E \]

Recovers \( x_{ij} = 0 \) for edges.

(Does not seem to capture clique constraints like Lasserre will do?)
Max-cut

Basic LP. $x_{ij}$ variables $i, j \in V$

Max $\sum_{(i,j) \in E} x_{ij}$

$x_{ij} + x_{jk} \leq x_{ik}$

$x_{ij} + x_{jk} + x_{ki} \leq 2$

Perhaps could add more valid inequalities
E.g. For every set of 5 vertices, at most 6 edges in cut.

[Kenyon-Mathieu, de la Vega 07] Gap $\frac{1}{2} + \epsilon$ after $\approx \log^{\Omega(1)} n$ rounds.

Idea: Construct consistent distribution $D_S$ on up to $2t + 3$ vertices (i.e. valid combination of cuts). But objective far from any cut.

[Charikar Makarychev Makarychev’ 09] Even after $n^\gamma$ rounds
Lasserre Hierarchy

$x_S$ variables as in Sherali Adams

Multiply by $\prod_{i \in I} x_i \prod_{j \in S \setminus I} (1 - x_j)$ for all $I \subset S$, $|S| \leq t$

Put the constraint that $M_t(x)$ is PSD
(previously $M_S(x)$ PSD)

Has the following nice implication
Recall: Y PSD iff \( y_{ij} = \langle v_i, v_j \rangle \) for some vectors \( v_1, ..., v_n \)
(\( Y = V V^{T} \), Cholesky decomposition)

\[ M_t(x) : \text{matrix} \quad \text{with entries are indexed by } S,T: \quad x_{\{S \cup T\}} \]

So \( x_{\{S \cup T\}} = \langle v_S, v_T \rangle \)

Stated as follows:

Linear constraints on \( x_S \) obtained from \( g_l \) and from constraint \( 1 \geq 0 \)
\( \langle v_S, v_S' \rangle = \langle v_T, v_T' \rangle \) if \( S \cup S' = T \cup T' \)
\[ |V_{\phi}|^2 = 1 \]
Interesting consequence

\[ x_i = < v_i, v_\phi > = < v_i, v_i > \]

In other words, \( \left| v_i - \frac{v_\phi}{2} \right|^2 = \frac{v_\phi^2}{4} \)

Could have also expressed things using \( x_{\{S, \alpha\}} \)

\[ x_{\{S, \alpha\}} = < v_\phi, v_{\{S, \alpha\}} > \]

\[ V_{\{S, \alpha\}} = \sum_{\{T \subset \alpha^{-1}(0)\}} \sum_{\{T \subset \alpha^{-1}(1) \cup T\}} (-1)^{|T|} v_{\{\alpha^{-1}(1) \cup T\}} \]
Knapsack

Let us look at the previous example
Max \( x_1 + \ldots + x_n \)
\[ x_1 + \ldots + x_n \leq 1.9 \]

Lasserre objective: \( \sum_i |v_i|^2 \)

Claim: Objective \( \leq 1 \)
As previously \( x_{ij} = 0 \). Now implies \( <v_i, v_j> = 0 \)

Suppose objective = \( t > 1 \).
Consider \( \left( \sum_i v_i - \sqrt{tv_0} \right)^2 \)
Max independent set

One round of Lasserre implies clique inequalities

\[ \sum_{i \in C} x_i \leq 1 \quad C: \text{clique} \]

As previously \( x_{ij} = 0 \). Now implies \( < v_i, v_j > = 0 \)

So, \( \sum_i \left( v_\phi, \frac{v_i}{|v_i|} \right)^2 \leq 1 \) \quad (as \( v_i/|v_i| \) orthonormal)

But \( x_i = < v_i, v_\phi > = |v_i|^2 \)
Max-cut

Goemans Williamson SDP get 0.878
(first level of Lasserre)
Mixed Hierarchy

Lasserre powerful, but Sherali adams also very useful.

Dense max-cut
Problems on bounded treewidth graphs
...

Mixed k-hierarchy: Sherali Adams k levels
But SDP only on level 1,
i.e. $X_{ij} = <v_i, v_j>$, $x_i = <v_i, v_\phi>$, $1 = <v_\phi, v_\phi>$

Raghavendra 08: Assuming UGC, best algorithm for any Max-k-CSP is one k-level mixed hierarchy.
Idea: Any integrality gap instance can be converted to NP-Hardness proof.
Partial Survey

How to use the power of hierarchies.
How to show that they do not help.

Progress on long-standing questions.
1. Sparsest Cut Find $S$, s.t. $\min \ E[S, V \setminus S]/(|S| |V \setminus S|)$

LP does not give better than $\log n$

$$\min \sum_{e \in E} x_e$$
$$\text{s.t. } \sum x_{ij} = 1 \text{ and } x_{ij} + x_{jk} \leq x_{ik}$$

ARV: Using SDPs ($2^{nd}$ level hierarchy) get $\sqrt{\log n}$

Natural formulation + triangle inequalities

$$x_{ij} = |v_i - v_j|^2$$ (embedding $l_2$ square metrics into $l_1$)

(known integrality gaps still far from this)
Coloring 3-colorable graphs

Using $n^{0.5}$ colors (Wigderson)
$n^{0.4}$ [Blum’91] (blum coloring tools)
$n^{0.25}$ [Karger Motwani Sudan]
$n^{3/14}$ [Blum Karger]

Sequence of improvements
Use ARV structure on space of solutions
Recent improvement on Blum coloring tools (Thorup Kawarabayashi’12)
Current best around $n^{0.203}$

Belief: $n^\epsilon$ approximation using $O\left(\frac{1}{\epsilon}\right)$ rounds of Lasserre

Recently Arora-Ge show ... Ratios for special classes
Use Local-Global Correlation ideas recently developed.
Dense $k$-subgraph improved best known bound $n^{1/4}$
Use Sherali Adams

Cannot beat $n^{1/4}$ using SA (SODA 12)
Strong Lasserre lower bounds: $n^\epsilon$ rounds but $n^\gamma$ hardness

Unique Games: Local-global correlations
Performance of Lasserre rounding related to spectrum of graph.

General Techniques: Decomposition Lemma in context of Knapsack
Directed Steiner Tree [Rothvoss]
How to show lower bounds.

Nice techniques developed

SA: Max-cut: any local constraint involves at most $2t+3$ vertices. If $x_S$ obtained from actual distribution on cuts on $V[S]$ no constraint can be violated. Goal: Design distributions s.t. locally good, but globally bad.

LS: $(1,x) \in N(P)$ if protection matrix $Y$ s.t. $(Y_0-Y_i)/(1-x_i) \in P$ and $\frac{Y_i}{x_i} \in P$ $Y_0 = (1, x), \quad Y_{ii} = x_i$ $x \in N^t(P)$ if terms above in $t-1$ th level.

Design a lower bound can be viewed as a game against adversary.