Solutions to exercises: week 1

Concep. 2.8 We have

$$X_{n+1} = \begin{cases} 1 & \text{with probability 1, if } X_n = 0 \\ 2 & \text{with probability 1, if } X_n = 1 \\ 2 & \text{with probability } p\text{, if } X_n = 2 \\ 0 & \text{with probability } 1-p\text{, if } X_n = 2. \end{cases}$$

Since X_{n+1} depends only on the current state and not on the past, $\{X_n, n \geq 0\}$ is a DTMC on state space $S = \{0, 1, 2\}$. The transition probability matrix is given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-p & 0 & p \end{bmatrix}.$$

Concep. 2.9 We have

$$X_n = \begin{cases} 1 & \text{if the item that arrives to the shop during} \\ & \text{the } (n-1)^{st} \text{ minute is non-defective} \\ 0 & \text{otherwise.} \end{cases}$$

Since X_{n+1} does not depend on the past, $\{X_n, n \geq 0\}$ is a DTMC on state space $S = \{0, 1\}$. The transition probability matrix is given by

$$P = \begin{bmatrix} 1-p & p \\ 1-p & p \end{bmatrix}.$$

Concep. 2.10 Let X_n

$$= \begin{cases} 1 & \text{if the machine is idle at the beginning of the } n^{th} \text{ minute} \\ & \text{and there are no items in the bin} \\ 2 & \text{if the machine is idle at the beginning of the } n^{th} \text{ minute} \\ & \text{and there is one item in the bin} \\ 3 & \text{if the machine has been busy for one minute at the beginning} \\ & \text{of the } n^{th} \text{ minute and there are no items in the bin} \\ 4 & \text{if the machine has been busy for one minute at the beginning} \\ & \text{of the } n^{th} \text{ minute and there is one item in the bin} \\ 5 & \text{if the machine has just started production at the beginning} \\ & \text{of the } n^{th} \text{ minute and there are no items in the bin.} \end{cases}$$

Then, $\{X_n, n \geq 0\}$ is a DTMC on $S = \{1, 2, 3, 4, 5\}$ with the transition probability matrix

$$P = \begin{bmatrix} 1-p & p & 0 & 0 & 0 \\ 0 & 1-p & 0 & 0 & p \\ 1-p & p & 0 & 0 & 0 \\ 0 & 1-p & 0 & 0 & p \\ 0 & 0 & 1-p & p & 0 \end{bmatrix}.$$

Concep. 2.11 Let X_n

$$= \begin{cases} 1 & \text{if the machine is idle at the beginning of the } n^{th} \text{ minute} \\ & \text{and there are no items in the bin} \\ 2 & \text{if the machine has been busy for one minute at the beginning} \\ & \text{of the } n^{th} \text{ minute and there are no items in the bin} \\ 3 & \text{if the machine has been busy for one minute at the beginning} \\ & \text{of the } n^{th} \text{ minute and there is one item in the bin.} \\ 4 & \text{if the machine has just started production at the beginning} \\ & \text{of the } n^{th} \text{ minute and there are no items in the bin} \end{cases}$$

Then, $\{X_n, n \geq 0\}$ is a DTMC on $S = \{1, 2, 3, 4\}$ with the transition probability matrix

$$P = \begin{bmatrix} 1-p & 0 & 0 & p \\ 1-p & 0 & 0 & p \\ 0 & 0 & 0 & 1 \\ 0 & 1-p & p & 0 \end{bmatrix}.$$

Concep. 2.12 Let

$$X_n = \begin{cases} 0 & \text{if day } n \text{ and } n-1 \text{ are both rainy} \\ 1 & \text{if day } n \text{ is sunny and day } n-1 \text{ is rainy} \\ 2 & \text{if day } n \text{ is rainy and day } n-1 \text{ is sunny} \\ 3 & \text{if day } n \text{ and } n-1 \text{ are both sunny.} \end{cases}$$

Then, $\{X_n, n \geq 0\}$ is a DTMC on state space $S = \{0, 1, 2, 3\}$ with the transition probability matrix

$$P = \begin{bmatrix} .4 & .6 & 0 & 0 \\ 0 & 0 & .2 & .8 \\ .3 & .7 & 0 & 0 \\ 0 & 0 & .1 & .9 \end{bmatrix}.$$

Concep. 2.15 Let A denote the event that the DTMC visits state 1 before it visits state N. Then

$$\begin{aligned} u_i = \mathsf{P}(A|X_0 = i) &= \sum_{j=1}^{N} \mathsf{P}(A|X_1 = j, X_0 = i)\mathsf{P}(X_1 = j|X_0 = i) \\ &= \sum_{j=1}^{N} \mathsf{P}(A|X_0 = j)\mathsf{P}(X_1 = j|X_0 = i) \\ &= \sum_{j=1}^{N} u_j p_{i,j}. \end{aligned}$$

Also, if $X_0 = 1$, then A occurs with probability 1, and if $X_0 = N$, then A occurs with probability 0. Hence

$$u_1 = 1, \quad u_N = 0.$$

Concep. 2.16 We have for $1 \leq X_n \leq (N-1)$

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } (N - X_n)/N \\ X_n - 1 & \text{with probability } X_n/N. \end{cases}$$

Further we have

$$\mathsf{P}(X_{n+1} = 1|X_n = 0) = 1, \ \mathsf{P}(X_{n+1} = N-1|X_n = N) = 1.$$

Since X_{n+1} depends only on the current state and not on the past, $\{X_n, n \geq 0\}$ is a DTMC on state space $S = \{0, \dots, N\}$ with transition probabilities

$$p_{i,i+1} = \frac{N-i}{N}, \ \ p_{i,i-1} = \frac{i}{N} \ (1 \leq i \leq (N-1)), \ p_{0,1} = 1, \ p_{N,N-1} = 1.$$

Comp. 2.1 a) $p_{5,\,j} = \mathsf{P}(X_1 = j|X_0 = 5) = 0$ $(j = 0, 1, 2, 3)$ because 1 packet is transmitted and the number of new packets A_1 is at least 0.
$p_{5,\,4} = \mathsf{P}(X_1 = 4|X_0 = 5) = \mathsf{P}(A_1 = 0) = 0.3679.$
$p_{5,\,5} = \mathsf{P}(X_1 = 5|X_0 = 5) = \mathsf{P}(A_1 = 1) = 0.3679.$
$p_{5,\,6} = \mathsf{P}(X_1 = 6|X_0 = 5) = \mathsf{P}(A_1 = 2) = 0.1839.$
$p_{5,\,7} = \mathsf{P}(X_1 = 7|X_0 = 5) = \mathsf{P}(A_1 \geq 3) = 0.0803.$

b) There are two paths from state 7 at time 0 to state 7 at time 2.
The path: $7 \to 7 \to 7$ with probability $(0.6321)^2$; in each time slot 1 packet is transmitted and at least 1 new packet arrives.
The path: $7 \to 6 \to 7$ with probability $(0.3679)(0.2642)$; in the first time slot 1 packet is transmitted and 0 new packets arrive, in the second time slot 1 packet is transmitted and at least 2 new packets arrive. So $p^{(2)}_{7,\,7} = 0.4968$.

c) $a^{(0)} = (0, 0, 0, 0, 0, 0, 1)$.
With MAXIM follows $a^{(2)} = (0, 0, 0, 0, 0, 0.1353, 0.3679, 0.4968)$, so $a^{(2)}_7 = 0.4968$.

d) Use the P matrix given in Equation (2.29). The desired quantity is

$$\mathsf{E}(X_5|X_3 = 7) = \mathsf{E}(X_2|X_0 = 7) = \sum_{j=0}^{7} j[P^{(2)}]_{7,j} = \sum_{j=0}^{7} j a^{(2)}_j = 6.3613.$$

Comp. 2.5 a) A_n = the number of arrivals of new packets during the nth time slot, the interval $(n-1, n]$: $A_n \sim \text{bin}(n = 5, p = 0.2)$.

$\mathsf{P}(A_n = 0) = \binom{5}{0}.2^0.8^{5-0} = 0.32768,$

$\mathsf{P}(A_n = 1) = \binom{5}{1}.2^1.8^{5-1} = 0.4096,$

$\mathsf{P}(A_n = 2) = \binom{5}{2}.2^2.8^{5-2} = 0.2048,$

$\mathsf{P}(A_n = 3) = \binom{5}{3}.2^3.8^{5-3} = 0.0512,$

$\mathsf{P}(A_n = 4) = \binom{5}{4}.2^4.8^{5-4} = 0.0064,$

$\mathsf{P}(A_n = 5) = \binom{5}{5}.2^5.8^{5-5} = 0.00032,$

X_n = the number of packets in the buffer at the end of the n-th time slot.
$\mathsf{P}(X_1 = 0|X_0 = 1) = \mathsf{P}(A_1 = 0) = 0.32768.$
$\mathsf{P}(X_1 = 1|X_0 = 1) = \mathsf{P}(A_1 = 1) = 0.4096.$
$\mathsf{P}(X_1 = 2|X_0 = 1) = \mathsf{P}(A_1 = 2) = 0.2048.$
$\mathsf{P}(X_1 = 3|X_0 = 1) = \mathsf{P}(A_1 = 3) = 0.0512.$

$\mathsf{P}(X_1 = 4|X_0 = 1) = \mathsf{P}(A_1 = 4) = 0.0064.$
$\mathsf{P}(X_1 = 5|X_0 = 1) = \mathsf{P}(A_1 = 5) = 0.00032.$
$\mathsf{P}(X_1 = j|X_0 = 1) = 0 \ (j = 6, 7, 8, 9, 10).$

$\mathsf{P}(X_1 = j|X_0 = 8) = 0 \ (j = 0, 1, 2, 3, 4, 5, 6).$
$\mathsf{P}(X_1 = 7|X_0 = 8) = \mathsf{P}(A_1 = 0) = 0.32768.$
$\mathsf{P}(X_1 = 8|X_0 = 8) = \mathsf{P}(A_1 = 1) = 0.4096.$
$\mathsf{P}(X_1 = 9|X_0 = 8) = \mathsf{P}(A_1 = 2) = 0.2048.$
$\mathsf{P}(X_1 = 10|X_0 = 8) = \mathsf{P}(A_1 \geq 3)) = 0.0579.$

The other rows of the transition probability matrix are found in a similar way. The transition probability matrix is given by

$$P = \begin{bmatrix} .3277 & .4096 & .2048 & .0512 & .0064 & .0003 & 0 & 0 & 0 & 0 & 0 \\ .3277 & .4096 & .2048 & .0512 & .0064 & .0003 & 0 & 0 & 0 & 0 & 0 \\ 0 & .3277 & .4096 & .2048 & .0512 & .0064 & .0003 & 0 & 0 & 0 & 0 \\ 0 & 0 & .3277 & .4096 & .2048 & .0512 & .0064 & .0003 & 0 & 0 & 0 \\ 0 & 0 & 0 & .3277 & .4096 & .2048 & .0512 & .0064 & .0003 & 0 & 0 \\ 0 & 0 & 0 & 0 & .3277 & .4096 & .2048 & .0512 & .0064 & .0003 & 0 \\ 0 & 0 & 0 & 0 & 0 & .3277 & .4096 & .2048 & .0512 & .0064 & .0003 \\ 0 & 0 & 0 & 0 & 0 & 0 & .3277 & .4096 & .2048 & .0512 & .0067 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3277 & .4096 & .2048 & .0579 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3277 & .4096 & .2627 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3277 & .6723 \end{bmatrix}$$

b) $a^{(0)} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$,
$a^{(1)} = a^{(0)}P = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0.327680, 0.672320)$, so the expected number of packets in the buffer at the end of the first time slot is equal to $\mathsf{E}(X_1|X_0 = 10) = \sum_{i=0}^{10} i a_i^{(1)} = 9.6723$.
In a similar way are calculated:
$a^{(2)}$ and $\mathsf{E}(X_2|X_0 = 10) = 9.4307$,
$a^{(5)}$ and $\mathsf{E}(X_5|X_0 = 10) = 8.8992$,
$a^{(10)}$ and $\mathsf{E}(X_{10}|X_0 = 10) = 8.2678$.

Comp. 2.6 a) The matrix P of Conceptual Problem 2.8 with $p = 0.95$ is: $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.05 & 0 & 0.95 \end{bmatrix}$.

With MATLAB follows:

$$P^5 = \begin{bmatrix} 0.045125 & 0.047500 & 0.907375 \\ 0.045369 & 0.045125 & 0.909506 \\ 0.045475 & 0.045369 & 0.909156 \end{bmatrix},$$

$a^{(5)} = a^{(0)}P^5 = (0,0,1)P^5 = (0.045475, 0.045369, 0.909156)$.

b) The set A of possible paths from state 2 at time $n = 0$ to state 2 at time $n = 5$ are given by:

$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$2 \rightarrow$	$2 \rightarrow$	$2 \rightarrow$	$2 \rightarrow$	$2 \rightarrow$	2,
$2 \rightarrow$	$0 \rightarrow$	$1 \rightarrow$	$2 \rightarrow$	$2 \rightarrow$	2,
$2 \rightarrow$	$2 \rightarrow$	$0 \rightarrow$	$1 \rightarrow$	$2 \rightarrow$	2,
$2 \rightarrow$	$2 \rightarrow$	$2 \rightarrow$	$0 \rightarrow$	$1 \rightarrow$	2.

The probability at the first path is $(0.95)^5$, at each of the other paths is $(0.95)^2(0.05)$, so $\mathsf{P}(A) = 0.909156$.

c) $p^{(5)}_{2,2}$ is defined as the probability te be in state 2 at time $n = 5$, starting in state 2 at time $n = 0$. The set A contains all possible paths to go in 5 state transitions from state 2 to state 2, so it is obvious that $\mathsf{P}(A) = p^{(5)}_{2,2}$. Because in this case $a^{(0)}_2 = 1$ it follows immediately that $p^{(5)}_{2,2} = a^{(5)}_2$.

d) With MATLAB or MAXIM calculate $a^{(5)}, a^{(10)}, a^{(15)}, a^{(20)}$, starting from $a^{(0)} = (0,0,1)$. The asked probabilities are $a^{(5)}_2 = 0.90915594$, $a^{(10)}_2 = 0.90909084$, $a^{(15)}_2 = 0.9090909091$, $a^{(20)}_2 = 0.9090909090$.

Comp. 2.7 The distribution of the grade of a randomly chosen employee at the beginning of week one is $a = [.50\ .25\ .15\ .10]$. Hence we study one employee with initial distribution a. Hence,

$$\mathsf{P}(X_3 = j) = \sum_{i=1}^{4} a_i p^{(3)}_{i,j}$$

is the probability that a randomly chosen employee ends up in grade j at the beginning of week 4. Hence, the expected number of employees in grade j is given by

$$100 * [.50\ .25\ .15\ .10] * P^3 = [47.40\ 28.01\ 14.67\ 9.92],$$

where P is from Example 2.6.

Comp. 2.8 Using the P matrix from Equation (2.7), the desired probability is obtained as $\mathsf{P}(X_3 = 0, X_2 = 0, X_1 = 1 | X_0 = 1)$

$$
\begin{aligned}
&= \mathsf{P}(X_3 = 0 | X_2 = 0)\mathsf{P}(X_2 = 0 | X_1 = 1)\mathsf{P}(X_1 = 1 | X_0 = 1) \\
&= p_{0,0}p_{1,0}p_{1,1} = (0.03)(0.02)(0.98) = 5.88 \times 10^{-4}.
\end{aligned}
$$

Comp. 2.9

a) Y_n = the number of machines "up" at the beginning of day n. State space is $S = \{0, 1, 2\}$. The second column of P contains the probabilities $p_{i,\ 1}$ $(i = 0, 1, 2)$.

$p_{0,\ 1} = \mathsf{P}(Y_{n+1} = 1 | Y_n = 0) =$
$\mathsf{P}($ 1 machine "up" at day $(n+1)|$ 2 machines "down" at day $n)$
$=$ 2*0.03*0.97 $=$ 0.0582,

$p_{1,\ 1} = \mathsf{P}(Y_{n+1} = 1 | Y_n = 1) =$
$\mathsf{P}($ 1 machine "up" at day $(n+1)|$ 1 machine "up" at day $n) =$
$\mathsf{P}($ machine "up" stays "up", machine "down" stays "down"$) +$
$\mathsf{P}($ machine "down" becomes "up", machine "up" goes "down"$) =$
$0.98 * 0.03 + 0.97 * 0.02 = 0.0488$,

$p_{2,\ 1} = \mathsf{P}(Y_{n+1} = 1 | Y_n = 2) =$
$\mathsf{P}($ 1 machine "up" at day $(n+1)|$ 2 machines "up" at day $n) =$
$\mathsf{P}($ 1 machine "up" goes "down", the other machine stays "up"$) =$
$0.98 * 0.02 * 2 = 0.0392$.

b) Using the P matrix from Equation (2.8), the desired probability is obtained as $\mathsf{P}(X_3 = 2, X_2 = 1, X_1 = 2 | X_0 = 2)$

$$
\begin{aligned}
&= \mathsf{P}(X_3 = 2 | X_2 = 1)\mathsf{P}(X_2 = 1 | X_1 = 2)\mathsf{P}(X_1 = 2 | X_0 = 2) \\
&= p_{1,2}p_{2,1}p_{2,2} = (0.9506)(0.0392)(0.9604) = 0.0358.
\end{aligned}
$$

Comp. 2.12

a) X_n is the number of PC's in stock at 8:00 a.m. Monday of the nth week. If there are 2 or more PC's in the store at 5:00 p.m. Friday of the nth week, then no more PC's will be ordered that weekend and then the number in stock at 8:00 a.m. Monday of the $(n+1)$th week is equal to the number in stock at 5:00 p.m. Friday of the nth week. Otherwise PC's are ordered that weekend and the number in stock at 8:00 a.m. Monday of the $(n+1)$th week is equal to 5. If the number in stock at 8:00 a.m. Monday is taken as state of the DTMC, then the state space is $S = \{2, 3, 4, 5\}$. The demand D_n in week n is Poisson distributed with an average demand of 3 PC's. The distribution is given in Table 2.1.

Only the probabilities in the second and the fourth row of P are explained below, the probabilities in the first and third row can be derived in an analogous way.
$p_{3,\ 2} = \mathsf{P}(X_{n+1} = 2|X_n = 3) = \mathsf{P}(D_n = 1) = 0.1494$,
$p_{3,\ 3} = \mathsf{P}(X_{n+1} = 3|X_n = 3) = \mathsf{P}(D_n = 0) = 0.0498$,
$p_{3,\ 4} = \mathsf{P}(X_{n+1} = 4|X_n = 3) = 0$,
$p_{3,\ 5} = \mathsf{P}(X_{n+1} = 5|X_n = 3) = \mathsf{P}(D_n \geq 2) = 0.8008$.

$p_{5,\ 2} = \mathsf{P}(X_{n+1} = 2|X_n = 5) = \mathsf{P}(D_n = 3) = 0.2240$,
$p_{5,\ 3} = \mathsf{P}(X_{n+1} = 3|X_n = 5) = \mathsf{P}(D_n = 2) = 0.2240$,
$p_{5,\ 4} = \mathsf{P}(X_{n+1} = 4|X_n = 5) = \mathsf{P}(D_n = 1) = 0.1494$,
$p_{5,\ 5} = \mathsf{P}(X_{n+1} = 5|X_n = 5) = \mathsf{P}(D_n \geq 4 \cup D_n = 0) = 0.0498 + 0.3528 = 0.4026$.

b) Using the P matrix of Example 2.4, the desired probability is obtained as $\mathsf{P}(X_4 = 3, X_3 = 5, X_2 = 2, X_1 = 4|X_0 = 5)$

$$= p_{5,4}p_{4,2}p_{2,5}p_{5,3} = (.1494)(.2240)(.9502)(.2240) = 0.0071.$$

Comp. 2.13 This implies that 4 or more PC's are sold during the previous week, hence the desired probability is

$$\mathsf{P}(D_n \geq 4) = 0.3528.$$

Comp. 2.16 Use the P matrix in Equation (2.10) to compute the following

$$M(51) = \begin{bmatrix} 10.1373 & 7.5681 & 4.5759 & 29.7188 \\ 9.2203 & 8.5956 & 4.5609 & 29.6232 \\ 9.2794 & 7.6673 & 5.5884 & 29.4649 \\ 9.2756 & 7.7264 & 4.6714 & 30.3266 \end{bmatrix}.$$

Thus, starting with full inventory, the store will have full inventories on the average 30.33 Mondays of the year.