2DD50 - Solutions to exercises week 8

Concep. 6.20 Consider the queue in front of server 1. The inter-arrival times are iid, each being a sum of two iid exp(λ) random variables. Hence it is a $G/M/1$ queue.

Concep. 6.21 The pdf of an U(a, b) random variable is

$$g(x) = \frac{1}{b-a}, \quad a \leq x \leq b.$$

Hence

$$\tilde{G}(s) = \int_a^b e^{-sx} g(x) dx = \frac{e^{-as} - e^{-bs}}{s(b-a)}.$$

The key functional equation is $u = \tilde{G}(\mu(1-u))$, which reduces to

$$u = \frac{e^{-a\mu(1-u)} - e^{-b\mu(1-u)}}{\mu(1-u)(b-a)}.$$

Comp. 6.32 a) $M/G/1$ queue. More specifically, it is an $M/E_2/1$ because the service times of customers are Erlang distributed with 2 phases.

b)
- The service times are iid Erl(2,12). Hence $\tau = 2/12 = 1/6$ hour, $\sigma^2 = 2/(12)^2 = 1/72$ hour^2. Thus $s^2 = 1/72 + 1/36 = 3/72 = 1/24$ hour^2. We also have $\lambda = 3$. Hence $L = 0.875$.
  $W = L/\lambda = 0.875/3 = 0.2917$ hour $= 17.5$ minutes.
- Mean number of occupied servers: $\rho = 1/2$.
  Mean service time: $\tau = 1/6$ hour $= 10$ minutes.
- Mean number of customers in the queue: $L_q = L - \rho = 0.375$.
  Mean queueing time of customers: $W_q = W - \tau = 7.5$ minutes.
- Fraction of time server is occupied: $\rho = 1/2$.
- Throughput: 3 customers per hour.

Comp. 6.33 This is an $M/G/1$ queue. Let T_1 and T_2 be two iid Exp(6) random variables. Then the service time (in hours) of a single customer is $T = \max(T_1, T_2)$. The cdf of T is given by

$$F(x) = \mathsf{P}(T \leq x) = \mathsf{P}(T_1 \leq x, T_2 \leq x) = (1 - e^{-6*x})^2.$$

Hence

$$\tau = \mathsf{E}(T) = \int_0^\infty xF'(x)dx = 0.25 \text{ hrs},$$

and

$$s^2 = \mathsf{E}(T^2) = \int_0^\infty x^2 F'(x)dx = 7/72 \text{ hrs}^2.$$

Using $\lambda = 3$ we get $L = 2.5$. Thus the congestion is more in this setup. This is because one of the two servers is forced to be idle part of the time.

Comp. 6.35 Let T be a typical service time. We are given

$$\mathsf{P}(T = 2) = 0.5, \quad \mathsf{P}(T = 3) = 0.2, \quad \mathsf{P}(T = 5) = 0.3.$$

Hence $\tau = 2 * 0.5 + 3 * 0.2 + 5 * 0.3 = 3.1$ minutes and $s^2 = 4 * 0.5 + 9 * 0.2 + 25 * 0.3 = 11.3$ minutes^2. Using $\lambda = 18$ per hour, we get $L = 8.194$.

Comp. 6.36 a)

- Consider the queue in front of server 1. It is a $G/M/1$ queue with iid Erl(2, 10) interarrival times, and Exp(6) service times. The traffic intensity is $\rho = 5/6 < 1$. The functional equation becomes

$$u = (\frac{10}{10 + 6(1 - u)})^2,$$

  with the required solution $\alpha = 0.7822$. Using Equation (6.42) we get $L = 3.826$ and $W = L/\lambda = 3.826/5 = 0.765$ hour $=$ 45.9 minutes.
- Mean number of occupied servers: $\rho = 5/6$.
  Mean service time: $1/\mu = 10$ minutes.
- Mean number of customers in queue: $L_q = L - \rho = 2.993$.
  Mean queueing time: $W_q = W - 1/\mu = 35.9$ minutes.
- Fraction of time server is occupied: $\rho = 5/6$.
- Throughput: 5 customers per hour.

b) Let $X(t)$ be the number of customers in the single line served by the two servers. Then $X(t)$ is an $M/M/2$ queue with $\lambda = 10, \quad \mu = 6$. The expected number of customers in this system is 5.4545.

c) The expected number of customers in the first system is 7.6518. The expected number of customers in the second system is 5.4545. Hence pooling the two servers will reduce congestion.

Comp. 6.40 This is a $G/M/1$ queue with arrival rate $\lambda = 12$ per hour (constant inter arrival times of 5 minutes), and service rate $\mu = 15$ per hour. The solution to the functional equation is $\alpha = 0.6286$. Hence $L = 2.1540$, and the fraction of the time the server is busy is given by $\lambda/\mu = 12/15 = 0.8$. Hence the cost rate is $0.8 * 40 + 2.1540 * 2 = 36.3080$ dollars/hour. If each customer is charged c, the revenue rate is $12c$ per hour. Hence we must have $c \geq 36.3080/12 = 3.0257$. Thus each customer should be charged at least 3.03 dollars in order for the system to break even.