2DD50 - Solutions to exercises week 6

Concep. 4.1 The state space of $\{X(t), t \geq 0\}$ is $\{0, 1, 2\}$. In state 2, both cables are up and each is subject to a load of $L/2$. Hence the lifetime of each cable is $\exp(\lambda L/2)$. The process jumps from state 2 to 1 when either cable breaks, hence $r_{2,1} = \lambda L/2 + \lambda L/2 = \lambda L$. In state 1, only one cable is working under a total load of L. Hence its lifetime is $\exp(\lambda L)$. Once it breaks, the process moves to state 0. Hence $r_{1,0} = \lambda L$. Once the process enters state 0, it stays there permanently. Hence $\{X(t), t \geq 0\}$ is a CTMC on state space $S = \{0, 1, 2\}$ with rate matrix

$$R = \begin{bmatrix} 0 & 0 & 0 \\ \lambda L & 0 & 0 \\ 0 & \lambda L & 0 \end{bmatrix}.$$

Concep. 4.2 The state space is $\{0, 1, \cdots, 10\}$. In state 0, the process jumps to 1 when there is an arrival, which occurs with rate λ. Hence $r_{0,1} = \lambda$. In state i, $1 \leq i \leq 3$, one customer is being served. The process jumps to state $i + 1$ with rate λ, and to state $i - 1$ with rate μ. Hence, $r_{i,i+1} = \lambda, \quad r_{i,i-1} = \mu$. In state i, $4 \leq i \leq 7$, two customers are in service. The process jumps to state $i + 1$ with rate λ, and to state $i - 1$ with rate 2μ, since any one of the two customers in service may complete service to trigger this transition. Proceeding in this fashion, we see that $\{X(t), t \geq 0\}$ is a Birth and Death Process with birth parameters

$$\lambda_i = \lambda, \quad 0 \leq i \leq 9,$$

and death parameters

$$\mu_i = \begin{cases} \mu & \text{for } 1 \leq i \leq 3 \\ 2\mu & \text{for } 4 \leq i \leq 7 \\ 3\mu & \text{for } 8 \leq i \leq 10. \end{cases}$$

Concep. 4.4 Let $A(t)$ be the set of busy servers at time t, $A(t) \subset \{1,2,3\}$. Let

$$X(t) = \begin{cases} 0 & \text{if } A(t) = \emptyset, \\ 1 & \text{if } A(t) = \{1\}, \\ 2 & \text{if } A(t) = \{2\}, \\ 3 & \text{if } A(t) = \{3\}, \\ 4 & \text{if } A(t) = \{1,2\}, \\ 5 & \text{if } A(t) = \{1,3\}, \\ 6 & \text{if } A(t) = \{2,3\}, \\ 7 & \text{if } A(t) = \{1,2,3\}. \end{cases}$$

Then, the usual triggering event analysis in each state shows that $\{X(t), t \geq 0\}$ is a CTMC with the following rate matrix:

$$R = \begin{bmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_1 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ \mu_3 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & \mu_2 & \mu_1 & 0 & 0 & 0 & 0 & \lambda \\ 0 & \mu_3 & 0 & \mu_1 & 0 & 0 & 0 & \lambda \\ 0 & 0 & \mu_3 & \mu_2 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & \mu_3 & \mu_2 & \mu_1 & 0 \end{bmatrix}.$$

Concep. 4.5 The state space of $\{X(t), t \geq 0\}$ is $\{0, 1, \cdots, K\}$. When it is in state i, $0 \leq i \leq M$, all customers are allowed to enter, hence it moves to state $i+1$ with rate $\lambda_1 + \lambda_2$. When it is in state i, $M < i < K$, only customers of type 1 are allowed to enter, hence it moves to state $i+1$ with rate λ_1. In state i, $1 \leq i \leq K$, one customer is in service, whose service time is $\exp(\mu)$, regardless of its type. Hence the system moves to state $i-1$ with rate μ. Thus, $\{X(t), t \geq 0\}$ is a birth and death process on state space $\{0, 1, \cdots, K\}$, with death parameters

$$\mu_i = \mu, \quad 1 \leq i \leq K,$$

and birth parameters

$$\lambda_i = \begin{cases} \lambda_1 + \lambda_2 & \text{for } 0 \leq i \leq M \\ \lambda_1 & \text{for } M+1 \leq i < K \\ 0 & \text{for } i = K. \end{cases}$$

Concep. 4.8 The state of the system $A(t)$ is a vector giving the state of the three machines at time t. Using w for "working", b for "blocked", and i for "idle", we get the following definition of $X(t)$,

$$X(t) = \begin{cases} 1 & \text{if } A(t) = \{w, w, w\} \\ 2 & \text{if } A(t) = \{w, b, w\} \\ 3 & \text{if } A(t) = \{w, i, w\} \\ 4 & \text{if } A(t) = \{w, w, i\} \\ 5 & \text{if } A(t) = \{w, i, i\} \\ 6 & \text{if } A(t) = \{b, w, w\} \\ 7 & \text{if } A(t) = \{b, b, w\} \\ 8 & \text{if } A(t) = \{b, w, i\}. \end{cases}$$

Then, performing the usual analysis of transition triggering events, we see that $\{X(t), t \geq 0\}$ is a CTMC with the following rate matrix:

$$R = \begin{bmatrix} 0 & \mu_2 & 0 & \mu_3 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & 0 & \mu_1 & 0 \\ \mu_1 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & \mu_3 \\ \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Concep. 4.9 Let $X(t)$ be the number of customers in the system at time t if the system is up, and let $X(t) = d$ if the system is down at time t. Thus the state-space is $S = \{d, 0, 1, \ldots, K\}$. Performing the usual analysis of transition triggering events, we see that $\{X(t), t \geq 0\}$ is a CTMC with transition rates

$$\begin{aligned} r_{i,d} &= \theta, \quad \text{for } 0 \leq i \leq K, \\ r_{d,0} &= \alpha, \\ r_{i,i-1} &= \mu, \quad \text{for } 1 \leq i \leq K, \\ r_{i,i+1} &= \lambda, \quad \text{for } 0 \leq i \leq K-1. \end{aligned}$$

All other transition rates are zero.

Comp. 4.21 a) The limiting distribution over $S = \{1, 2, 3, 4, 5\}$ is $(p_1, p_2, p_3, p_4, p_5)$. A set of balance equations becomes

$$
\begin{array}{lclclclclcl}
8p_1 & = & & & 5p_2 & + & 5p_3, & & & & \\
15p_2 & = & 4p_1 & & & + & 5p_3 & + & 5p_4, & & \\
18p_3 & = & 4p_1 & + & 5p_2 & & & + & 5p_4 & + & 5p_5, \\
14p_4 & = & & & 5p_2 & + & 4p_3 & & & + & 5p_5, \\
10p_5 & = & & & & & 4p_3 & + & 4p_4, & & \\
1 & = & p_1 & + & p_2 & + & p_3 & + & p_4 & + & p_5.
\end{array}
$$

b) By filling in $p = (p_1, p_2, p_3, p_4, p_5) = (0.2528,\ 0.1981,\ 0.2064,\ 0.1858,\ 0.1569)$ in the set of equations above, we see that p is a solution of this set. Since the chain is irreducible, p must be the unique limiting distribution.

Comp. 4.26 a) The phone-reservation system has s reservation agents and can put a maximum of H callers on hold. Each incoming call is directly handled by one agent if available, otherwise the caller is put on hold if possible. The completion rate of an individual call is μ. A caller who is served,is handled by one agent. Hence the transition rate μ_i from state i to state $i-1$ in this birth and death is equal to the product of the number of callers i and μ if $i \leq s$ and is equal to the product of s and μ if $i > s$, otherwise formulated: $\mu_i = \min(i, s)\mu$.

b) In Comp. 4.9 is given: $s = 8, H = 4, K = s + H = 12$. The arrival rate is 60 callers per hour, each call takes an average of 6 minutes to handle. The limiting distribution $p = (p_0, p_1, \ldots, p_{12})$ is given by the solution to Equations (4.43) and (4.44). These equations are, using the R matrix from Example 4.10, and the data from Computational Problem 4.9:

$$\begin{array}{rcll}
60p_0 & = & 10p_1 & \\
70p_1 & = & 60p_0 & +20p_2 \\
80p_2 & = & 60p_1 & +30p_3 \\
90p_3 & = & 60p_2 & +40p_4 \\
100p_4 & = & 60p_3 & +50p_5 \\
110p_5 & = & 60p_4 & +60p_6 \\
120p_6 & = & 60p_5 & +70p_7 \\
130p_7 & = & 60p_6 & +80p_8 \\
140p_8 & = & 60p_7 & +80p_9 \\
140p_9 & = & 60p_8 & +80p_{10} \\
140p_{10} & = & 60p_9 & +80p_{11} \\
140p_{11} & = & 60p_{10} & +80p_{12}
\end{array} \quad p_0 + p_1 + \ldots + p_{12} = 1$$

c) By filling in $p = (p_0, p_1, \ldots, p_{12})$ in the set of equations above, we see that p is a solution of this set. Since the chain is irreducible, p must be the unique limiting distribution.

d) This probability is equal to the limiting probability $p_0 = 0.0023$.

e) i. All clerks are busy in states 8, 9, 10, 11, 12. Hence the desired answer is $p_8 + p_9 + p_{10} + p_{11} + p_{12} = .2975$.

ii. The system has to turn away calls when it is in state 12. Hence the desired answer is $p_{12} = 0.0309$.

iii. The expected number of busy clerks in steady state is given by:

$$\sum_{i=0}^{12} \min(i, 8)p_i = 5.8149.$$

f) The average number of occupied agents is equal to

$0*p_0+1*p_1+2*p_2+\ldots+7*p_7+8*(p_8+p_9+\ldots+p_{12}) = 5.8146$ agents. Hence the fraction of time a reservation agent is busy is equal to $\frac{5.8146}{8} = 0.7268$.

g) The throughput is equal to $60*(1-p_{12}) = 58.15$ callers per hour. The throughput is also equal to the product of the average number of busy agents (5.8146) and the service rate of a agent (10 callers per hour), so is equal to 5.8146*10 = 58.15 callers.

Comp. 4.27 a) The capacity of the switch is 6 calls. Calls arrive at a rate of 4 calls per minute. Calls that arrive when the switch is full, are lost. A call has an average duration of 2 minutes. Calls are handled simultaneously. In the state $i(1 \leq i \leq 6)$ calls in the switch, the rate of a transition to state $i-1$ is equal to $\mu_i = \frac{i}{2}$.

b) The limiting distribution $p = (p_0, p_1, \ldots, p_6)$ is given by the solution to Equations (4.43) and (4.44). Using the parameter values from Example 4.31, we get

$$R = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2.0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2.5 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 3.0 & 0 \end{bmatrix}.$$

The equations become:

$$\begin{array}{rcll} 4p_0 & = & & 0.5p_1 \\ 4.5p_1 & = & 4p_0 & +p_2 \\ 5p_2 & = & 4p_1 & +1.5p_3 \\ 5.5p_3 & = & 4p_2 & +2.0p_4 \\ 6.0p_4 & = & 4p_3 & +2.5p_5 \\ 6.5p_5 & = & 4p_4 & +3.0p_6 \\ 3.0p_6 & = & 4p_5 & \end{array} \qquad p_0 + p_1 + \ldots + p_6 = 1$$

The solution is

$$p = [0.0011,\ 0.0086,\ 0.0343,\ 0.0913,\ 0.1827,\ 0.2923,\ 0.3898]$$

c) The probability that al lines are busy is equal to $p_6 = 0.3898$.

d) The desired answer is

$$\sum_{i=0}^{6} i p_i = 4.882.$$

e) The fraction of time a line is occupied is equal to $\frac{4.882}{6} = 0.8137$.

f) The throughput is equal to 2.44 calls per minute (= 6 lines * $\frac{1}{2}$ call per line per minute * 0.8137 fraction of time a line is occupied).

Comp. 4.38 The cost rates in each state are given by

$$\begin{aligned}
&c(1) = 105, \quad c(2) = 70, \quad c(3) = 70, \\
&c(4) = 70, \quad c(5) = 35, \quad c(6) = 70, \\
&c(7) = 35, \quad c(8) = 35.
\end{aligned}$$

The limiting distribution p is given in Computational Problem 4.25

$$p = [0.1540,\ 0.0642,\ 0.0868,\ 0.1314,\ 0.0868,\ 0.0840,\ 0.1342,\ 0.2585].$$

The long run net revenue is given by

$$\sum_{i=1}^{8} c(i)p_i = 58.6079.$$

Comp. 4.44 Consider the system using server 1. We have $\lambda = 5$ per hour $\mu = 7.5$ per hour, and capacity = 10. Let $X(t)$ be the number of customers in the system at time t. Then $\{X(t), t \geq 0\}$ is a CTMC with state space $\{0, 1, \ldots, 10\}$ as described in Example 4.7. Each customer spends 8 minutes in service on the average, and pays $10 for it. This means that the system earns money at rate of $75 per hour that the server is busy. It has to pay the server at a rate of $20 per hour whether it is busy or not. This revenue structure can be accounted for by the following revenue vector:

$$c(i) = \begin{cases} -20 & \text{if } i = 0, \\ 75 - 20 & \text{otherwise.} \end{cases}$$

Let p be the limiting distribution of $\{X(t), t \geq 0\}$. The long run revenue rate per hour is then given by

$$\sum_{i=0}^{10} c(i)p_i = 75(1 - p_0) - 20.$$

Using the results of Example 4.25, we get

$$p_0 = 0.3372.$$

Hence the long run revenue rate is

$$75(1 - 0.3372) - 20 = \ 29.71 \ \text{ dollars/hour.}$$

Similar analysis with server 2 yields the long run revenue rate to be 33.44 dollars/hour. Hence it is more profitable to use the slower but cheaper server 2.

Comp. 4.45 a) Let m_i be the expected time to reach state 5 starting from state i. The equations of Theorem 4.11 are given by

$$\begin{aligned}
m_1 &= \frac{1}{8} + \frac{4}{8}m_2 + \frac{4}{8}m_3, \\
m_2 &= \frac{1}{15} + \frac{5}{15}m_1 + \frac{5}{15}m_3 + \frac{5}{15}m_4, \\
m_3 &= \frac{1}{18} + \frac{5}{18}m_1 + \frac{5}{18}m_2 + \frac{4}{18}m_4, \\
m_4 &= \frac{1}{14} + \frac{5}{14}m_2 + \frac{5}{14}m_3,
\end{aligned}$$

or, alternatively,

$$\begin{aligned}
8m_1 &= 1 + 4m_2 + 4m_3, \\
15m_2 &= 1 + 5m_1 + 5m_3 + 5m_4, \\
18m_3 &= 1 + 5m_1 + 5m_2 + 4m_4, \\
14m_4 &= 1 + 5m_2 + 5m_3.
\end{aligned}$$

b) The solution is given by

$$m_1 = 0.7425, \quad m_2 = 0.6725, \quad m_3 = 0.5625, \quad m_4 = 0.5125.$$

The desired answer is $m_1 = 0.7425$.

Comp. 5.1 The lifetime of a battery is given to be $Erl(k, \lambda)$ with $k = 3$, and $\lambda = 1$. Hence, the mean lifetime of a battery is $\tau = k/\lambda = 3$. From Theorem 5.2, the long run replacement rate is given by

$$\lim_{t\to\infty} = \frac{N(t)}{t} = \frac{1}{\tau} = \frac{1}{3}.$$

Comp. 5.2 Let $L \sim Erl(3,1)$ be the lifetime of a battery. Then, the expected inter replacement time is given by

$$\begin{aligned} \mathsf{E}(T) &= \mathsf{E}(\min(3, L)) \\ &= \int_0^3 x f_L(x)dx + \int_3^\infty 3f_L(x)dx \\ &= \int_0^3 xe^{-x}\frac{x^2}{2}dx + 3\mathsf{P}(L > 3) \\ &= 3 - 13.5e^{-3} = 2.3279. \end{aligned}$$

Thus, the long run replacement rate is given by

$$\lim_{t\to\infty} = \frac{N(t)}{t} = \frac{1}{\mathsf{E}(T)} = 0.4296.$$

Comp. 5.3 Let T_1 be the time of first planned replacement. Using the argument in Example 5.7, we get

$$\tau = \mathsf{E}(T_1) = \mathsf{E}(\min(L_1, 3)) + \tau\mathsf{P}(L_1 < 3) = 2.3279 + 0.5768\tau.$$

(Here we have used $\mathsf{E}(\min(L_1, 3)) = 2.3279$ from Computational Problem 5.2.) Solving for τ we get $\tau = 5.5008$. Hence, the long run planned replacement rate is given by

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mathsf{E}(T_1)} = 0.1818.$$

Alternative solution:

An arbitrary replacement is with probability $P(L_1 > 3) = 0.4232$ a planned replacement. Hence the long-run planned replacement rate is $(0.4296) \cdot (0.4232) = 0.1818$.

Comp. 5.9 Follow the computations in Example 5.14. Let L_1 be the lifetime of the first battery, T_1 be the time of first replacement, and C_1 be the cost of that replacement. From the solution to Computational Problem 5.2,

$$\mathsf{E}(T_1) = \mathsf{E}(\min(L_1, 3)) = 2.3279.$$

Furthermore,

$$\mathsf{E}(C_1) = 75 + 75\mathsf{P}(L_1 < 3) = 75 + 75 * .5768 = 118.26.$$

The long run cost rate under the "planned replacement" policy is given by

$$\lim_{t\to\infty} \frac{C(t)}{t} = \frac{\mathsf{E}(C_1)}{\mathsf{E}(T_1)} = \frac{118.26}{2.3279} = 50.80 \text{ dollars/year.}$$

Under the "replace upon failure" policy, we have

$$\mathsf{E}(T_1) = \mathsf{E}(L_1) = 3, \quad \mathsf{E}(C_1) = 75 + 75 = 150.$$

Hence the long run cost per year is given by

$$\lim_{t\to\infty} \frac{C(t)}{t} = \frac{\mathsf{E}(C_1)}{\mathsf{E}(T_1)} = \frac{150}{3} = 50 \text{ dollars/year.}$$

Hence replacement upon failure is cheaper.