# 2DD50 - Solutions to exercises week 7

**Concep. 6.8** Let $X(t)$ be the number of items in the warehouse at time $t$. The state space of $\{X(t), t \geq 0\}$ is $\{0, 1, 2, \cdots, K\}$. Assume that the machine always produces items, and if there is no space in the warehouse for a produced item, it is lost. Since the exponential distribution has memoryless property, this has the same effect as turning the machine off when the warehouse is full. Then the production process is $PP(\lambda)$ and acts as an arrival process to the warehouse. Assume that the items form a queue in the warehouse. Then the first item in the warehouse has to wait an Exp($\mu$) amount of time for the next demand before it is removed. Again, memoryless property of exponential distribution implies that lost demands do not have any effect. Hence the "service times" of the items are iid Exp($\mu$) random variables. Hence the number of items in the warehouse forms an $M/M/1/K$ queue.

**Concep. 6.10** $\{X(t), t \geq 0\}$ is a stochastic process with state space $\{0, 1, 2, \cdots, K\}$. When $X(t) = i > 0$, a repair takes place with rate $\mu$ (in which case the state decreases by 1), and a failure occurs at rate $(K - i)\lambda$ (in which case the state increases by 1). If $X(t) = 0$, no repairs take place, while a failure occurs at rate $K\lambda$. Hence $\{X(t), t \geq 0\}$ is a birth and death process with birth rates

$$\lambda_i = (K - i)\lambda, \quad 0 \leq i \leq K,$$

and death rates

$$\mu_i = \mu, \quad 1 \leq i \leq K.$$

**Concep. 6.11** $\{X(t), t \geq 0\}$ is a birth and death process with birth rates

$$\lambda_i = (K - i)\lambda, \quad 0 \leq i \leq K,$$

and death rates

$$\mu_i = \min(i, s)\mu, \quad 0 \leq i \leq K.$$

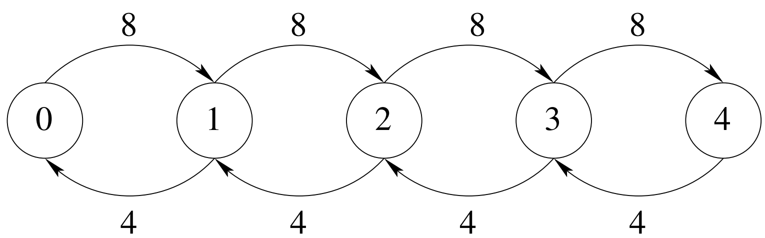
**Comp. 6.1** $\lambda = 5, \quad \tau = 1.3$. Let $s$ be the number of servers. From Theorem 6.4, the queue is stable if $s > \lambda\tau = 5 * 1.3 = 6.5$. Hence the minimum number of servers needed for stability is 7.

**Comp. 6.2** From Example 6.5, the expected number of busy servers is given by $B = \min(\lambda\tau, s) = \min(6.5, s)$. Thus $B = s$ if $1 \leq s \leq 6$, and $B = 6.5$ if $s \geq 7$.

**Comp. 6.6** Time unit: hour. $\lambda = 8, \mu = 4$.

a) This is an $M/M/1/K$ queue with $K = 4$.

b)




c) Balance equations

$$\begin{aligned}
8p_0(4) &= 4p_1(4), \\
12p_1(4) &= 8p_0(4) + 4p_2(4), \\
12p_2(4) &= 8p_1(4) + 4p_3(4), \\
12p_3(4) &= 8p_2(4) + 4p_4(4), \\
4p_4(4) &= 8p_3(4),
\end{aligned}$$

or, alternatively,

$$\begin{aligned}
8p_0(4) &= 4p_1(4), \\
8p_1(4) &= 4p_2(4), \\
8p_2(4) &= 4p_3(4), \\
8p_3(4) &= 4p_4(4).
\end{aligned}$$

Normalizing equation: $p_0(4) + p_1(4) + p_2(4) + p_3(4) + p_4(4) = 1$.

d) The limiting distribution is given by

$$p(4) = \left[\frac{1}{31}, \frac{2}{31}, \frac{4}{31}, \frac{8}{31}, \frac{16}{31}\right].$$

e) $p_4(4) = \frac{16}{31}$

f) $1 - p_0(4) = \frac{30}{31}$

g) The arrival rate of entering customers is $\lambda(1 - p_4(4)) = \frac{120}{31}$ customers per hour. Furthermore, $L = \sum_{i=0}^{4} ip_i(4) = \frac{98}{31}$. Hence

$$W = \frac{L}{\lambda(1 - p_4(4))} = 0.8167 \text{ hours} = 49 \text{ minutes}$$

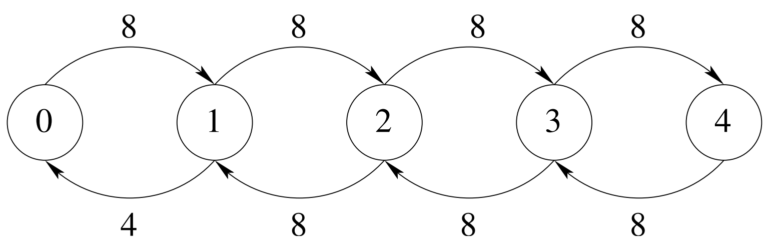
h) $W^q = W - \frac{1}{4} = 0.5667$ hours $= 34$ minutes.

**Comp. 6.7** The fraction of the customers lost is given by $p_4(4) = 0.5161$. Hence the fraction of the customers that enter is given by $1 - p_4(4) = 0.4839$. Hence the rate at which customers enter is $0.4839 * 8 = 3.8712$ per hour. Each entering customer pays 12 dollars. Hence the long run revenue rate is

$$3.8712 * 12 = 46.4544 \quad \text{dollars/hour.}$$

**Comp. 6.8** a) This is an $M/M/s/K$ queue with $s = 2$ and $K = 4$.

b)




c) Balance equations

$$\begin{aligned} 8p_0(4) &= 4p_1(4), \\ 12p_1(4) &= 8p_0(4) + 8p_2(4), \\ 16p_2(4) &= 8p_1(4) + 8p_3(4), \\ 16p_3(4) &= 8p_2(4) + 8p_4(4), \\ 8p_4(4) &= 8p_3(4), \end{aligned}$$

or, alternatively,

$$\begin{aligned} 8p_0(4) &= 4p_1(4), \\ 8p_1(4) &= 8p_2(4), \\ 8p_2(4) &= 8p_3(4), \\ 8p_3(4) &= 8p_4(4). \end{aligned}$$

Normalizing equation: $p_0(4) + p_1(4) + p_2(4) + p_3(4) + p_4(4) = 1$.

d) The limiting distribution is given by

$$p(4) = \left[\frac{1}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}\right].$$

e) $p_4(4) = \frac{2}{9}$

f) $0 \cdot p_0(4) + \frac{1}{2} \cdot p_1(4) + 1 \cdot (p_2(4) + p_3(4) + p_4(4)) = \frac{7}{9}$

g) $\lambda(1 - p_4(4)) = \frac{56}{9}$ (alternatively: $\frac{7}{9} \cdot 2 \cdot 4 = \frac{56}{9}$)

h) The arrival rate of entering customers is $\lambda(1 - p_4(4)) = \frac{56}{9}$ customers per hour. Furthermore, $L = \sum_{i=0}^{4} ip_i(4) = \frac{20}{9}$. Hence

$$W = \frac{L}{\lambda(1 - p_4(4))} = 0.3571 \text{ hours} = 21.43 \text{ minutes}$$

and

$$W^q = W - \frac{1}{4} = 0.1071 \text{ hours} = 6.43 \text{ minutes}$$

i) The new rate of revenue is given by

$$12 * \lambda * (1 - p_4(4)) = 74.6667 \text{ dollars/hour}.$$

**Comp. 6.10** This is an $M/M/1/K$ queue with $\lambda = 1$ per hour, $\mu = 20/24 = 5/6$ per hour, $K = 10$. The machine is off whenever the warehouse is full. The long run fraction of the time the machine is off is given by $p_{10}(10) = 0.1926$.

**Comp. 6.11** This is the same queue as in Computational Problem 6.10. The demands are lost when the warehouse is empty. The demands occur according to a Poisson process. Hence, according to PASTA, the long run probability that a demand sees the warehouse empty is given by $p_0(10)$. Hence the long run fraction of the demands lost are given by $p_0(10) = 0.0311$.

**Comp. 6.12** This is the same queue as in Computational Problem 6.10. Let $W$ be the expected time an item spends in the warehouse in steady state. Then the expected revenue from the sale is $100 - W$ dollars. Using the parameters given in Computational Problem 6.10, we get $W = 8.3114$

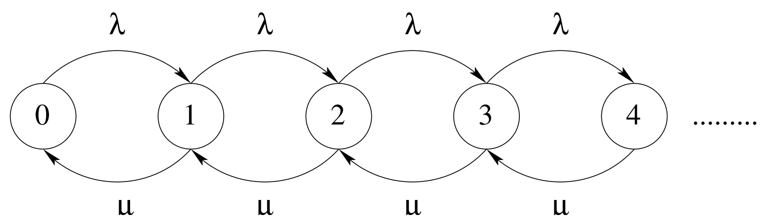
hours. Hence the expected sale price is \$91.6886. Now the rate at which items enter the warehouse is

$$\lambda \cdot (1 - p_K(K)) = 1 \cdot (1 - 0.1926) = 0.8074 \text{ per hour.}$$

Hence the revenue rate is $0.8074 \cdot 91.6886 = 74.0294$ dollars/hour.
*Alternative solution*: If there are $i$ items on stock, the warehouse looses $i$ dollar per hour. So the expected loss per hour is $L$ dollar, where $L$ is the expected number of items on stock. Hence the revenue rate is

$$\lambda \cdot (1 - p_K(K)) \cdot 100 - L = 80.74 - 6.71 = 74.03 \text{ dollar per hour.}$$

**Comp. 6.21** a) $M/M/1$ queue.

b) 


c) The balance equations are

$$\begin{aligned} \lambda p_0 &= \mu p_1, \\ (\lambda + \mu) p_n &= \lambda p_{n-1} + \mu p_{n+1}, \quad n = 1, 2, 3, \ldots, \end{aligned}$$

or, alternatively,

$$\lambda p_n = \mu p_{n+1}, \quad n = 0, 1, 2, \ldots.$$

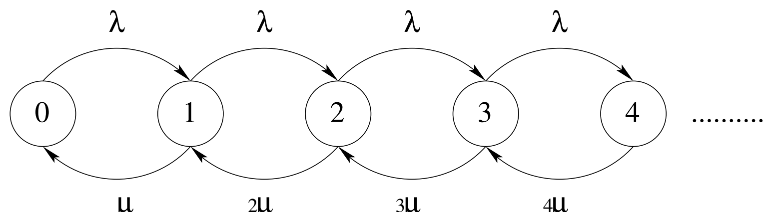
Normalizing equation: $\sum_{n=0}^{\infty} p_n = 1$.

d) $\lambda < \mu$.

e) We have $\mu = 12$ items per hour. The fraction of demands lost is $1 - \rho$, where $\rho = \lambda/\mu$. Hence we must have $\rho \geq 0.9$, i.e., $\lambda \geq 0.9\mu = 10.8$ per hour. Thus, the mean production time is at most $60/10.8 = 5.556$ minutes. The mean number in the warehouse is then $\rho/(1 - \rho) = 0.9/0.1 = 9$.

**Comp. 6.28** a) This is an $M/M/\infty$ queue with $\lambda = 40$ and $\mu = 1/3$.

b)




c) The balance equations are

$$\begin{aligned} \lambda p_0 &= \mu p_1, \\ (\lambda + i\mu)p_i &= \lambda p_{i-1} + (i+1)\mu p_{i+1}, \quad i = 1, 2, 3, \ldots, \end{aligned}$$

or, alternatively,

$$\lambda p_i = (i+1)\mu p_{i+1}, \quad i = 0, 1, 2, \ldots.$$

Normalizing equation: $\sum_{i=0}^{\infty} p_i = 1$.

d) Use that

$$\begin{aligned} p_i &= \frac{\lambda}{i\mu} p_{i-1} \\ &= \frac{120}{i} p_{i-1} \\ &\vdots \\ &= (\frac{120^i}{i!}) p_0. \end{aligned}$$

From the normalizing equation we obtain $p_0 = e^{-120}$.

e)
- Mean number of cars in parking lot: $L = \lambda/\mu = 120$. (in steady state the number of cars in the lot is a Poisson random variable with mean $40/(1/3) = 120$).
  Mean waiting time of cars: $W = L/\lambda = 3$ minutes.
- Mean number of occupied parking places: 120.
  Mean parking time of cars: 3 minutes.
- Mean number of cars in the queue: 0.
  Mean queueing time of cars: 0 minutes.
- Fraction of time a certain parking place is occupied: 0 (due to the assumption that in the model there are infinitely many parking places)

- Probability that all places are occupied: 0 (due to the assumption that in the model there are infinitely many parking places)
- Throughput is 40 cars per hour.