Homogenization

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Overview (wishful thinking!):

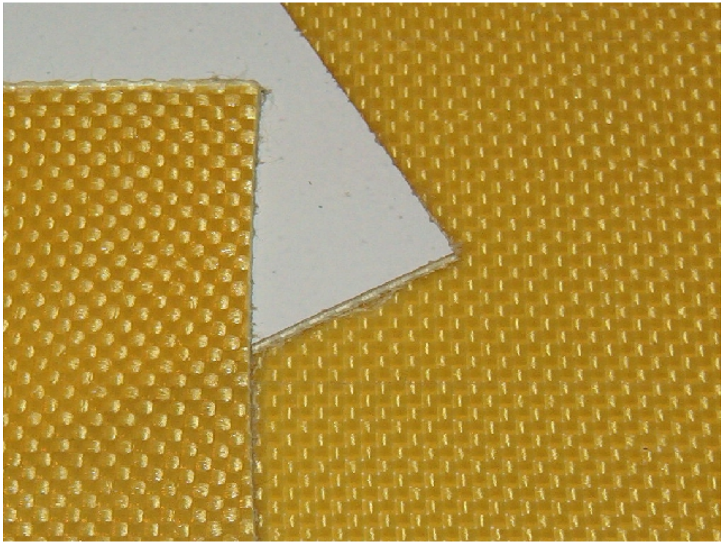
- Formal approach
  - Asymptotic expansions (composite, perforated materials)
  - Flow in porous media
  - Reactive porous media flow models
  - Double porosity models
- Mathematical techniques: IAM students
  - Energy methods
  - Two-scale convergence

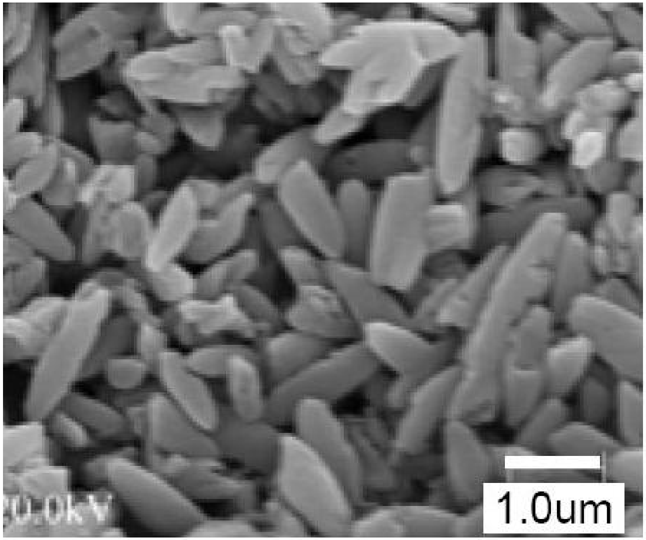
Accompanying material:
U. Hornung, *Homogenization and Porous Media*, Springer, 1997
D. Cioranescu, P. Donato, *An Introduction to Homogenization*, Oxford University Press, 2000
Lecture notes (under development)
Schedule:
2 hours/week (lectures) & 2 hours/week (lecture + exercises)

Tue, 8:45-10:30, Potentiaal 2.19

Thu, 10:45-12:30, Potentiaal 2.19

1. Introduction & Basic idea

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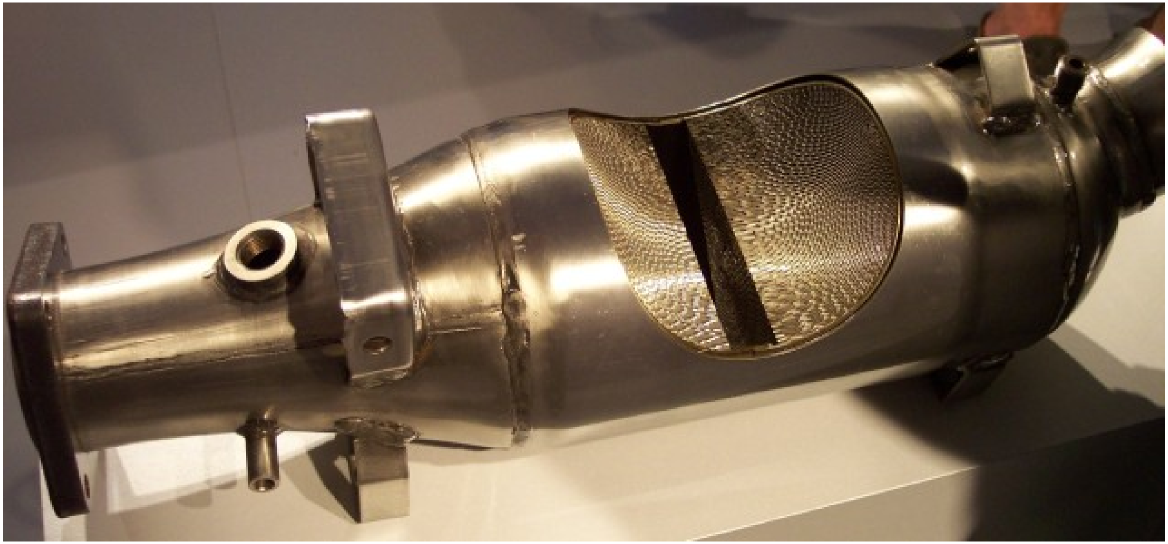

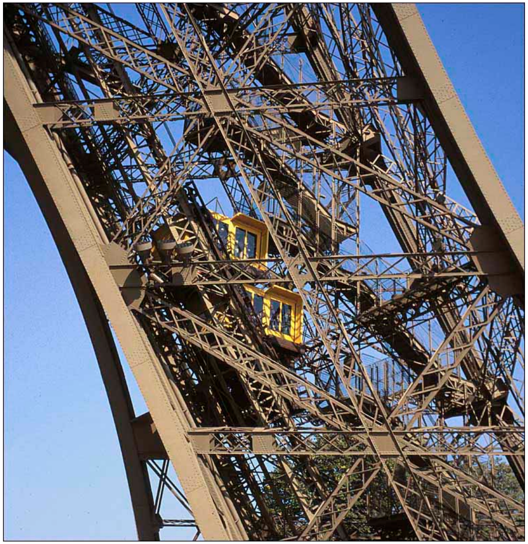

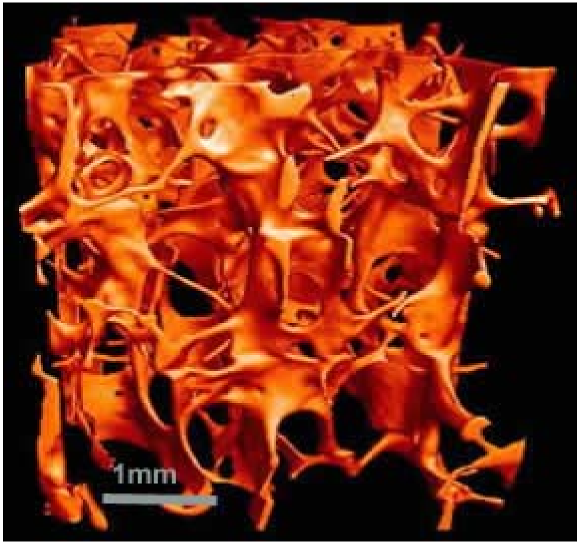
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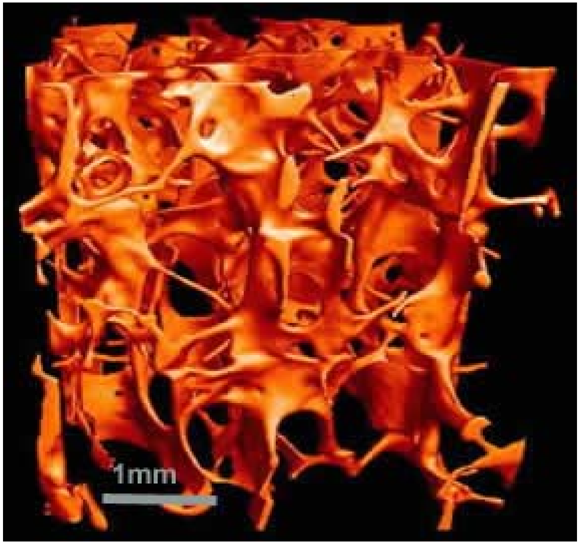
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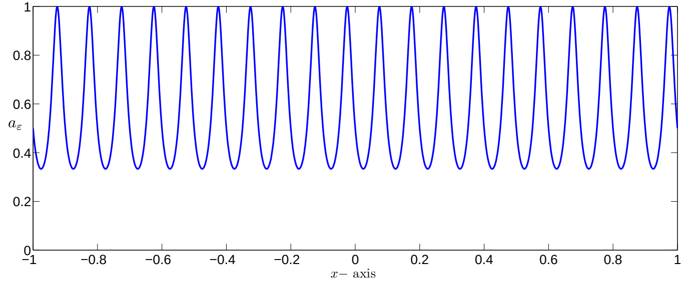
Sweet dream

A one dimensional example: oscillations



Let $a : \mathbb{R} \to \mathbb{R}$ be s.t. $0 < m \leq a(y) \leq M$; assume a 1-periodic:

$$a(y) = a(y+1) \qquad \text{for all} \qquad y \in [0,1), \qquad .$$

With $1 >> \varepsilon > 0$, define

$$a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right), \qquad \text{for all} \qquad x \in \mathbb{R},$$

and consider

$$(P^\varepsilon) \quad \begin{cases} -\frac{d}{dx}\left(a^\varepsilon(x)\frac{d}{dx}u^\varepsilon(x)\right) = 0, & \text{for } x \in (0,1), \\ u^\varepsilon(0) = 0, \quad u^\varepsilon(1) = 1. \end{cases}$$

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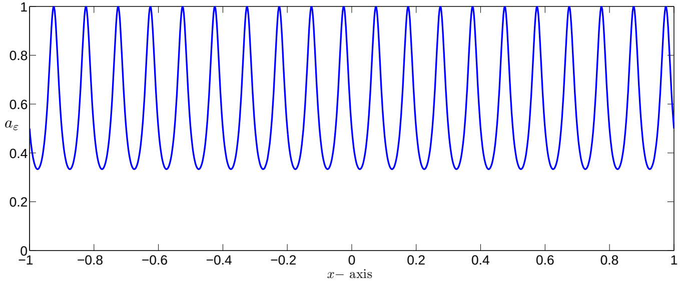

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Pb: Find an *averaged* u^* approximating u^ε, but including no oscillations.

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Rem: Alternatively, find an upscaled/averaged equation satisfied by u^*.

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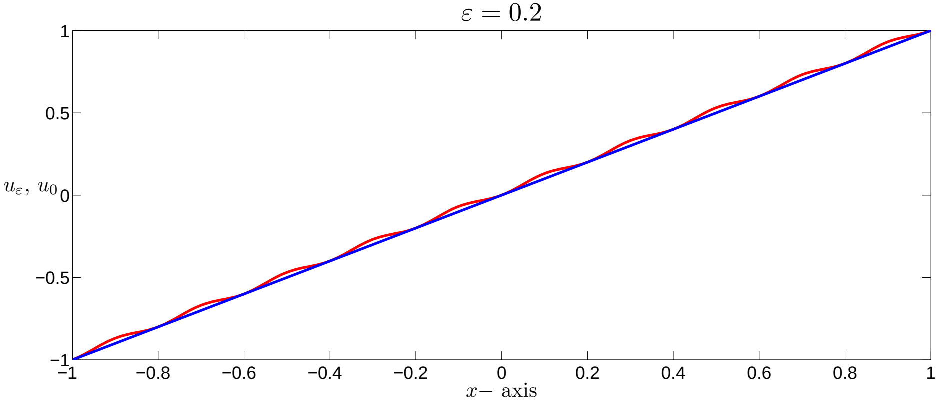
where $a^\varepsilon(x) = a(\frac{x}{\varepsilon})$ with $a(y) = a(y+1)$.

Solution:

$$u^\varepsilon(x) = \frac{\int_0^x \frac{1}{a^\varepsilon(z)}dz}{\int_0^1 \frac{1}{a^\varepsilon(z)}dz} = \frac{\int_0^{\frac{x}{\varepsilon}} \frac{1}{a(y)}dy}{\int_0^{\frac{1}{\varepsilon}} \frac{1}{a(y)}dy}.$$



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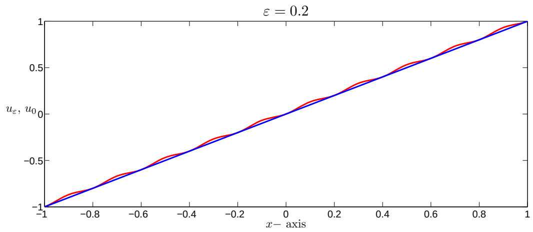
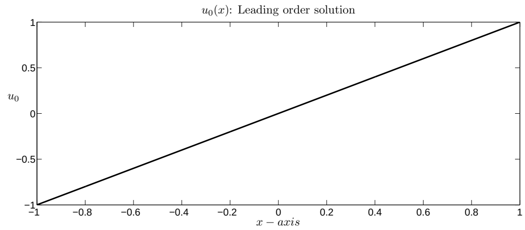


Effective quantities:

$$u^*(x) = x, \quad \text{and}$$

$$a^* = \frac{1}{\int_0^1 \frac{1}{a^\varepsilon(z)}dz} = \frac{1}{\int_0^1 \frac{1}{a(y)}dy}.$$



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Effective quantities:

$u^*(x) = x,$ and

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Then:

$$\begin{aligned} u^\varepsilon(x) &= \varepsilon a^* \int_0^{\frac{x}{\varepsilon}} \frac{1}{a(y)}dy \\ &= x + \varepsilon\int_0^{\frac{x}{\varepsilon}} \left(\frac{a^*}{a(y)} - 1\right)dy = u^*(x) + \varepsilon u_1\left(\tfrac{x}{\varepsilon}\right), \quad \text{with} \quad u_1(s) = \int_0^s \left(\frac{a^*}{a(y)} - 1\right)dy. \end{aligned}$$

Note: u^* - effective approximtion, u_1 - corrector (bounded!)

$$|u^\varepsilon(x) - u^*(x)| \le C\varepsilon$$

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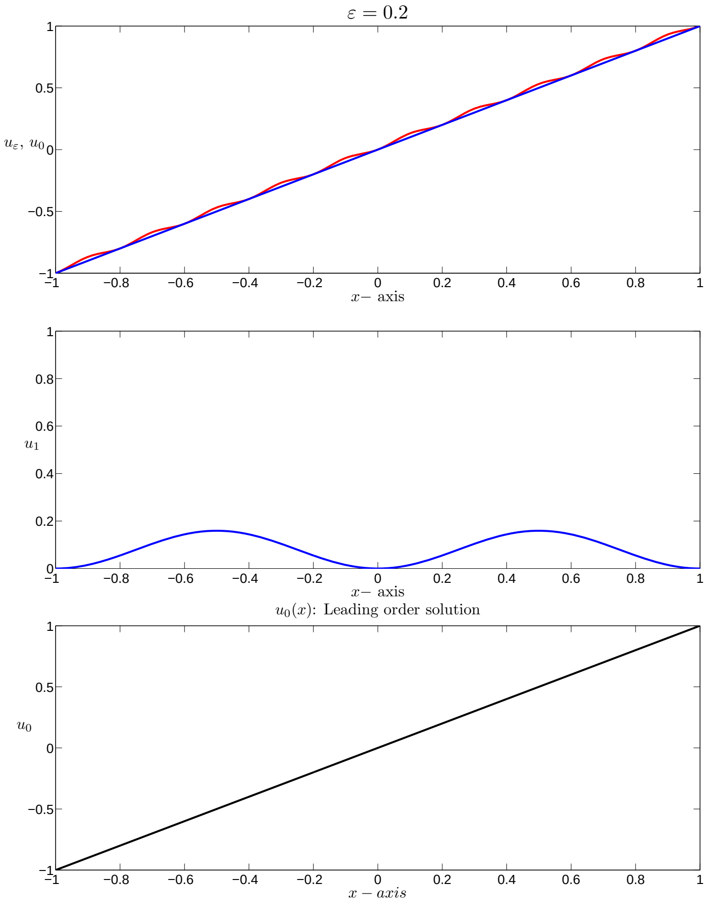
$\varepsilon = 0.2$
$u_\varepsilon,\ u_0$
$x-$ axis
u_1
$x-$ axis
$u_0(x)$: Leading order solution
u_0
$x - axis$

2. The asymptotic expansion method

Let $\varepsilon > 0$ (small), $\Omega \subset \mathbb{R}^d, (d \geq 1)$ - bounded domain ($\partial\Omega$ - the boundary), $Y = [0,1]^d$ - unit cube,
$a : \mathbb{R}^d \to \mathbb{R}$ s.t. $0 < m \leq a(y_1, \ldots, y_d) \leq M < \infty$, , and Y-periodic: for all $y = (y_1, \ldots, y_d) \in Y$,

$$a(y_1, y_2, \ldots, y_d) = a(y_1 + 1, y_2, \ldots, y_d) = a(y_1, y_2 + 1, \ldots, y_d) = \cdots = a(y_1, y_2, \ldots, y_d + 1).$$

With $a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$, consider

$$(P^\varepsilon) \quad \begin{cases} -\nabla \cdot (a^\varepsilon \nabla u^\varepsilon) = f, & \text{for all } x \in \Omega, \\ u^\varepsilon = 0, & \text{on } \partial\Omega. \end{cases}$$

Q: How to approximate u^ε?

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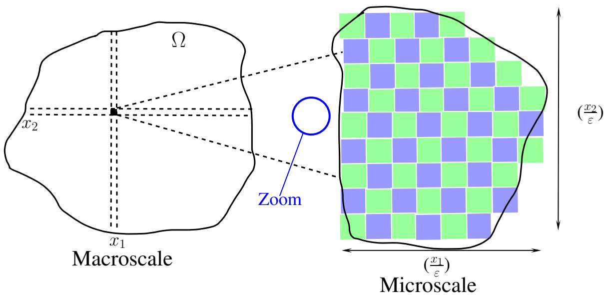
Q: How to approximate u^ε?

Idea: Multiple scales!

$$x \longrightarrow (x, y), \text{ with } \boxed{y = \frac{x}{\varepsilon}}$$



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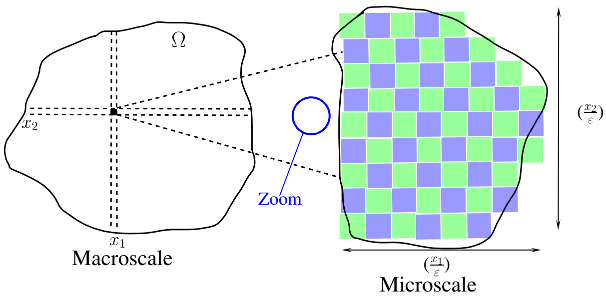

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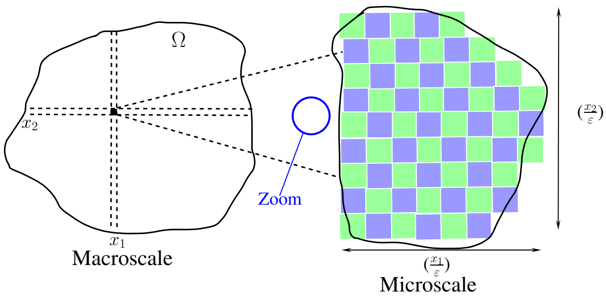
Homogenization ansatz:

$$u^\varepsilon(x) = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \ldots,$$

with u_k being Y-periodic.



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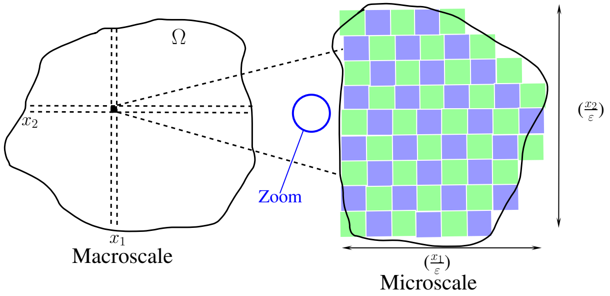
$$u^\varepsilon(x) = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \ldots,$$

with u_k being Y-periodic.

Note: (1) $u_0 = \lim_{\varepsilon \searrow 0} u^\varepsilon$, $u_1 = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}(u^\varepsilon - u_0)$, etc.



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Homogenization ansatz:

$$u^\varepsilon(x) = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \ldots,$$

with u_k being Y-periodic.

Note: (2) In fact, for any function f we can define $\tilde{f}(x) := f\left(x, \frac{x}{\varepsilon}\right) = f(x, y)$, implying that

$$\frac{\partial f}{\partial x_i}(x, y) \quad \text{becomes} \quad \frac{d\tilde{f}}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(x, y) + \frac{\partial y_i}{\partial x_i}\frac{\partial f}{\partial y_i}(x, y) = \frac{\partial f}{\partial x_i}(x, y) + \frac{1}{\varepsilon}\frac{\partial f}{\partial y_i}(x, y)$$

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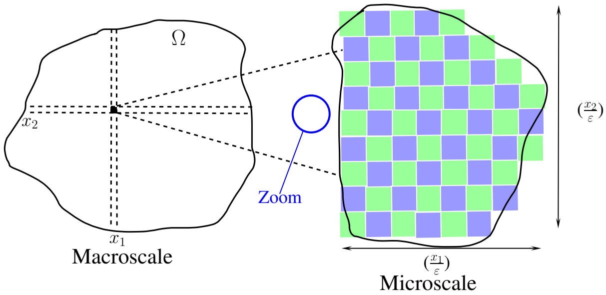
2.1. The diffusion problem



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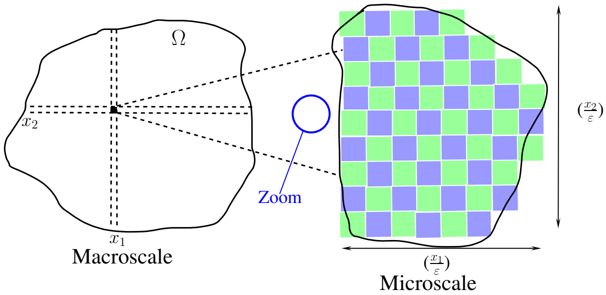

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Recall:

$u^\varepsilon(x) = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \ldots$, **with u_k being Y-periodic.**

$$a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right) = a(y)$$

$$\nabla \longrightarrow \nabla_x + \frac{1}{\varepsilon}\nabla_y$$

2.1. The diffusion problem

$$-\left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right) \cdot \left[a(y)\left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right)\left(u_0(x,y) \;\; + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \ldots\right)\right] = f.$$

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$$-\left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right)\cdot\left[a(y)\left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right)(u_0(x,y) \; + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \ldots)\right] = f.$$

Rewrites

$$-\frac{1}{\varepsilon^2}\,\nabla_y\cdot(a(y)\nabla_y u_0(x,y)) - \frac{1}{\varepsilon}\{\nabla_x\cdot[a(y)\nabla_y u_0(x,y)] + \nabla_y\cdot[a(y)(\nabla_x u_0(x,y) + \nabla_y u_1(x,y))]\}$$
$$-\{\nabla_x\cdot[a(y)(\nabla_x u_0(x,y) + \nabla_y u_1(x,y))] \; + \nabla_y\cdot[a(y)(\nabla_x u_1(x,y) + \nabla_y u_2(x,y))]\} - O(\varepsilon) = f.$$

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Next: equate terms of the same ε order

2.1. The diffusion problem

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Rewrites

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$$(Problem\ P^{-2}) \quad \begin{cases} -\nabla_y \cdot (a(y)\nabla_y u_0(x,y)) = 0, & \text{for all } y \in Y, \\ u_0(x,y) \text{ is } Y-\text{periodic.} \end{cases}$$

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$$-\left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right) \cdot \left[a(y)\left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right)(u_0(x,y) \ +\varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \dots)\right] = f.$$

Rewrites

$$-\frac{1}{\varepsilon^2}\,\nabla_y \cdot (a(y)\nabla_y u_0(x,y)) - \frac{1}{\varepsilon}\{\nabla_x \cdot [a(y)\nabla_y u_0(x,y)] + \nabla_y \cdot [a(y)(\nabla_x u_0(x,y) + \nabla_y u_1(x,y))]\}$$
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$$(Problem\ P^{-2}) \quad \begin{cases} -\nabla_y \cdot (a(y)\nabla_y u_0(x,y)) &= 0, \quad \text{for all } y \in Y, \\ u_0(x,y) & \text{is } \ Y - \text{periodic.} \end{cases}$$

Gives: $u_0(x,y) = u_0(x)$ and thus $\nabla_y u_0(x,y) = 0$ for all $y \in Y$.

Q: $u_0(x) =?$

$$-\frac{1}{\varepsilon^2}\nabla_y\cdot(a(y)\nabla_y u_0(x,y))-\frac{1}{\varepsilon}\{\nabla_x\cdot[a(y)\nabla_y u_0(x,y)]+\nabla_y\cdot[a(y)(\nabla_x u_0(x,y)+\nabla_y u_1(x,y))]\}$$
$$-\{\nabla_x\cdot[a(y)(\nabla_x u_0(x,y)+\nabla_y u_1(x,y))]+\nabla_y\cdot[a(y)(\nabla_x u_1(x,y)+\nabla_y u_2(x,y))]\}-O(\varepsilon)=f.$$

$$(Problem\ P^{-1})\quad \begin{cases} -\nabla_y\cdot\big(a(y)\nabla_y u_1(x,y)\big) = \nabla_y\cdot\big(a(y)\nabla_x u_0(x)\big), & \text{for all } y\in Y,\\ u_1(x,y) \text{ is } Y-\text{periodic}. \end{cases}$$

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$$-\frac{1}{\varepsilon^2}\nabla_y\cdot(a(y)\nabla_y u_0(x,y)) - \frac{1}{\varepsilon}\{\nabla_x\cdot[a(y)\nabla_y u_0(x,y)] + \nabla_y\cdot[a(y)(\nabla_x u_0(x,y)+\nabla_y u_1(x,y))]\}$$
$$-\{\nabla_x\cdot[a(y)(\nabla_x u_0(x,y)+\nabla_y u_1(x,y))] + \nabla_y\cdot[a(y)(\nabla_x u_1(x,y)+\nabla_y u_2(x,y))]\} - O(\varepsilon) = f.$$

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Note: Solution u_1 depends on u_0! With $\mathbf{e}_j = (0,\ldots,0,1,0,\ldots,0)$,

$$\nabla_x u_0(x) = \sum_{j=1}^{d}\mathbf{e}_j\,\partial_{x_j}u_0(x).$$

Suggestion: solve first the *cell problems* with e_j replacing $\nabla_x u_0(x)$!

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$$-\frac{1}{\varepsilon^2}\nabla_y\cdot(a(y)\nabla_y u_0(x,y)) - \frac{1}{\varepsilon}\{\nabla_x\cdot[a(y)\nabla_y u_0(x,y)] + \nabla_y\cdot[a(y)(\nabla_x u_0(x,y)+\nabla_y u_1(x,y))]\}$$
$$-\{\nabla_x\cdot[a(y)(\nabla_x u_0(x,y)+\nabla_y u_1(x,y))] + \nabla_y\cdot[a(y)(\nabla_x u_1(x,y)+\nabla_y u_2(x,y))]\} - O(\varepsilon) = f.$$

$$(Problem\ P^{-1}) \quad \begin{cases} -\nabla_y\cdot\big(a(y)\nabla_y u_1(x,y)\big) = \nabla_y\cdot\big(a(y)\nabla_x u_0(x)\big), & \text{for all } y\in Y,\\ u_1(x,y) \text{ is } Y-\text{periodic}. \end{cases}$$

Note: Solution u_1 depends on u_0! With $\mathbf{e}_j = (0,\dots,0,1,0,\dots,0)$,

$$\nabla_x u_0(x) = \sum_{j=1}^{d}\mathbf{e}_j\,\partial_{x_j}u_0(x).$$

Suggestion: solve first the *cell problems* with $\mathbf{e}_j$ replacing $\nabla_x u_0(x)$! For all $j=1,\dots,d$, consider:

$$(Problem\ P_j^{-1}) \quad \begin{cases} -\nabla_y\cdot\big(a(y)\nabla_y w^j(y)\big) = \nabla_y\cdot\big(a(y)\,\mathbf{e}_j\big), & \text{for all } y\in Y,\\ w^j(y) \text{ is } Y-\text{periodic},\\ \int_Y w^j(y)dy = 0. \end{cases}$$

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$$-\frac{1}{\varepsilon^2}\nabla_y\cdot(a(y)\nabla_y u_0(x,y)) - \frac{1}{\varepsilon}\{\nabla_x\cdot[a(y)\nabla_y u_0(x,y)] + \nabla_y\cdot[a(y)(\nabla_x u_0(x,y)+\nabla_y u_1(x,y))]\}$$
$$-\{\nabla_x\cdot[a(y)(\nabla_x u_0(x,y)+\nabla_y u_1(x,y))] + \nabla_y\cdot[a(y)(\nabla_x u_1(x,y)+\nabla_y u_2(x,y))]\} - O(\varepsilon) = f.$$

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$$\nabla_x u_0(x) = \sum_{j=1}^{d} \mathbf{e}_j\,\partial_{x_j} u_0(x).$$

This gives ($\tilde{u}_1(x)$ plays actually no role)

$$u_1(x,y) = \tilde{u}_1(x) + \sum_{j=1}^{d} w^j(y)\,\partial_{x_j} u_0(x),$$

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$$-\{\nabla_x\cdot[a(y)(\nabla_x u_0(x,y)+\nabla_y u_1(x,y))] + \nabla_y\cdot[a(y)(\nabla_x u_1(x,y)+\nabla_y u_2(x,y))]\} - O(\varepsilon) = f.$$

$$(Problem\ P^0)\quad \begin{cases} -\nabla_x\cdot[a(y)(\nabla_x u_0(x)+\nabla_y u_1(x,y))] - \nabla_y\cdot[a(y)(\nabla_x u_1(x,y)+\nabla_y u_2(x,y))] = f, \\ u_2(x,y) \text{ is } Y - \text{periodic.} \end{cases}$$

$$-\frac{1}{\varepsilon^2}\nabla_y \cdot (a(y)\nabla_y u_0(x,y)) - \frac{1}{\varepsilon}\{\nabla_x \cdot [a(y)\nabla_y u_0(x,y)] + \nabla_y \cdot [a(y)(\nabla_x u_0(x,y) + \nabla_y u_1(x,y))]\}$$
$$-\{\nabla_x \cdot [a(y)(\nabla_x u_0(x,y) + \nabla_y u_1(x,y))] + \nabla_y \cdot [a(y)(\nabla_x u_1(x,y) + \nabla_y u_2(x,y))]\} - O(\varepsilon) = f.$$

$$(\textit{Problem } P^0) \quad \begin{cases} -\nabla_x \cdot [a(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y))] - \nabla_y \cdot [a(y)(\nabla_x u_1(x,y) + \nabla_y u_2(x,y))] = f, \\ u_2(x,y) \text{ is } Y-\text{periodic}. \end{cases}$$

Option: Find u_2 in terms of u_0 and u_1... still gives no u_0!

$$-\frac{1}{\varepsilon^2}\nabla_y\cdot(a(y)\nabla_y u_0(x,y))-\frac{1}{\varepsilon}\{\nabla_x\cdot[a(y)\nabla_y u_0(x,y)]+\nabla_y\cdot[a(y)(\nabla_x u_0(x,y)+\nabla_y u_1(x,y))]\}$$
$$-\{\nabla_x\cdot[a(y)(\nabla_x u_0(x,y)+\nabla_y u_1(x,y))]+\nabla_y\cdot[a(y)(\nabla_x u_1(x,y)+\nabla_y u_2(x,y))]\}-O(\varepsilon)=f.$$

$$(Problem\ P^0)\quad\begin{cases}-\nabla_x\cdot[a(y)(\nabla_x u_0(x)+\nabla_y u_1(x,y))]-\nabla_y\cdot[a(y)(\nabla_x u_1(x,y)+\nabla_y u_2(x,y))]=f,\\ u_2(x,y)\text{ is }Y-\text{periodic}.\end{cases}$$

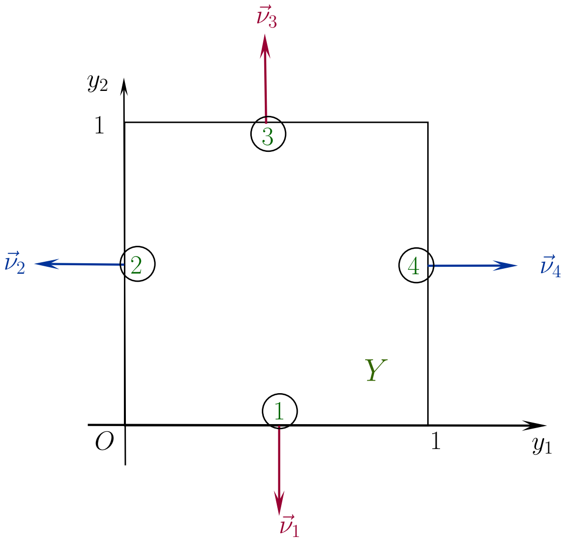
Alternative: Eliminate u_2 by integration!

$$-\nabla_x\cdot\left[\int_Y a(y)(\nabla_x u_0(x)+\nabla_y u_1(x,y))dy\right]-\int_Y\nabla_y\cdot[a(y)(\nabla_x u_1(x,y)+\nabla_y u_2(x,y))]dy=f(x),$$

$$-\nabla_x \cdot \left[\int_Y a(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y))dy\right] - \int_Y \nabla_y \cdot [a(y)(\nabla_x u_1(x,y) + \nabla_y u_2(x,y))]dy = f(x)$$

We have

$$\begin{aligned}\int_Y \nabla_y \cdot [a(y)(\nabla_x u_1(x,y) + \nabla_y u_2(x,y))]dy &= \int_{\partial Y} \nu \cdot [a(y)(\nabla_x u_1(x,y) + \nabla_y u_2(x,y))]d\sigma_y, \\ &= \sum_{k=1}^{4} \int_{\partial Y_k} \nu_k \cdot [a(y)(\nabla_x u_1(x,y) + \nabla_y u_2(x,y))]d\sigma_y = 0\end{aligned}$$



$$-\nabla_x \cdot \left[\int_Y a(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y))dy\right] = f(x)$$

Since

$$\nabla_y u_1(x,y) = \sum_{j=1}^{d} \partial_{x_j} u_0(x)\ \nabla_y w^j(y),$$

we have

$$-\nabla_x \cdot \left[\int_Y a(y) \left(\nabla_x u_0(x) + \sum_{j=1}^{d} \partial_{x_j} u_0(x)\ \nabla_y w^j(y)\right) dy\right] = f(x).$$

or, equivalently,

$$(P) \quad \begin{cases} -\nabla \cdot (A^* \nabla U) & = & f, & \text{for all } x \in \Omega, \\ U & = & 0, & \text{on } \partial\Omega, \end{cases}$$

with $A^* \in \mathbb{R}^{d \times d}$, $a^*_{ij} = \displaystyle\int_Y a(y)\big(\delta_{ij} + \partial_{y_i} w^j(y)\big)dy = \int_Y a(y)\big(\mathbf{e}_j + \nabla_y w^j(y)\big)\mathbf{e}_i dy$

$$-\nabla_x \cdot \left[\int_Y a(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y))dy\right] = f(x)$$

Since

$$\nabla_y u_1(x,y) = \sum_{j=1}^{d} \partial_{x_j} u_0(x)\, \nabla_y w^j(y),$$

we have

$$-\nabla_x \cdot \left[\int_Y a(y) \left(\nabla_x u_0(x) + \sum_{j=1}^{d} \partial_{x_j} u_0(x)\, \nabla_y w^j(y)\right) dy\right] = f(x).$$

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Lemma

a) A^* is symmetric, i.e. $a^*_{ij} = a^*_{ji}$ for all $i, j = 1, \ldots, d$.

b) A^* is positive definite, i.e. there exist a constant $C > 0$ s.t. for all (column vectors) $z \in \mathbb{R}^d$, $z^T(A^* z) \geq C(z^T z)$.