Solutions to exercises: week 2

Comp. 2.19 a. irreducible, b. irreducible, c. reducible, d. reducible.

Comp. 2.20 a. aperiodic, b. periodic with period 4, c. periodic with period 2, d. periodic with period 3.

Comp. 2.21 a) The normalized balance equations of Theorem 2.5 are as follows:

$$[\pi_1\ \pi_2\ \pi_3\ \pi_4] = [\pi_1\ \pi_2\ \pi_3\ \pi_4] \cdot \begin{bmatrix} .10 & .30 & .20 & .40 \\ .10 & .30 & .40 & .20 \\ .30 & .10 & .10 & .50 \\ .15 & .25 & .35 & .25 \end{bmatrix},$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1.$$

Otherwise:

$$\begin{array}{lcllllllll} \pi_1 & = & 0.10\pi_1 & + & 0.10\pi_2 & + & 0.30\pi_3 & + & 0.15\pi_4, \\ \pi_2 & = & 0.30\pi_1 & + & 0.30\pi_2 & + & 0.10\pi_3 & + & 0.25\pi_4, \\ \pi_3 & = & 0.20\pi_1 & + & 0.40\pi_2 & + & 0.10\pi_3 & + & 0.35\pi_4, \\ \pi_4 & = & 0.40\pi_1 & + & 0.20\pi_2 & + & 0.50\pi_3 & + & 0.25\pi_4, \\ 1 & = & \pi_1 & + & \pi_2 & + & \pi_3 & + & \pi_4. \end{array}$$

b)
1. The DTMC is irreducible and aperiodic. Hence the limiting distribution exists, and is given by $\pi = [.1703\ .2297\ .2687\ .3313]$.
2. By Corollary 2.3, the stationary distribution is given by $\pi^* = \pi$.
3. By Theorem 2.7 or 2.9, the occupancy distribution is given by $\hat{\pi} = \pi^*$.

Comp. 2.24 The questions a) and b) are solved simultaneously. We solve the normalized balance equations of Theorem 2.5 "by hand".

1. The limiting distribution does not exist since the DTMC is periodic.
2. The stationary distribution π^* is given by the solution to (see Theorem 2.6)

$$[\pi_1^*\ \pi_2^*\ \pi_3^*\ \pi_4^*] = [\pi_1^*\ \pi_2^*\ \pi_3^*\ \pi_4^*] \cdot \begin{bmatrix} 0 & 0 & .40 & .60 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\pi_1^* + \pi_2^* + \pi_3^* + \pi_4^* = 1.$$

Otherwise:

$$\begin{array}{lclclclcl}
\pi_1^* & = & & & \pi_2^*, & & & & \\
\pi_2^* & = & & & & & \pi_3^* & + & \pi_4^*, \\
\pi_3^* & = & 0.40\pi_1^*, & & & & & & \\
\pi_4^* & = & 0.60\pi_1^*, & & & & & & \\
1 & = & \pi_1^* & + & \pi_2^* & + & \pi_3^* & + & \pi_4^*.
\end{array}$$

Solving "by hand", we get $\pi^* = (\frac{1}{3},\ \frac{1}{3},\ \frac{2}{15},\ \frac{1}{5})$.

3. By Theorem 2.7 or 2.9, the occupancy distribution is given by $\hat{\pi} = \pi^*$.

Comp. 2.25 Consider the DTMC $\{X_n, n \geq 0\}$ with the transition probability matrix as given in the solution to Computational Problem 2.5. Solve the normalized balance equation numerically to obtain

$$\pi = [.0393\ .0805\ .0960\ .0981\ .0981\ .0981\ .0980\ .0980\ .0980\ .0980\ .0979].$$

The DTMC is irreducible and aperiodic. Hence π is the limiting distribution, which also equals occupancy distribution $\hat{\pi}$. Hence, the long run fraction of time that the buffer is full is given by

$$\hat{\pi}_{10} = \pi_{10} = 0.0979.$$

The expected number of packets in the buffer in the long run is given as

$$\sum_{i=0}^{10} i\pi_i = 5.3686.$$

Comp. 2.26 Using the P matrix of Example 2.6, we can compute the corresponding limiting distribution as

$$\pi = [.2727\ .4545\ .1818\ .0909].$$

Since the hundred employees are independent of each other, the expected number of employees in grades $i = 1, 2, 3, 4$ in steady state is given by

$$100\pi = [27.27\ 45.45\ 18.18\ 9.09].$$

Comp. 2.33 Let $c(i)$ be the expected number of items finished by the end of the nth minute if the machine is in state i at the beginning of the nth minute. Referring to the solution of Conceptual Problem 2.10, we get $c = [0\ 0\ 2\ 2\ 0]'$. over the state space $S = \{1, 2, 3, 4, 5\}$.

a) If the machine is idle at the beginning of the first minute, i.e. $a^{(0)} = (1, 0, 0, 0, 0)$ then $a^{(4)} = a^{(0)}P^4$ gives the probabilitiy distribution at the beginning of the fifth minute. In this case $a^{(4)}$ equals the first row of the matrix P^4.
MAXIM gives $a^{(4)} = (0.0023, 0.0862, 0.0045, 0.0857, 0.8213)$.
The expected number of visits to state $i \in S$ during the fifth minute equals $1{*}a_i^{(4)}+0{*}(1-a_i^{(4)}) = a_i^{(4)}$. Hence the expected number of processed items in the fifth minute equals $\sum_{i=1}^{5} a_i^{(4)}c(i) = 2 * (0.0045 + 0.0857) = 0.1804$ items.

b) In an analogous way the expected number of items processed in the fifth and sixth minute is equal to
$\sum_{i=1}^{5} (a_i^{(4)} + a_i^{(5)})c(i) = 2 * (0.0045 + 0.0857 + 0.0411 + 0.7802) = 1.823$ items.

This last number 1.823 items is also equal to the sum of the number of items processed in the fifth minute and the expected number of items for which the processing starts in the fifth minute. This sum is 0.1804 + 2 * 0.8213 = 1.823 items.

c) We are interested in computing $g(1, 9)$. Using Theorem 2.11, we get

$$g(9) = M(9) * c = [7.3152\ 7.8848\ 9.3152\ 9.8848\ 9.3152]'.$$

Thus the machine produces 7.3152 items on the average in the first 10 minutes if it idle and the bin is empty at time 0.

Comp. 2.37 a) In state i (number of PC's in stock at 8:00 a.m. Monday) the storage cost during a week is -50 $*\ i$. Buying cost (-1500) and selling revenue (1750) are considered in the week in which a PC is sold. The demand in a week is D_n, the expected number of PC's sold is $\mathsf{E}(\min(i, D_n))$. Hence the net revenue becomes $c(i) = -50 * i + (1750 - 1500) * \mathsf{E}(\min(i, D_n))$ $(i = 2, 3, 4, 5)$.

b) Starting with $a^{(0)} = (0, 0, 0, 1)$ the expected revenue in the third week is $\sum_{i=2}^{5} a_i^{(2)} c(i) = 440.76$.

c) The long run expected cost per week of operating the system is given by (using Theorem 2.12) πc where c is given in Example 2.27 as

$$c = [337.76\ 431.96\ 470.15\ 466.325]'.$$

Using the transition probability matrix P given in Equation (2.10), we compute the limiting distribution as given below

$$\pi = [.1812\ .1504\ .0908\ .5775].$$

Hence, it costs \$438.20 per week to operate this system.

Comp. 2.40

a) X_n is the number of packets in the buffer at the end of the nth time slot. If $X_n = 0$ then the expected number of transmitted items in the $(n+1)$th time slot is 0, if $X_n \geq 1$ then the expected number of transmitted packets in the $(n+1)$th time slot is 1, so $c = (0, 1, 1, 1, 1, 1, 1, 1)$.

b) The expected number of transmitted packets in the fifth minute, if the buffer is full at the beginning of the first minute, is (of course) equal to 1.
The expected number of transmitted packets in the eighth minute, if the buffer is full at the beginning of the first minute, is equal to $\sum_{i=0}^{7} a_i^{(7)} c(i) = 0 * 0.0009 + 1 * 0.9991 = 0.9991$.

c) We are interested in $g(3, 9)$, the third component of the vector $g(9) = M(9) * c$. With MAXIM follows $g(3, 9) = 9.4296$ transmitted packets in the first 10 time slots.

d) Let $\{X_n, n \geq 0\}$ be the DTMC of Example 2.8. The limiting distribution of the DTMC is given in Example 2.24 to be

$$\pi = [.0682\ .1172\ .1331\ .1361\ .1363\ .1364\ .1363\ .1364].$$

Let $c(i)$ be the expected number of packets transmitted during the nth slot if the DTMC is in state i a the beginning of the n the slot. We have

$$c = [0\ 1\ 1\ 1\ 1\ 1\ 1\ 1].'$$

From Theorem 2.12, the expected number of packets transmitted per unit time is given by

$$\pi c = 0.9318.$$

The expected number of packets transmitted per unit time plus the expected number of packets lost per unit time must be equal to the expected number of packets that arrive to the system per unit time (in this case it is 1, since the distribution of the number of packets that come to the system in a time slot is a $P(1)$ random variable). In Example 2.29, the fraction of packets lost is computed to be .0682. Now, .0682 + .9318 = 1, thus verifying the above assertion.

Handout section 1

Exercise 2 (a) End classes $E_1 = \{4,5,6,7\}$, $E_2 = \{8\}$, $E_3 = \{9,10\}$. $C = \{1,2,3\}$.

(b) The limiting distribution over $S = \{1,2,3,4,5,6,7,8,9,10\}$ is $\pi = (0,0,0,\frac{14}{255},\frac{14}{85},\frac{14}{85},\frac{7}{85},\frac{1}{6},\frac{11}{150},\frac{22}{75})$.

(c) We have

$$P^\infty = \begin{pmatrix} 0 & 0 & 0 & \frac{8}{255} & \frac{8}{85} & \frac{8}{85} & \frac{4}{85} & \frac{1}{6} & \frac{17}{150} & \frac{34}{75} \\ 0 & 0 & 0 & \frac{14}{255} & \frac{14}{85} & \frac{14}{85} & \frac{7}{85} & \frac{1}{6} & \frac{11}{150} & \frac{22}{75} \\ 0 & 0 & 0 & \frac{2}{51} & \frac{2}{17} & \frac{2}{17} & \frac{1}{17} & \frac{1}{3} & \frac{1}{15} & \frac{4}{15} \\ 0 & 0 & 0 & \frac{2}{17} & \frac{6}{17} & \frac{6}{17} & \frac{3}{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{17} & \frac{6}{17} & \frac{6}{17} & \frac{3}{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{17} & \frac{6}{17} & \frac{6}{17} & \frac{3}{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{17} & \frac{6}{17} & \frac{6}{17} & \frac{3}{17} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \end{pmatrix}.$$

Exercise 3 (a) The state of a student is registrated at the end of a week. During the presence of a student in the training, this is done immediately before taking the exam belonging to the course of that week. The state space is $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ with

1: (C_1, f) = Course 1, first trial,
2: (C_1, s) = Course 1, second trial,
3: (C_2, f) = Course 2, first trial,
4: (C_2, s) = Course 2, second trial,
5: (C_3, f) = Course 3, first trial,
6: (C_3, s) = Course 3, second trial,
7: "left training without diploma",
8: "left training with diploma".

The matrix of transition probabilities P at S is:

$$P = \begin{pmatrix} 0 & \frac{15}{100} & \frac{7}{10} & 0 & 0 & 0 & \frac{15}{100} & 0 \\ 0 & 0 & \frac{5}{10} & 0 & 0 & 0 & \frac{5}{10} & 0 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{8}{10} & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 & \frac{6}{10} & 0 & \frac{4}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{100} & \frac{5}{100} & \frac{9}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{10} & \frac{7}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) the possible paths through the state space S are

$1 \to 2 \to 3 \to 5 \to 7$,
$1 \to 2 \to 3 \to 4 \to 7$,
$1 \to 3 \to 4 \to 5 \to 7$,
$1 \to 3 \to 5 \to 6 \to 7$.

The total probability is $\dfrac{33}{2000} = 0.0165$.

(c) The probabilities are $\dfrac{249271}{400000} = 0.6232$ en $\dfrac{150729}{400000} = 0.3768$

Exercise 4 (a) The state of an employee is registrated at the end of a calendar year n. For an employee in this job, the state is the salary level in the salary scale for next year $(n+1)$. For an employee who has left this job, the state is characterised by "LC = Left Company" or by "AF = Another Function". In the model is assumed that new employees in this job always start on January 1 of a year on salary level 1.
The state space is $S = \{1, 2, 3, 4, AF, LC\}$.

The matrix of transition probabilities P at this S is:

$$P = \begin{pmatrix} 0 & 0.8 & 0 & 0 & 0 & 0.2 \\ 0 & 0.3 & 0.5 & 0 & 0 & 0.2 \\ 0 & 0 & 0.4 & 0.3 & 0.1 & 0.2 \\ 0 & 0 & 0 & 0.5 & 0.4 & 0.1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) There are several paths to state AF after 4 years.

year 1	year 2	year 3	year 4	year 5
1	2	2	3	AF
1	2	3	3	AF
1	2	3	4	AF
1	2	3	AF	AF

The total probability on these paths is: $(0.8)(0.3)(0.5)(0.1) + (0.8)(0.5)(0.4)(0.1) + (0.8)(0.5)(0.3)(0.4) + (0.8)(0.5)(0.1)(1) = 0.116$.

(c) The probability is $\dfrac{71}{105} = 0.6762$.
The percentage is 32.38%.