Introduction to Homogenization (2WA25)

Exercises Week 1, WS 2013/14

1. **A one-dimensional example**
   Let $a : \mathbb{R} \to \mathbb{R}$ be a periodic function satisfying $a(x) = a(x+1)$ for all $x \in \mathbb{R}$, and for which there exists $m, M > 0$ such that $m \leq a(x) \leq M$ for all $x \in \mathbb{R}$. Further, let $\epsilon = \frac{1}{n}$ ($n \in \mathbb{N}$) be a small, positive number and consider the boundary value problem (BVP)

$$\left(P^\epsilon\right) \begin{cases} -\left(a_\epsilon(x) u_x^\epsilon(x)\right)_x = 0, & x \in (0,1), \\ u^\epsilon(0) = 0, & u^\epsilon(1) = 1, \end{cases}$$

   where $a_\epsilon : \mathbb{R} \to \mathbb{R}$ satisfies $a_\epsilon(x) = a\left(\frac{x}{\epsilon}\right)$ for all $x \in \mathbb{R}$. Apply the asymptotic expan- sion method to derive the homogenized problem corresponding to Problem P^ϵ. Then determine explicitly the homogenized diffusion coefficient for the case $a_\epsilon(x) = \left(2+\sin(2\pi x/\epsilon)\right)^{-1}$.

2. **A layered medium**
   Consider a *layered medium* occupying the two-dimensional domain $\Omega = (0,1)^2$. With $\epsilon > 0$, let $a_\epsilon : \Omega \to \mathbb{R}$, $a_\epsilon(x) = a(x_2/\epsilon)$ be an ϵ - periodic function depending only on x_2, satisfying $0 < m \leq a_\epsilon(x) \leq M < \infty$ for some constants m, M, uniformly in Ω. Consider the ϵ dependent (micro scale) equation

$$-\nabla \cdot \left(a_\epsilon(x) \nabla u^\epsilon(x)\right) = f, \quad x \in \Omega,$$

   where $f : \Omega \to \mathbb{R}$ is a function that does not depend on ϵ. Assume further that $u^\epsilon = 0$ on the boundary of Ω. Apply the asymptotic expansion method to determine the effective (macro scale) equation.

   Theoretically flavoured problems

3. When applying the asymptotic expansion method to the diffusion problem with strongly oscillating diffusion coefficients we have identified the ϵ^{-2} order problem

$$\left(P_x^{-2}\right) \begin{cases} -\nabla_y \cdot [a(y) \nabla_y u_0(x,y)] = 0, & y \in Y = (0,1)^d, \\ u_0(x,\cdot) - Y-\text{periodic}. \end{cases}$$

   Show that any $C^2(Y) \cap C^1(\bar{Y})$ solution to this problem does not depend on y.

4. Assuming that $0 < \alpha \leq a(y) \leq \beta < \infty$ for all $y \in Y$, show that the effective diffusion tensor $\bar{A} = \left(\bar{a}_{ij}\right)_{i,j=\overline{1,d}}$ is symmetric and positive definite, i.e. for any column vector $z \in \mathbb{R}^d$ one has $z^T(\bar{A}z) \geq 0$, where z^T is the transposed vector.

Hint: Show first that $\bar{a}_{ij} = \int_Y a(y)(\vec{e}_j + \nabla_y w_j) \cdot (\vec{e}_i + \nabla_y w_i) dy$ for all i, j.

For the mathematics freaks:

5. Do the calculations justifying the explicit expressions of u^ϵ, u_0 and u_1, as presented in the lecture. Further, show that the function u_1 is *uniformly bounded*, i.e. there exist a constant $C > 0$ such that, for all $s \in \mathbb{R}$, $|u_1(s)| \leq C$. Find a uniform estimate for the difference $|u^\epsilon - u_0|$.

6. As before let $\epsilon = \frac{1}{n}$ and consider the boundary value problem (BVP)

$$\left(P^\epsilon\right) \begin{cases} -u^\epsilon_{xx} + \frac{1}{\epsilon} b_\epsilon(x) u^\epsilon_x = 0, & x \in (0,1), \\ u^\epsilon(0) = 0, & u^\epsilon(1) = 1, \end{cases}$$

where $b_\epsilon(x) = b(x/\epsilon)$. Here b is a 1-periodic function satisfying $\int_0^1 b(y)dy = 0$. For $\epsilon \searrow 0$, show that u^ϵ converges to $\bar{u} : [0,1] \to \mathbb{R}$, $\bar{u}(x) = x$. Evaluate the first order corrector as well as the difference $u^\epsilon - \bar{u}$.

7. Show that the effective diffusion tensor $\bar{A}$ is strictly positive definite, i.e. there exists $\alpha > 0$ s.t. $z^T(\bar{A}z) \geq \alpha$ for all $z \in \mathbb{R}^d$ with $||z|| = 1$.

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