2DD50 - Solutions to exercises week 4

Concep. 2.8 We have

$$X_{n+1} = \begin{cases} 1 & \text{with probability } 1, \text{ if } X_n = 0 \\ 2 & \text{with probability } 1, \text{ if } X_n = 1 \\ 2 & \text{with probability } p, \text{ if } X_n = 2 \\ 0 & \text{with probability } 1-p, \text{ if } X_n = 2. \end{cases}$$

Since X_{n+1} depends only on the current state and not on the past, $\{X_n, n \geq 0\}$ is a DTMC on state space $S = \{0, 1, 2\}$. The transition probability matrix is given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-p & 0 & p \end{bmatrix}.$$

Concep. 2.9 We have

$$X_n = \begin{cases} 1 & \text{if the item that arrives to the shop during} \\ & \text{the } (n-1)^{st} \text{ minute is non-defective} \\ 0 & \text{otherwise.} \end{cases}$$

Since X_{n+1} does not depend on the past, $\{X_n, n \geq 0\}$ is a DTMC on state space $S = \{0, 1\}$. The transition probability matrix is given by

$$P = \begin{bmatrix} 1-p & p \\ 1-p & p \end{bmatrix}.$$

Concep. 2.10 Let X_n

$$= \begin{cases} 1 & \text{if the machine is idle at the beginning of the } n^{th} \text{ minute} \\ & \text{and there are no items in the bin} \\ 2 & \text{if the machine is idle at the beginning of the } n^{th} \text{ minute} \\ & \text{and there is one item in the bin} \\ 3 & \text{if the machine has been busy for one minute at the beginning} \\ & \text{of the } n^{th} \text{ minute and there are no items in the bin} \\ 4 & \text{if the machine has been busy for one minute at the beginning} \\ & \text{of the } n^{th} \text{ minute and there is one item in the bin} \\ 5 & \text{if the machine has just started production at the beginning} \\ & \text{of the } n^{th} \text{ minute and there are no items in the bin.} \end{cases}$$

Then, $\{X_n, n \geq 0\}$ is a DTMC on $S = \{1, 2, 3, 4, 5\}$ with the transition probability matrix

$$P = \begin{bmatrix} 1-p & p & 0 & 0 & 0 \\ 0 & 1-p & 0 & 0 & p \\ 1-p & p & 0 & 0 & 0 \\ 0 & 1-p & 0 & 0 & p \\ 0 & 0 & 1-p & p & 0 \end{bmatrix}.$$

Concep. 2.11 Let X_n

$$= \begin{cases} 1 & \text{if the machine is idle at the beginning of the } n^{th} \text{ minute} \\ & \text{and there are no items in the bin} \\ 2 & \text{if the machine has been busy for one minute at the beginning} \\ & \text{of the } n^{th} \text{ minute and there are no items in the bin} \\ 3 & \text{if the machine has been busy for one minute at the beginning} \\ & \text{of the } n^{th} \text{ minute and there is one item in the bin.} \\ 4 & \text{if the machine has just started production at the beginning} \\ & \text{of the } n^{th} \text{ minute and there are no items in the bin} \end{cases}$$

Then, $\{X_n, n \geq 0\}$ is a DTMC on $S = \{1, 2, 3, 4\}$ with the transition probability matrix

$$P = \begin{bmatrix} 1-p & 0 & 0 & p \\ 1-p & 0 & 0 & p \\ 0 & 0 & 0 & 1 \\ 0 & 1-p & p & 0 \end{bmatrix}.$$

Concep. 2.12 Let

$$X_n = \begin{cases} 0 & \text{if day } n \text{ and } n-1 \text{ are both rainy} \\ 1 & \text{if day } n \text{ is sunny and day } n-1 \text{ is rainy} \\ 2 & \text{if day } n \text{ is rainy and day } n-1 \text{ is sunny} \\ 3 & \text{if day } n \text{ and } n-1 \text{ are both sunny.} \end{cases}$$

Then, $\{X_n, n \geq 0\}$ is a DTMC on state space $S = \{0, 1, 2, 3\}$ with the transition probability matrix

$$P = \begin{bmatrix} .4 & .6 & 0 & 0 \\ 0 & 0 & .2 & .8 \\ .3 & .7 & 0 & 0 \\ 0 & 0 & .1 & .9 \end{bmatrix}.$$

Concep. 2.16 We have for $1 \le X_n \le (N-1)$

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } (N - X_n)/N \\ X_n - 1 & \text{with probability } X_n/N. \end{cases}$$

Further we have

$$\mathsf{P}(X_{n+1} = 1 | X_n = 0) = 1, \ \mathsf{P}(X_{n+1} = N-1 | X_n = N) = 1.$$

Since X_{n+1} depends only on the current state and not on the past, $\{X_n, n \ge 0\}$ is a DTMC on state space $S = \{0, \ldots, N\}$ with transition probabilities

$$p_{i,i+1} = \frac{N-i}{N}, \ \ p_{i,i-1} = \frac{i}{N} \ (1 \le i \le (N-1)), \ p_{0,1} = 1, \ p_{N,N-1} = 1.$$

Comp. 2.7 The distribution of the grade of a randomly chosen employee at the beginning of week one is $a = [.50\ .25\ .15\ .10]$. Hence we study one employee with initial distribution a. Hence,

$$\mathsf{P}(X_3 = j) = \sum_{i=1}^{4} a_i p_{i,j}^{(3)}$$

is the probability that a randomly chosen employee ends up in grade j at the beginning of week 4. Hence, the expected number of employees in grade j is given by

$$100 * [.50\ .25\ .15\ .10] * P^3 = [47.40\ 28.01\ 14.67\ 9.92],$$

where P is from Example 2.6.

Comp. 2.8 Using the P matrix from Equation (2.7), the desired probability is obtained as $\mathsf{P}(X_3 = 0, X_2 = 0, X_1 = 1 | X_0 = 1)$

$$\begin{aligned} &= \mathsf{P}(X_3 = 0 | X_2 = 0) \mathsf{P}(X_2 = 0 | X_1 = 1) \mathsf{P}(X_1 = 1 | X_0 = 1) \\ &= p_{0,0} p_{1,0} p_{1,1} = (0.03)(0.02)(0.98) = 5.88 \times 10^{-4}. \end{aligned}$$

Comp. 2.9 1) Y_n = the number of machines "up" at the beginning of day n. State space is $S = \{0, 1, 2\}$. The second column of P contains the probabilities $p_{i,\ 1}$ $(i = 0, 1, 2)$.

$p_{0,\ 1} = \mathsf{P}(Y_{n+1} = 1|Y_n = 0) =$
$\mathsf{P}($ 1 machine "up" at day $(n+1)|$ 2 machines "down" at day $n)$
$= 2$*0.03*0.97 $= 0.0582$,

$p_{1,\ 1} = \mathsf{P}(Y_{n+1} = 1|Y_n = 1) =$
$\mathsf{P}($ 1 machine "up" at day $(n+1)|$ 1 machine "up" at day $n) =$
$\mathsf{P}($ machine "up" stays "up", machine "down" stays "down"$) +$
$\mathsf{P}($ machine "down" becomes "up", machine "up" goes "down"$) =$
$0.98 * 0.03 + 0.97 * 0.02 = 0.0488$,

$p_{2,\ 1} = \mathsf{P}(Y_{n+1} = 1|Y_n = 2) =$
$\mathsf{P}($ 1 machine "up" at day $(n+1)|$ 2 machines "up" at day $n) =$
$\mathsf{P}($ 1 machine "up" goes "down", the other machine stays "up"$) =$
$0.98 * 0.02 * 2 = 0.0392$.

2) Using the P matrix from Equation (2.8), the desired probability is obtained as $\mathsf{P}(X_3 = 2, X_2 = 1, X_1 = 2|X_0 = 2)$

$$\begin{aligned} &= \mathsf{P}(X_3 = 2|X_2 = 1)\mathsf{P}(X_2 = 1|X_1 = 2)\mathsf{P}(X_1 = 2|X_0 = 2) \\ &= p_{1,2}p_{2,1}p_{2,2} = (0.9506)(0.0392)(0.9604) = 0.0358. \end{aligned}$$

Comp. 2.12 1) X_n is the number of PC's in stock at 8:00 a.m. Monday of the nth week. If there are 2 or more PC's in the store at 5:00 p.m. Friday of the nth week, then no more PC's will be ordered that weekend and then the number in stock at 8:00 a.m. Monday of the $(n+1)$th week is equal to the number in stock at 5:00 p.m. Friday of the nth week. Otherwise PC's are ordered that weekend and the number in stock at 8:00 a.m. Monday of the $(n+1)$th week is equal to 5. If the number in stock at 8:00 a.m. Monday is taken as state of the DTMC, then the state space is $S = \{2, 3, 4, 5\}$.
The demand D_n in week n is Poisson distributed with an average demand of 3 PC's. The distribution is given in Table 2.1.

Only the probabilities in the second and the fourth row of P are explained below, the probabilities in the first and third row can be derived in an analogous way.
$p_{3,\ 2} = \mathsf{P}(X_{n+1} = 2|X_n = 3) = \mathsf{P}(D_n = 1) = 0.1494$,
$p_{3,\ 3} = \mathsf{P}(X_{n+1} = 3|X_n = 3) = \mathsf{P}(D_n = 0) = 0.0498$,

$p_{3,\ 4} = \mathsf{P}(X_{n+1} = 4|X_n = 3) = 0,$
$p_{3,\ 5} = \mathsf{P}(X_{n+1} = 5|X_n = 3) = \mathsf{P}(D_n \geq 2) = 0.8008.$

$p_{5,\ 2} = \mathsf{P}(X_{n+1} = 2|X_n = 5) = \mathsf{P}(D_n = 3) = 0.2240,$
$p_{5,\ 3} = \mathsf{P}(X_{n+1} = 3|X_n = 5) = \mathsf{P}(D_n = 2) = 0.2240,$
$p_{5,\ 4} = \mathsf{P}(X_{n+1} = 4|X_n = 5) = \mathsf{P}(D_n = 1) = 0.1494,$
$p_{5,\ 5} = \mathsf{P}(X_{n+1} = 5|X_n = 5) = \mathsf{P}(D_n \geq 4 \cup D_n = 0) = 0.0498 + 0.3528 = 0.4026.$

2) Using the P matrix of Example 2.4, the desired probability is obtained as $\mathsf{P}(X_4 = 3, X_3 = 5, X_2 = 2, X_1 = 4|X_0 = 5)$

$$= p_{5,4}p_{4,2}p_{2,5}p_{5,3} = (.1494)(.2240)(.9502)(.2240) = 0.0071.$$

Comp. 2.13 This implies that 4 or more PC's are sold during the previous week, hence the desired probability is

$$\mathsf{P}(D_n \geq 4) = 0.3528.$$

Comp. 2.19 a. irreducible, b. irreducible, c. reducible, d. reducible.

Comp. 2.20 a. aperiodic, b. periodic with period 4, c. periodic with period 2, d. periodic with period 3.

Comp. 2.24 The questions 1) and 2) are solved simultaneously. We solve the normalized balance equations of Theorem 2.5 "by hand".

1. The limiting distribution does not exist since the DTMC is periodic.
2. The stationary distribution π^* is given by the solution to (see Theorem 2.6)

$$[\pi_1^*\ \pi_2^*\ \pi_3^*\ \pi_4^*] = [\pi_1^*\ \pi_2^*\ \pi_3^*\ \pi_4^*] \cdot \begin{bmatrix} 0 & 0 & .40 & .60 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\pi_1^* + \pi_2^* + \pi_3^* + \pi_4^* = 1.$$

Otherwise:

$$\begin{array}{rcl} \pi_1^* & = & \pi_2^*, \\ \pi_2^* & = & \pi_3^* + \pi_4^*, \\ \pi_3^* & = & 0.40\pi_1^*, \\ \pi_4^* & = & 0.60\pi_1^*, \\ 1 & = & \pi_1^* + \pi_2^* + \pi_3^* + \pi_4^*. \end{array}$$

Solving "by hand", we get $\pi^* = (\frac{1}{3}, \frac{1}{3}, \frac{2}{15}, \frac{1}{5})$.

3. By Theorem 2.7 or 2.9, the occupancy distribution is given by $\hat{\pi} = \pi^*$.

Exercise X1 1) We have that

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

so that

$$P^3 = \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{3}{8} & \frac{1}{8} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8}. \end{pmatrix}$$

As such, $P(X_3 = 1|X_0 = 1) = 0$, $P(X_3 = 2|X_0 = 1) = 3/4$ and $P(X_3 = 3|X_0 = 1) = 1/4$.

2) Note that the exercise asks for the occupancy time $m_{2,2}(3)$. We have that $M(3) = P^0 + P^1 + P^2 + P^3$. Working this out yields

$$M(3) = \begin{pmatrix} \frac{3}{2} & \frac{7}{4} & \frac{3}{4} \\ \frac{7}{8} & \frac{15}{8} & \frac{5}{4} \\ \frac{3}{8} & \frac{5}{4} & \frac{19}{8} \end{pmatrix}$$

As such, $m_{2,2}(3) = \frac{15}{8}$.

c) The limiting distribution is given by $\pi = (\pi_1, \pi_2, \pi_3) = (1/5, 2/5, 2/5)$.

Exercise X2 1)

2) 0.26

3) Irreducible, aperiodic

4, 5, 6) The stationary, occupancy and limiting distributions are given by $\pi = (\pi_1, \pi_2, \pi_3, \pi_4) = (5/11, 5/22, 2/11, 3/22)$.

Handout section 1

Exercise 2 (a) End classes $E_1 = \{4,5,6,7\}$, $E_2 = \{8\}$, $E_3 = \{9,10\}$. $C = \{1,2,3\}$.

(b) The limiting distribution over $S = \{1,2,3,4,5,6,7,8,9,10\}$ is $\pi = (0,0,0,\frac{14}{255},\frac{14}{85},\frac{14}{85},\frac{7}{85},\frac{1}{6},\frac{11}{150},\frac{22}{75})$.

(c) We have

$$P^{\infty} = \begin{pmatrix} 0 & 0 & 0 & \frac{8}{255} & \frac{8}{85} & \frac{8}{85} & \frac{4}{85} & \frac{1}{6} & \frac{17}{150} & \frac{34}{75} \\ 0 & 0 & 0 & \frac{14}{255} & \frac{14}{85} & \frac{14}{85} & \frac{7}{85} & \frac{1}{6} & \frac{11}{150} & \frac{22}{75} \\ 0 & 0 & 0 & \frac{2}{51} & \frac{2}{17} & \frac{2}{17} & \frac{1}{17} & \frac{1}{3} & \frac{1}{15} & \frac{4}{15} \\ 0 & 0 & 0 & \frac{2}{17} & \frac{6}{17} & \frac{6}{17} & \frac{3}{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{17} & \frac{6}{17} & \frac{6}{17} & \frac{3}{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{17} & \frac{6}{17} & \frac{6}{17} & \frac{3}{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{17} & \frac{6}{17} & \frac{6}{17} & \frac{3}{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \end{pmatrix}.$$

Exercise 3 (a) The state of a student is registrated at the end of a week. During the presence of a student in the training, this is done immediately before taking the exam belonging to the course of that week. The state space is $S = \{1,2,3,4,5,6,7,8\}$ with

1: (C_1, f) = Course 1, first trial,
2: (C_1, s) = Course 1, second trial,
3: (C_2, f) = Course 2, first trial,
4: (C_2, s) = Course 2, second trial,
5: (C_3, f) = Course 3, first trial,
6: (C_3, s) = Course 3, second trial,
7: "left training without diploma",
8: "left training with diploma".

The matrix of transition probabilities P at S is:

$$P = \begin{pmatrix} 0 & \frac{15}{100} & \frac{7}{10} & 0 & 0 & 0 & \frac{15}{100} & 0 \\ 0 & 0 & \frac{5}{10} & 0 & 0 & 0 & \frac{5}{10} & 0 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{8}{10} & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 & \frac{6}{10} & 0 & \frac{4}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{100} & \frac{5}{100} & \frac{9}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{10} & \frac{7}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) the possible paths through the state space S are

$1 \to 2 \to 3 \to 5 \to 7$,
$1 \to 2 \to 3 \to 4 \to 7$,
$1 \to 3 \to 4 \to 5 \to 7$,
$1 \to 3 \to 5 \to 6 \to 7$.

The total probability is $\frac{33}{2000} = 0.0165$.

(c) The probabilities are $\frac{249271}{400000} = 0.6232$ en $\frac{150729}{400000} = 0.3768$

Exercise 4 (a) The state of an employee is registrated at the end of a calendar year n. For an employee in this job, the state is the salary level in the salary scale for next year $(n+1)$. For an employee who has left this job, the state is characterised by "LC = Left Company" or by "AF = Another Function". In the model is assumed that new employees in this job always start on January 1 of a year on salary level 1.
The state space is $S = \{1, 2, 3, 4, AF, LC\}$.

The matrix of transition probabilities P at this S is:

$$P = \begin{pmatrix} 0 & 0.8 & 0 & 0 & 0 & 0.2 \\ 0 & 0.3 & 0.5 & 0 & 0 & 0.2 \\ 0 & 0 & 0.4 & 0.3 & 0.1 & 0.2 \\ 0 & 0 & 0 & 0.5 & 0.4 & 0.1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) There are several paths to state AF after 4 years.

year 1	year 2	year 3	year 4	year 5
1	2	2	3	AF
1	2	3	3	AF
1	2	3	4	AF
1	2	3	AF	AF

The total probability on these paths is: $(0.8)(0.3)(0.5)(0.1) + (0.8)(0.5)(0.4)(0.1) + (0.8)(0.5)(0.3)(0.4) + (0.8)(0.5)(0.1)(1) = 0.116$.

(c) The probability is $\dfrac{71}{105} = 0.6762$.
The percentage is 32.38%.