2DD50 - Solutions to exercises week 5

Concep. 2.15 Let A denote the event that the DTMC visits state 1 before it visits state N. Then

$$\begin{aligned} u_i = \mathsf{P}(A|X_0 = i) &= \sum_{j=1}^{N} \mathsf{P}(A|X_1 = j, X_0 = i)\mathsf{P}(X_1 = j|X_0 = i) \\ &= \sum_{j=1}^{N} \mathsf{P}(A|X_0 = j)\mathsf{P}(X_1 = j|X_0 = i) \\ &= \sum_{j=1}^{N} u_j p_{i,j}. \end{aligned}$$

Also, if $X_0 = 1$, then A occurs with probability 1, and if $X_0 = N$, then A occurs with probability 0. Hence

$$u_1 = 1, \quad u_N = 0.$$

Concep. 2.18 The total expected cost incurred by the DTMC until it hits state N is zero if it starts in state N. Hence, $g(N) = 0$. Let C be the total cost incurred by the DTMC until it visits state N. Then, for $1 \le i \le N-1$,

$$\begin{aligned} g(i) = \mathsf{E}(C|X_0 = i) &= \sum_{j=1}^{N} \mathsf{E}(C|X_0 = i, X_1 = j)\mathsf{P}(X_1 = j|X_0 = i) \\ &= \sum_{j=1}^{N} \big(\mathsf{E}(C|X_1 = j) + c(i)\big)\mathsf{P}(X_1 = j|X_0 = i) \\ &= c(i) + \sum_{j=1}^{N} g(j) p_{i,j}. \end{aligned}$$

The first line is due to conditioning and the law of total probability, the second line uses the Markov property and the definition of $c(i)$, and the last line uses again conditioning and the law of total probability (to extract $c(i)$ from the sum) and the definition of $g(j)$.

Concep. 2.19 In the cost model of Section 2.6, the DTMC incurs a random cost $C(i)$ every time it visits state i. In the cost per transition model, this random cost is $C(i,j)$ with probability $p_{i,j}$. Hence, the expected cost incurred upon a visit to state i is given by

$$c(i) = \mathsf{E}(C(i)) = \sum_{j=1}^{N} \mathsf{E}(C(i,j))p_{i,j}.$$

Thus the total expected cost in the cost per transition model can be computed using Theorem 2.11 with this $c(i)$. Similarly the long run cost rate for the cost per transition model can be computed by using Theorem 2.12 with this $c(i)$.

Comp. 2.33 Let $c(i)$ be the expected number of items finished by the end of the nth minute if the machine is in state i at the beginning of the nth minute. Referring to the solution of Conceptual Problem 2.10 (see solutions to exercises week 4), we get $c = [0\ 0\ 2\ 2\ 0]'$. over the state space $S = \{1, 2, 3, 4, 5\}$.

a) If the machine is idle at the beginning of the first minute, i.e. $a^{(0)} = (1, 0, 0, 0, 0)$ then $a^{(4)} = a^{(0)}P^4$ gives the probability distribution at the beginning of the fifth minute. In this case $a^{(4)}$ equals the first row of the matrix P^4.
Thus, $a^{(4)} = (0.0023, 0.0862, 0.0045, 0.0857, 0.8213)$.
The expected number of visits to state $i \in S$ during the fifth minute equals $1{*}a_i^{(4)}+0{*}(1-a_i^{(4)}) = a_i^{(4)}$. Hence the expected number of processed items in the fifth minute equals $\sum_{i=1}^{5} a_i^{(4)}c(i) = 2 * (0.0045 + 0.0857) = 0.1804$ items.

b) In an analogous way the expected number of items processed in the fifth and sixth minute is equal to
$\sum_{i=1}^{5} (a_i^{(4)} + a_i^{(5)})c(i) = 2 * (0.0045 + 0.0857 + 0.0411 + 0.7802) =$ 1.823 items.

c) We want to compute $g(1, 9)$. Using Theorem 2.11, we get

$$g(9) = M(9){*}c = (\sum_{i=1}^{9} P^i){*}c = [7.3152\ 7.8848\ 9.3152\ 9.8848\ 9.3152]'.$$

Thus the machine produces 7.3152 items on the average in the first 10 minutes if it idle and the bin is empty at time 0. Observe that we do not use the matrix M(10).

Comp. 2.37

a) In state i (number of PC's in stock at 8:00 a.m. Monday) the storage cost during a week is -50 $* i$. Buying cost (-1500) and selling revenue (1750) are considered in the week in which a PC is sold. The demand in a week is D_n, the expected number of PC's sold is $\mathsf{E}(\min(i, D_n))$. Hence the net revenue becomes $c(i) = -50 * i + (1750 - 1500) * \mathsf{E}(\min(i, D_n))$ $(i = 2, 3, 4, 5)$.

b) Starting with $a^{(0)} = (0, 0, 0, 1)$ the expected revenue in the third week is $\sum_{i=2}^{5} a_i^{(2)} c(i) = 440.76$.

c) The long run expected cost per week of operating the system is given by (using Theorem 2.12) πc where c is given in Example 2.27 as

$$c = [337.76\ 431.96\ 470.15\ 466.325]'.$$

Normalising the solution of the vector q as given in the exercise, we obtain the limiting distribution as given below

$$\pi = [.1812\ .1504\ .0908\ .5775].$$

Hence, it costs \$438.20 per week to operate this system.

Exercise X3

a) Recall from the solution to Exercise X1 that

$$M(3) = \begin{pmatrix} \frac{3}{2} & \frac{7}{4} & \frac{3}{4} \\ \frac{7}{8} & \frac{15}{8} & \frac{5}{4} \\ \frac{3}{8} & \frac{5}{4} & \frac{19}{8} \end{pmatrix}.$$

Novak pays 20 euro to Roger if the chain visits state 1. Hence, $c = (20, 0, 0)$. Using Theorem 2.11, we thus have that the expected amount of money that Novak pays to Roger equals $20 * \frac{3}{8} = 7.5$. Likewise, Roger wins 20 euro per visit to state 1, and loses ten euro per visit to state 2. As such, by taking $c = (-20, 10, 0)$ and Theorem 2.11, Roger expects to lose $-20 * \frac{3}{8} + 10\frac{5}{4} = 5$ euro.

b) Recall from the solution to Exercise X1 that the occupancy distribution is given by $\pi = (1/5, 2/5, 2/5)$. Novak pays 20 euro to Roger whenever the chain visits state 1, but gets 10 euro back whenever the chain visits state 2. Thus, by taking $c = (20, -10, 0)$ and using Theorem 2.12, Novak loses on average an amount of $20 * 1/5 - 10 * 2/5 + 0 * 2/5 = 0$ euro to Roger per transition in the long run. As such, no-one is better off in the long run.

c) We are looking for the first passage time into state 3. Solving the set of equations

$$m_1 = 1 + m_2$$
$$m_2 = 1 + 0.5 * m_1$$

leads to $m_1 = 4$, $m_2 = 3$. As such, the answer is three transitions.

Exercise X4

a) Recall that the occupancy distribution of this Markov chain satisfies $\pi = (5/11, 5/22, 2/11, 3/22)$. Using $c = (1400, 1400, 1050, 0)$, Theorem 2.12 gives us 1145.45 units of work per week.

b) Using $c = (0, 0, 0, 500)$, we have by Theorem 2.12 that the repairman earns 68.18 euro per week on average from this processor, which is 3545.45 euro per year.

c) The expected time between two repairs equals the expected time for a perfect processor to become bad. Solving the set of equations

$$m_1 = 1 + 0.7 * m_1 + 0.2 * m_2 + 0.1 * m_3$$
$$m_2 = 1 + 0.6 * m_2 + 0.2 * m_3$$
$$m_3 = 1 + 0.5 * m_3$$

leads to $m_1 = 6.33$, $m_2 = 3.5$ and $m_3 = 2$. As such, the answer is 6.33 weeks.

Handout section 2

Exercise 1

(a) -

(b)
   i. 13, 5, 2.1
   ii. 18 trainees, 1 junior mechanic, 0 senior mechanics

Exercise 2

(a) The state space is $S = \{1, 2, 3, 4, LIC\}$, where i ($i = 1, 2, 3, 4$) stands for premium level i and LIC stands for "Left Insurance Company".
The matrix of transition probabilities P at this S is:

$$P = \begin{pmatrix} 0.5 & 0.3 & 0 & 0 & 0.2 \\ 0.4 & 0 & 0.5 & 0 & 0.1 \\ 0 & 0.25 & 0 & 0.7 & 0.05 \\ 0 & 0 & 0.125 & 0.85 & 0.025 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) The expected numbers on the four premium levels are: 31500, 24500, 39375, 58750.

(c) The long-run expected numbers on the four premium levels are: $\approx 23478, \approx 29348, \approx 49217, \approx 236348$.