# Solutions to exercises: week 3

**Comp. 2.45** a) The set of equations is

$$
\begin{aligned}
t_0 &= 0,\\
t_1 &= 1 + 0.3679t_0 + 0.3679t_1 + 0.1839t_2 + 0.0613t_3\\
&\quad + 0.0153t_4 + 0.0031t_5 + 0.0005t_6 + 0.0001t_7,\\
t_2 &= 1 + 0.3679t_1 + 0.3679t_2 + 0.1839t_3\\
&\quad + 0.0613t_4 + 0.0153t_5 + 0.0031t_6 + 0.0006t_7,\\
t_3 &= 1 + 0.3679t_2 + 0.3679t_3\\
&\quad + 0.1839t_4 + 0.0613t_5 + 0.0153t_6 + 0.0037t_7,\\
t_4 &= 1 + 0.3679t_3\\
&\quad + 0.3679t_4 + 0.1839t_5 + 0.0613t_6 + 0.0190t_7,\\
t_5 &= 1\\
&\quad + 0.3679t_4 + 0.3679t_5 + 0.1839t_6 + 0.0803t_7,\\
t_6 &= 1\\
&\quad + 0.3679t_5 + 0.3679t_6 + 0.2642t_7,\\
t_7 &= 1\\
&\quad + 0.3679t_6 + 0.6321t_7.
\end{aligned}
$$

b) Let $\{X_n, n \geq 0\}$ be the DTMC in Example 2.12. Let $T = \min\{n \geq 0 : X_n = 0\}$ and $m_i = \mathsf{E}(T|X_0 = i)$. We are asked to compute $m_7$. Using the $P$ matrix from Equation (2.29) in Theorem 2.13, we get $m_7 = 60.7056$.

## Handout section 2

**Exercise 1** (a) -

(b) i. 13, 5, 2.1

ii. 18 trainees, 1 junior mechanic, 0 senior mechanics

**Exercise 2** (a) The state space is $S = \{1, 2, 3, 4, LIC\}$, where $i$ ($i = 1, 2, 3, 4$) stands for premium level i and $LIC$ stands for "Left Insurance Company".

The matrix of transition probabilities $P$ at this $S$ is:

$$
P = \begin{pmatrix}
0.5 & 0.3 & 0 & 0 & 0.2\\
0.4 & 0 & 0.5 & 0 & 0.1\\
0 & 0.25 & 0 & 0.7 & 0.05\\
0 & 0 & 0.125 & 0.85 & 0.025\\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

(b) The expected numbers on the four premium levels are: 31500, 24500, 39375, 58750.

(c) The long-run expected numbers on the four premium levels are: $\approx 23478, \approx 29348, \approx 49217, \approx 236348$.

**Comp. 3.4** The lifetime of the satellite is given by $\max(T_1, T_2)$ where $T_1, T_2 \sim Exp(\lambda)$ (with $\lambda = .2$) random variables. The cdf is given by

$$\begin{aligned} \mathsf{P}(\max(T_1, T_2) < x) &= \mathsf{P}(T_1 < x, T_2 < x) \\ &= \mathsf{P}(T_1 < x)\mathsf{P}(T_2 < x) = (1 - e^{-0.2x})^2. \end{aligned}$$

The expected lifetime can be found either by integrating the complementary cdf or by using the result of Conceptual Problem 3.12. Here, we show the second approach. Let$X_2$ be the time interval during which both computers are working, and $X_1$ be the time interval during which only one computer is working. Thus the first failure occurs at time $X_2$ and the second at time $X_2 + X_1$. Hence, $\max\{T_1, T_2\} = X_2 + X_1$. Now, $X_2 = \min(T_1, T_2)$, hence $X_2 \sim exp(2\lambda)$. Also, $X_1$ is the remaining lifetime of the computer that did not fail at time $X_2$. From the memoryless property of the exponential distribution it follows that this remaining lifetime is $exp(\lambda)$. Hence

$$\mathsf{E}(\max\{T_1, T_2\}) = \mathsf{E}(X_2 + X_1) = \frac{1}{2\lambda} + \frac{1}{\lambda} = 7.5 \text{ years}.$$

**Comp. 3.9** The remaining service times at the tellers are exponentially distributed random variables with mean 5 minutes, i.e., with parameter .2 $(\text{min})^{-1}$. Thus, the amount of time the customer has to wait is the minimum of three $Exp(0.2)$ random variables, which is $Exp(0.6)$. The mean is 1/.6 = 1.667 minutes.

**Comp. 3.10** The expected waiting time at the ATM is 2 minutes, while the expected waiting time inside is 1.667 minutes (from Computational Problem 3.9). Thus he should wait inside.

**Concep. 4.1** The state space of $\{X(t), t \geq 0\}$ is $\{0, 1, 2\}$. In state 2, both cables are up and each is subject to a load of $L/2$. Hence the lifetime of each cable is $Exp(\lambda L/2)$. The process jumps from state 2 to 1 when either cable breaks, hence $r_{2,1} = \lambda L/2 + \lambda L/2 = \lambda L$. In state 1, only one cable is working under a total load of $L$. Hence its lifetime is $Exp(\lambda L)$. Once it breaks, the process moves to state 0. Hence $r_{1,0} = \lambda L$. Once the process enters state 0, it stays there permanently. Hence $\{X(t), t \geq 0\}$ is a CTMC on state space $S = \{0, 1, 2\}$ with rate matrix

$$R = \begin{bmatrix} 0 & 0 & 0 \\ \lambda L & 0 & 0 \\ 0 & \lambda L & 0 \end{bmatrix}.$$

**Concep. 4.2** The state space is $\{0, 1, \cdots, 10\}$. In state 0, the process jumps to 1 when there is an arrival, which occurs with rate $\lambda$. Hence $r_{0,1} = \lambda$. In state $i$, $1 \leq i \leq 3$, one customer is being served. The process jumps to state $i + 1$ with rate $\lambda$, and to state $i - 1$ with rate $\mu$. Hence, $r_{i,i+1} = \lambda, \quad r_{i,i-1} = \mu$. In state $i$, $4 \leq i \leq 7$, two customers are in service. The process jumps to state $i + 1$ with rate $\lambda$, and to state $i - 1$ with rate $2\mu$, since any one of the two customers in service may complete service to trigger this transition. Proceeding in this fashion we see that $\{X(t), t \geq 0\}$ is a Birth and Death Process with birth parameters

$$\lambda_i = \lambda, \quad 0 \leq i \leq 9,$$

and death parameters

$$\mu_i = \begin{cases} \mu & \text{for } 1 \leq i \leq 3 \\ 2\mu & \text{for } 4 \leq i \leq 7 \\ 3\mu & \text{for } 8 \leq i \leq 10. \end{cases}$$

**Concep. 4.4** Let $A(t)$ be the set of busy servers at time $t$, $A(t) \subset \{1, 2, 3\}$. Let

$$X(t) = \begin{cases} 0 & \text{if } A(t) = \emptyset, \\ 1 & \text{if } A(t) = \{1\}, \\ 2 & \text{if } A(t) = \{2\}, \\ 3 & \text{if } A(t) = \{3\}, \\ 4 & \text{if } A(t) = \{1, 2\}, \\ 5 & \text{if } A(t) = \{1, 3\}, \\ 6 & \text{if } A(t) = \{2, 3\}, \\ 7 & \text{if } A(t) = \{1, 2, 3\}. \end{cases}$$

Then, the usual triggering event analysis in each state shows that $\{X(t), t \geq 0\}$ is a CTMC with the following rate matrix:

$$R = \begin{bmatrix} 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_1 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ \mu_3 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & \mu_2 & \mu_1 & 0 & 0 & 0 & 0 & \lambda \\ 0 & \mu_3 & 0 & \mu_1 & 0 & 0 & 0 & \lambda \\ 0 & 0 & \mu_3 & \mu_2 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & \mu_3 & \mu_2 & \mu_1 & 0 \end{bmatrix}.$$

**Concep. 4.5** The state space of $\{X(t), t \geq 0\}$ is $\{0, 1, \cdots, K\}$. When it is in state $i$, $0 \leq i \leq M$, all customers are allowed to enter, hence it moves to state $i+1$ with rate $\lambda_1 + \lambda_2$. When it is in state $i$, $M < i < K$, only customers of type 1 are allowed to enter, hence it moves to state $i+1$ with rate $\lambda_1$. In state $i$, $1 \leq i \leq K$, one customer is in service, whose service time is $Exp(\mu)$, regardless of its type. Hence the system moves to state $i-1$ with rate $\mu$. Thus, $\{X(t), t \geq 0\}$ is a birth and death process on state space $\{0, 1, \cdots, K\}$, with death parameters

$$\mu_i = \mu, \quad 1 \leq i \leq K,$$

and birth parameters

$$\lambda_i = \begin{cases} \lambda_1 + \lambda_2 & \text{for } 0 \leq i \leq M \\ \lambda_1 & \text{for } M+1 \leq i < K \\ 0 & \text{for } i = K. \end{cases}$$

**Concep. 4.8** The state of the system $A(t)$ is a vector giving the state of the three machines at time $t$. Using $w$ for "working", $b$ for "blocked", and $i$ for "idle", we get the following definition of $X(t)$,

$$X(t) = \begin{cases} 1 & \text{if } A(t) = \{w, w, w\} \\ 2 & \text{if } A(t) = \{w, b, w\} \\ 3 & \text{if } A(t) = \{w, i, w\} \\ 4 & \text{if } A(t) = \{w, w, i\} \\ 5 & \text{if } A(t) = \{w, i, i\} \\ 6 & \text{if } A(t) = \{b, w, w\} \\ 7 & \text{if } A(t) = \{b, b, w\} \\ 8 & \text{if } A(t) = \{b, w, i\}. \end{cases}$$

Then, performing the usual analysis of transition triggering events, we see that $\{X(t), t \geq 0\}$ is a CTMC with the following rate matrix:

$$R = \begin{bmatrix} 0 & \mu_2 & 0 & \mu_3 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & 0 & \mu_1 & 0 \\ \mu_1 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & \mu_3 \\ \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Concep. 4.9** Let $X(t)$ be the number of customers in the system at time $t$ if the system is up, and let $X(t) = d$ if the system is down at time $t$. Thus the state-space is $S = \{d, 0, 1, \ldots, K\}$. Performing the usual analysis of transition triggering events, we see that $\{X(t), t \geq 0\}$ is a CTMC with transition rates

$$\begin{aligned} r_{i,d} &= \theta, & \text{for } 0 \leq i \leq K, \\ r_{d,0} &= \alpha, & \\ r_{i,i-1} &= \mu, & \text{for } 1 \leq i \leq K, \\ r_{i,i+1} &= \lambda, & \text{for } 0 \leq i \leq K-1. \end{aligned}$$

All other transition rates are zero.